Invariant and Coinvariant Spaces for the Algebra of Symmetric Polynomials in Non-Commuting Variables

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Invariant and coinvariant spaces for the algebra of symmetric polynomials in non-commuting variables

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Abstract

We analyze the structure of the algebra $\mathbb{K}\langle x \rangle^{S_n}$ of symmetric polynomials in non-commuting variables in so far as it relates to $\mathbb{K}[x]^{S_n}$, its commutative counterpart. Using the “place-action” of the symmetric group, we are able to realize the latter as the invariant polynomials inside the former. We discover a tensor product decomposition of $\mathbb{K}\langle x \rangle^{S_n}$ analogous to the classical theorems of Chevalley, Shephard-Todd on finite reflection groups.

Résumé. Nous analysons la structure de l’algèbre $\mathbb{K}\langle x \rangle^{S_n}$ des polynômes symétriques en des variables non-commutatives pour obtenir des analogues des résultats classiques concernant la structure de l’anneau $\mathbb{K}[x]^{S_n}$ des polynômes symétriques en des variables commutatives. Plus précisément, au moyen de “l’action par positions”, on réalise $\mathbb{K}[x]^{S_n}$ comme sous-module de $\mathbb{K}\langle x \rangle^{S_n}$. On découvre alors une nouvelle décomposition de $\mathbb{K}\langle x \rangle^{S_n}$ comme produit tensorial, obtenant ainsi un analogues des théorèmes classiques de Chevalley et Shephard-Todd.

1 Introduction

One of the more striking results of invariant theory is certainly the following: if $W$ is a finite group of $n \times n$ matrices (over some field $\mathbb{K}$ containing $\mathbb{Q}$), then there is a $W$-module decomposition of the polynomial ring $S = \mathbb{K}[x]$, in variables $x = \{x_1, x_2, \ldots, x_n\}$, as a tensor product

\[ S \simeq S_W \otimes S^W \] (1)

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if and only if \( W \) is a group generated by (pseudo) reflections. As usual, \( S \) is afforded a natural \( W \)-module structure by considering it as the symmetric space on the defining vector space \( X^* \) for \( W \), e.g., \( w \cdot f(x) = f(x \cdot w) \). It is customary to denote by \( S^W \) the ring of \( W \)-invariant polynomials for this action. To finish parsing (1), recall that \( S_W \) stands for the **coinvariant space**, i.e., the \( W \)-module

\[
S_W := S/\langle S_W^+ \rangle
\]

defined as the quotient of \( S \) by the ideal generated by constant-term free \( W \)-invariant polynomials. We give \( S \) an \( N \)-grading by degree in the variables \( x \). Since the \( W \)-action on \( S \) preserves degrees, both \( S^W \) and \( S_W \) inherit a grading from the one on \( S \), and (1) is an isomorphism of graded \( W \)-modules. One of the motivations behind the quotient in (2) is to eliminate trivially redundant copies of irreducible \( W \)-modules inside \( S \). Indeed, if \( V \) is such a module and \( f \) is any \( W \)-invariant polynomial with no constant term, then \( Vf \) is an isomorphic copy of \( V \) living within \( \langle S_W^+ \rangle \). Thus, the coinvariant space \( S_W \) is the more interesting part of the story.

The context for the present paper is the algebra \( T = \mathbb{k}\langle x \rangle \) of noncommutative polynomials, with \( W \)-module structure on \( T \) obtained by considering it as the tensor space on the defining space \( X^* \) for \( W \). In the special case when \( W \) is the symmetric group \( \mathfrak{S}_n \), we elucidate a relationship between the space \( S^W \) and the subalgebra \( T^W \) of \( W \)-invariants in \( T \). The subalgebra \( T^W \) was first studied in [4, 20] with the aim of obtaining noncommutative analogs of classical results concerning symmetric function theory. Recent work in [2, 15] has extended a large part of the story surrounding (1) to this noncommutative context. In particular, there is an explicit \( \mathfrak{S}_n \)-module decomposition of the form \( T \simeq T_{\mathfrak{S}_n} \otimes T^{\mathfrak{S}_n} \) [2, Theorem 8.7]. See [7] for a survey of other results in noncommutative invariant theory.

By contrast, our work proceeds in a somewhat complementary direction. We consider \( N = T^{\mathfrak{S}_n} \) as a tower of \( \mathfrak{S}_d \)-modules under the “place-action” and realize \( S^{\mathfrak{S}_n} \) inside \( N \) as a subspace \( \Lambda \) of invariants for this action. This leads to a decomposition of \( N \) analogous to (1). More explicitly, our main result is as follows.

**Theorem 1.** There is an explicitly constructed subspace \( \mathcal{C} \) of \( N \) so that \( \mathcal{C} \) and the place-action invariants \( \Lambda \) exhibit a graded vector space isomorphism

\[
N \simeq \mathcal{C} \otimes \Lambda.
\]

An analogous result holds in the case \( |x| = \infty \). An immediate corollary in either case is the Hilbert series formula

\[
\text{Hilb}_t(\mathcal{C}) = \text{Hilb}_t(N) \prod_{i=1}^{\lfloor x \rfloor} (1 - t^i).
\]

Here, the **Hilbert series** of a graded space \( V = \bigoplus_{d \geq 0} V_d \) is the formal power series defined as

\[
\text{Hilb}_t(V) = \sum_{d \geq 0} \dim V_d t^d.
\]
where \( V_d \) is the **homogeneous degree \( d \) component** of \( V \). The fact that (4) expands as a series in \( \mathbb{N}[t] \) is not at all obvious, as one may check that the Hilbert series of \( \mathbb{N} \) is

\[
\text{Hilb}_t(\mathbb{N}) = 1 + \sum_{k=1}^{\lfloor |x| \rfloor} \frac{t^k}{(1-t)(1-2t) \cdots (1-kt)}.
\] (5)

In Sections 2 and 3, we recall the relevant structural features of \( S \) and \( T \). Section 4 describes the place-action structure of \( T \) and the original motivation for our work. Our main results are proven in Sections 5 and 6. We underline that the harder part of our work lies in working out the case \( |x| < \infty \). This is accomplished in Section 6. If we restrict ourselves to the case \( |x| = \infty \), both \( \mathbb{N} \) and \( \Lambda \) become Hopf algebras and our results are then consequences of a general theorem of Blattner, Cohen and Montgomery. As we will see in Section 5, stronger results hold in this simpler context. For example, (4) may be refined to a statement about “shape” enumeration.

## 2 The algebra \( S^\mathfrak{S} \) of symmetric functions

### 2.1 Vector space structure of \( S^\mathfrak{S} \)

We specialize our introductory discussion to the group \( W = \mathfrak{S}_n \) of permutation matrices (writing \( |x| = n \)). The action on \( S = \mathbb{K}[x] \) is simply the **permutation action** \( \sigma \cdot x_i = x_{\sigma(i)} \) and \( S^\mathfrak{S}_n \) comprises the familiar symmetric polynomials. We suppress \( n \) in the notation and denote the subring of symmetric polynomials by \( S^\mathfrak{S} \). (Note that upon sending \( n \) to \( \infty \), the elements of \( S^\mathfrak{S} \) become formal series in \( \mathbb{K}[x] \) of bounded degree; we call both finite and infinite versions “functions” in what follows to affect a uniform discussion.) A monomial in \( S \) of degree \( d \) may be written as follows: given an \( r \)-subset \( y = \{y_1, y_2, \ldots, y_r\} \) of \( x \) and a **composition** of \( d \) into \( r \) parts, \( \mathbf{a} = (a_1, a_2, \ldots, a_r) \) \((a_i > 0)\), we write \( y^\mathbf{a} \) for \( y_1^{a_1} y_2^{a_2} \cdots y_r^{a_r} \).

We assume that the variables \( y_i \) are naturally ordered, so that whenever \( y_i = x_j \) and \( y_{i+1} = x_k \) we have \( j < k \). Reordering the entries of a composition \( \mathbf{a} \) in decreasing order results in a partition \( \lambda(\mathbf{a}) \) called the **shape** of \( \mathbf{a} \). Summing over monomials \( y^\mathbf{a} \) with the same shape leads to the monomial symmetric function

\[
m_\mu = m_\mu(x) := \sum_{\lambda(\mathbf{a})=\mu} y^\mathbf{a}.
\]

Letting \( \mu = (\mu_1, \mu_2, \ldots, \mu_r) \) run over all partitions of \( d = |\mu| = \mu_1 + \mu_2 + \cdots + \mu_r \) gives a basis for \( S^\mathfrak{S}_d \). As usual, we set \( m_0 := 1 \) and agree that \( m_\mu = 0 \) if \( \mu \) has too many parts (i.e., \( n < r \)).

### 2.2 Dimension enumeration

A fundamental result in the invariant theory of \( \mathfrak{S}_n \) is that \( S^\mathfrak{S} \) is generated by a family \( \{ f_k \}_{1 \leq k \leq n} \) of algebraically independent symmetric functions, having respective degrees
\( \deg f_k = k. \) (One may choose \( \{m_k\}_{1 \leq k \leq n} \) for such a family.) It follows that the Hilbert series of \( S^e \) is

\[
\text{Hilb}_t(S^e) = \prod_{i=1}^{n} \frac{1}{1 - t^i}.
\]

(6)

Recalling that the Hilbert series of \( S \) is \( (1 - t)^{-n} \), we see from (1) and (6) that the Hilbert series for the coinvariant space \( S^e \) is the well-known \( t \)-analog of \( n! \):

\[
\prod_{i=1}^{n} \frac{1 - t^i}{1 - t} = \prod_{i=1}^{n} (1 + t + \cdots + t^{i-1}).
\]

(7)

In particular, contrary to the situation in (4), the series \( \text{Hilb}_t(S)/\text{Hilb}_t(S^e) \) in \( \mathbb{Q}[t] \) obviously belongs to \( \mathbb{N}[t] \).

\section{Algebra and coalgebra structures of \( S^e \)}

Given partitions \( \mu \) and \( \nu \), there is an explicit multiplication rule for computing the product \( m_\mu \cdot m_\nu \). In lieu of giving the formula, see [2, §4.1], we simply give an example

\[
m_{21} \cdot m_{11} = 3 m_{2111} + 2 m_{221} + 2 m_{311} + m_{32}
\]

(8)

and highlight two features relevant to the coming discussion.

First, we note that if \( n < 4 \), then the first term is equal to zero. However, if \( n \) is sufficiently large then analogs of this term always appear with positive integer coefficients. If \( \mu = (\mu_1, \mu_2, \ldots, \mu_r) \) and \( \nu = (\nu_1, \nu_2, \ldots, \nu_s) \) with \( r \leq s \), then the partition indexing the left-most term in \( m_\mu m_\nu \) is denoted by \( \mu \cup \nu \) and is given by sorting the list \( (\mu_1, \ldots, \mu_r, \nu_1, \ldots, \nu_s) \) in increasing order; the right-most term is indexed by \( \mu + \nu := (\mu_1 + \nu_1, \ldots, \mu_r + \nu_r, \nu_{r+1}, \ldots, \nu_s) \). Taking \( \mu = 31 \) and \( \nu = 221 \), we would have \( \mu \cup \nu = 32211 \) and \( \mu + \nu = 531 \).

Second, we point out that the leftmost term (indexed by \( \mu \cup \nu \)) is indeed a leading term in the following sense. An important partial order on partitions takes

\[
\lambda \leq \mu \quad \text{iff} \quad \sum_{i=1}^{k} \lambda_i \leq \sum_{i=1}^{k} \mu_i \quad \text{for all} \quad k.
\]

With this ordering, \( \mu \cup \nu \) is the least partition occuring with nonzero coefficient in the product of \( m_\mu m_\nu \). That is, \( S^e \) is \textbf{shape-filtered}: \( (S^e)_\lambda \cdot (S^e)_\mu \subseteq \bigoplus_{\lambda \cup \mu = \nu} (S^e)_\nu \). Here \( (S^e)_\lambda \) denotes the subspace of \( S^e \) indexed by partitions of shape \( \lambda \) (the linear span of \( m_\lambda \)), which we point out in preparation for the noncommutative analog.

The ring \( S^e \) is afforded a coalgebra structure with counit \( \varepsilon : S^e \rightarrow \mathbb{K} \) and coproduct \( \Delta : S^e_{d} \rightarrow \bigoplus_{k=0}^{d} S^e_{k} \otimes S^e_{d-k} \) given, respectively, by

\[
\varepsilon(m_\mu) = \delta_{\mu,0} \quad \text{and} \quad \Delta(m_\nu) = \sum_{\lambda \cup \mu = \nu} m_\lambda \otimes m_\mu.
\]

If \( |x| = \infty \), \( \Delta \) and \( \varepsilon \) are algebra maps, making \( S^e \) a graded connected Hopf algebra.
3 The algebra $\mathcal{N}$ of noncommutative symmetric functions

3.1 Vector space structure of $\mathcal{N}$

Suppose now that $\mathbf{x}$ denotes a set of non-commuting variables. The algebra $T = \mathbb{K}\langle \mathbf{x} \rangle$ of noncommutative polynomials is graded by degree. A degree $d$ noncommutative monomial $z \in T_d$ is simply a length $d$ “word”:

$$z = z_1 z_2 \cdots z_d, \quad \text{with each } z_i \in \mathbf{x}.$$

In other terms, $z$ is a function $z : [d] \to \mathbf{x}$, with $[d]$ denoting the set $\{1, 2, \ldots, d\}$. The permutation-action on $\mathbf{x}$ clearly extends to $T$, giving rise to the subspace $\mathcal{N} = T^\mathfrak{S}$ of noncommutative $\mathfrak{S}$-invariants. With the aim of describing a linear basis for the homogeneous component $\mathcal{N}_d$, we next introduce set partitions of $[d]$ and the type of a monomial $z : [d] \to \mathbf{x}$. Let $A = \{A_1, A_2, \ldots, A_r\}$ be a set of subsets of $[d]$. Say $A$ is a set partition of $[d]$, written $A \vdash [d]$, iff $A_1 \cup A_2 \cup \ldots \cup A_r = [d]$, $A_i \neq \emptyset$ ($\forall i$), and $A_i \cap A_j = \emptyset$ ($\forall i \neq j$). The type $\tau(z)$ of a degree $d$ monomial $z : [d] \to \mathbf{x}$ is the set partition

$$\tau(z) := \{z^{-1}(x) : x \in \mathbf{x}\} \setminus \{\emptyset\} \quad \text{of} \quad [d],$$

whose parts are the non-empty fibers of the function $z$. For instance,

$$\tau(x_1 x_3 x_5 x_8) = \{\{1, 3\}, \{2, 5\}, \{4\}\}.$$

Note that the type of a monomial is a set partition with at most $n$ parts. In what follows, we lighten the heavy notation for set partitions, writing, e.g., the set partition $\{\{1, 3\}, \{2, 5\}, \{4\}\}$ as $13254$. We also always order the parts in increasing order of their minimum elements. The shape $\lambda(A)$ of a set partition $A = \{A_1, A_2, \ldots, A_r\}$ is the (integer) partition $\lambda(|A_1|, |A_2|, \ldots, |A_r|)$ obtained by sorting the part sizes of $A$ in increasing order, and its length $\ell(A)$ is its number of parts ($r$). Observing that the permutation-action is type preserving, we are led to index the monomial linear basis for the space $\mathcal{N}_d$ by set partitions:

$$m_A = m_A(\mathbf{x}) := \sum_{\tau(z) = A, \ z \in \mathbf{x}^{[d]}} z.$$

For example, with $n = 2$, we have $m_1 = x_1 + x_2$, $m_{12} = x_1^2 + x_2^2$, $m_{123} = x_1^3 + x_2^3$, $m_{123} = x_1^2x_2 + x_2^2x_1$, $m_{132} = x_1x_2x_1 + x_2x_1x_2$, $m_{12} = 0$, and so on. (We set $m_{\emptyset} := 1$, taking $\emptyset$ as the unique set partition of the empty set, and we agree that $m_A = 0$ if $A$ is a set partition with more than $n$ parts.)

3.2 Dimension enumeration and shape grading

Above, we determined that $\dim \mathcal{N}_d$ is the number of set partitions of $d$ into at most $n$ parts. These are counted by the (length restricted) Bell numbers $B_d^{(n)}$. Consequently,
(5) follows from the fact that its right-hand side is the ordinary generating function for length restricted Bell numbers. See [10, §2]. We next highlight a finer enumeration, where we grade \( N \) by shape rather than degree.

For each partition \( \mu \), we may consider the subspace \( N_\mu \) spanned by those \( m_A \) for which \( \lambda(A) = \mu \). This results in a direct sum decomposition \( N_d = \bigoplus_{\mu | d} N_\mu \). A simple dimension description for \( N_d \) takes the form of a shape Hilbert series in the following manner.

View commuting variables \( q_i \) as marking parts of size \( i \) and set \( q_\mu := q_{\mu_1} q_{\mu_2} \cdots q_{\mu_r} \). Then

\[
\text{Hilb}_q(N_d) = \sum_{\mu | d} \dim N_\mu q_\mu = \sum_{A | [d]} q_{\lambda(A)}. \tag{9}
\]

Here, \( q_\mu \) is a marker for set partitions of shape \( \lambda(A) = \mu \) and the sum is over all partitions into at most \( n \) parts. Such a shape grading also makes sense for \( S_d^\text{SS} \). Summing over all \( d \geq 0 \) and all \( \mu \), we get

\[
\text{Hilb}_q(S_d^\text{SS}) = \sum_\mu q_\mu = \prod_{i \geq 1} \frac{1}{1 - q_i}. \tag{10}
\]

Using classical combinatorial arguments, one finds the enumerating polynomials \( \text{Hilb}_q(N_d) \) are naturally collected in the exponential generating function

\[
\sum_{d=0}^\infty \text{Hilb}_q(N_d) \frac{t^d}{d!} = \sum_{m=0}^n \frac{1}{m!} \left( \sum_{k=1}^\infty q_k \frac{t^k}{k!} \right)^m. \tag{11}
\]

See [1, Chap. 2.3], Example 13(a). For instance, with \( n = 3 \), we have

\[
\text{Hilb}_q(N_6) = q_6 + 6 q_5 q_1 + 15 q_4 q_2 + 15 q_4 q_1^2 + 10 q_3^2 + 60 q_3 q_2 q_1 + 15 q_2^3,
\]

thus \( \dim N_{222} = 15 \) when \( n \geq 3 \). Evidently, the \( q \)-polynomials \( \text{Hilb}_q(N_d) \) specialize to the length restricted Bell numbers \( B_d^{(n)} \) when we set all \( q_k \) equal to 1.

In view of (10), (11), and Theorem 1, we claim the following refinement of (4).

**Corollary 2.** Sending \( n \) to \( \infty \), the shape Hilbert series of the space \( \mathcal{C} \) is given by

\[
\text{Hilb}_q(\mathcal{C}) = \sum_{d \geq 0} d! \exp \left( \sum_{k=1}^\infty q_k \frac{t^k}{k!} \right) \left|_{t^d} \prod_{i \geq 1} (1 - q_i) \right., \tag{12}
\]

with \( (-)^d \) standing for the operation of taking the coefficient of \( t^d \).

This refinement of (4) will follow immediately from the isomorphism \( \mathcal{C} \otimes \Lambda \to N \) in Section 5, which is shape-preserving in an appropriate sense. Thus we have the expansion

\[
\text{Hilb}_q(\mathcal{C}) = 1 + 2 q_2 q_1 + (3 q_3 q_1 + 2 q_2^2 + 3 q_2 q_1^2) + (4 q_4 q_1 + 9 q_3 q_2 + 6 q_3 q_1^2 + 10 q_2^2 q_1 + 4 q_2 q_1^3) + \cdots
\]
3.3 Algebra and coalgebra structures of $\mathcal{N}$

Since the action of $\mathcal{S}$ on $T$ is multiplicative, it is straightforward to see that $\mathcal{N}$ is a subalgebra of $T$. The *multiplication rule* in $\mathcal{N}$, expressing a product $m_A \cdot m_B$ as a sum of basis vectors $\sum_C m_C$, is easy to describe. Since we make heavy use of the rule later, we develop it carefully here. We begin with an example (digits corresponding to $B = 1.2$ appear in bold):

$$m_{13.2} \cdot m_{1.2} = m_{13.2 \cdot 4.5} + m_{134.2.5} + m_{135.2.4} + m_{132.4.5} + m_{132.5.4} + m_{135.2.4} + m_{134.25}$$

(13)

Notice that the shapes indexing the first and last terms in (13) are the partitions $\lambda(13.2) \cup \lambda(1.2)$ and $\lambda(13.2) + \lambda(1.2)$. As was the case in $S^\mathcal{S}$, one of these shapes, namely $\lambda(A) + \lambda(B)$, will always appear in the product, while appearance of the shape $\lambda(A) \cup \lambda(B)$ depends on the cardinality of $x$.

Let us now describe the multiplication rule. Given any $D \subseteq \mathbb{N}$ and $k \in \mathbb{N}$, we write $D^{+k}$ for the set

$$D^{+k} := \{a + k : a \in D\}.$$ 

By extension, for any set partition $A = \{A_1, A_2, \ldots, A_r\}$ we set $A^{+k} := \{A_1^{+k}, A_2^{+k}, \ldots, A_r^{+k}\}$. Also, we set $A_1 := A \setminus \{A_i\}$. Next, if $X$ is a collection of set partitions of $D$, and $A$ is a set disjoint from $D$, we extend $X$ to partitions of $A \cup D$ by the rule

$$A \circ X := \bigcup_{B \in X} \{A\} \cup B.$$ 

Finally, given partitions $A = \{A_1, A_2, \ldots, A_r\}$ of $C$ and $B = \{B_1, B_2, \ldots, B_s\}$ of $D$ (disjoint from $C$), their *quasi-shuffles* $A \circ B$ are the set partitions of $C \cup D$ recursively defined by the rules:

- $A \circ \emptyset = \emptyset \circ A := A$, where $\emptyset$ is the unique set partition of the empty set;
- $A \circ B := \bigcup_{i=0}^s (A_1 \cup B_i) \circ (A_1 \cup (B_1))$, taking $B_0$ to be the empty set.

If $A \vdash [c]$ and $B \vdash [d]$, we abuse notation and write $A \circ B$ for $A \circ B^{+c}$. As shown in [2, Prop. 3.2], the multiplication rule for $m_A$ and $m_B$ in $\mathcal{N}$ is

$$m_A \cdot m_B = \sum_{C \subseteq A \circ B} m_C.$$  

(14)

The subalgebra $\mathcal{N}$, like its commutative analog, is freely generated by certain monomial symmetric functions $\{m_A\}_{A \in \mathcal{A}}$, where $\mathcal{A}$ is some carefully chosen collection of set partitions. This is the main theorem of Wolf [20]. We use two such collections later, our choice depending on whether or not $|x| < \infty$.

The operation $(-)^{+k}$ has a left inverse called the **standardization** operator and denoted by “$(-)^{1n}$”. It maps set partitions $A$ of any cardinality $d$ subset $D \subseteq \mathbb{N}$ to set
partitions of $[d]$, by defining $A^\downarrow$ as the pullback of $A$ along the unique increasing bijection from $[d]$ to $D$. For example, $(18.4)^\downarrow = 13.2$ and $(18.4.67)^\downarrow = 15.2.34$. The coproduct $\Delta$ and counit $\varepsilon$ on $N$ are given, respectively, by

$$\Delta(m_A) = \sum_{B \cup C = A} m_B^\downarrow \otimes m_C^\downarrow \quad \text{and} \quad \varepsilon(m_A) = \delta_{A,0},$$

where $B \cup C = A$ means that $B$ and $C$ form complementary subsets of $A$. In the case $|x| = \infty$, the maps $\Delta$ and $\varepsilon$ are algebra maps, making $N$ a graded connected Hopf algebra.

4 The place-action of $S$ on $N$

4.1 Swapping places in $T_d$ and $N_d$

On top of the permutation-action of the symmetric group $S_x$ on $T$, we also consider the “place-action” of $S_d$ on the degree $d$ homogeneous component $T_d$. Observe that the permutation-action of $\sigma \in S_x$ on a monomial $z$ corresponds to the functional composition

$$\sigma \circ z : [d] \xrightarrow{z} x \xrightarrow{\sigma} x$$

(notation as in Section 3.1). By contrast, the place-action of $\rho \in S_d$ on $z$ gives the monomial

$$z \circ \rho : [d] \xrightarrow{\rho} [d] \xrightarrow{z} x,$$

composing $\rho$ on the right with $z$. In the linear extension of this action to all of $T_d$, it is easily seen that $N_d$ (even each $N_\mu$) is an invariant subspace of $T_d$. Indeed, for any set partition $A = \{A_1, A_2, \ldots, A_r\} \vdash [d]$ and any $\rho \in S_d$, one has

$$m_A \cdot \rho = m_{\rho^{-1}A}$$

(see [15, §2]), where as usual $\rho^{-1} \cdot A := \{\rho^{-1}(A_1), \rho^{-1}(A_2), \ldots, \rho^{-1}(A_r)\}$.

4.2 The place-action structure of $N$

Notice that the action in (15) is shape-preserving and transitive on set partitions of a given shape (i.e., $N_\mu$ is an $S_d$-submodule of $N_d$ for each $\mu \vdash d$). It follows that there is exactly one copy of the trivial $S_d$-module inside $N_\mu$ for each $\mu \vdash d$, that is, a basis for the place-action invariants in $N_d$ is indexed by partitions. We choose as basis the functions

$$m_\mu := \frac{1}{(\dim N_\mu) \mu!} \sum_{\lambda(A) = \mu} m_A,$$

with $\mu! = a_1! a_2! \cdots$ whenever $\mu = 1^{a_1} 2^{a_2} \cdots$. The rationale for choosing this normalizing coefficient will be revealed in (20).

To simplify our discussion of the structure of $N$ in this context, we will say that $S$ acts on $N$ rather than being fastidious about underlying in each situation that individual
$N_d$'s are being acted upon on the right by the corresponding group $\mathfrak{S}_d$. We denote the set $N^S$ of place-invariants by $\Lambda$ in what follows. To summarize,

$$\Lambda = \text{span}\{ m_{\mu} : \mu \text{ a partition of } d, d \in \mathbb{N} \}.$$  

(17)

The pair $(N, \Lambda)$ begins to look like the pair $(S, S^S)$ from the introduction. This was the observation that originally motivated our search for Theorem 1.

We next decompose $N$ into irreducible place-action representations. Although this can be worked out for any value of $n$, the results are more elegant when we send $n$ to infinity. Recall that the Frobenius characteristic of a $\mathfrak{S}_d$-module $V$ is a symmetric function $\text{Frob}(V) = \sum_{\mu \vdash d} v_{\mu} s_{\mu}$, where $s_{\mu}$ is a Schur function (the character of "the" irreducible $\mathfrak{S}_d$ representation $V_{\mu}$ indexed by $\mu$) and $v_{\mu}$ is the multiplicity of $V_{\mu}$ in $V$. To reveal the $\mathfrak{S}_d$-module structure of $N_{\mu}$, we use (15) and techniques from the theory of combinatorial species.

**Proposition 3.** For a partition $\mu = 1^{a_1} 2^{a_2} \cdots k^{a_k}$, having $a_i$ parts of size $i$, we have

$$\text{Frob}(N_{\mu}) = h_{a_1} [h_1] h_{a_2} [h_2] \cdots h_{a_k} [h_k],$$

(18)

with $f[g]$ denoting plethysm of $f$ and $g$, and $h_i$ denoting the $i$th homogeneous symmetric function.

Recall that the plethysm $f[g]$ of two symmetric functions is obtained by linear and multiplicative extension of the rule $p_k[p_\ell] := p_k \ell$, where the $p_k$'s denote the usual power sum symmetric functions (see [12, I.8] for notation and details).

Let $\text{Par}$ denote the combinatorial species of set partitions. So $\text{Par}[n]$ denotes the set partitions of $[n]$ and permutations $\sigma : [n] \to [n]$ are transferred in a natural way to permutations $\text{Par}[\sigma] : \text{Par}[n] \to \text{Par}[n]$.) The number $\text{fix} \text{Par}[\sigma]$ of fixed points of this permutation is the same as the character $\chi_{\text{Par}[n]}(\sigma)$ of the $\mathfrak{S}_n$-representation given by $\text{Par}[n]$. Given a partition $\mu = 1^{a_1} 2^{a_2} \cdots k^{a_k}$, put $z_{\mu} := 1^{a_1} a_1! 2^{a_2} a_2! \cdots k^{a_k} a_k!$. (There are $n!/z_{\mu}$ permutations in $\mathfrak{S}_n$ of cycle type $\mu$.) The cycle index series for $\text{Par}$ is defined by

$$Z_{\text{Par}} = \sum_{n \geq 0} \sum_{\mu \vdash n} \text{fix} \text{Par}[\sigma_{\mu}] \frac{p_{\mu}}{z_{\mu}},$$

where $\sigma_{\mu}$ is any permutation with cycle type $\mu$ and $p_{\mu} := p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ (taking $p_i$ as the $i$-th power sum symmetric function).

**Proof.** Recall that the Schur and power sum symmetric functions are related by

$$s_{\lambda} = \sum_{\mu \vdash |\lambda|} \chi_{\lambda}(\sigma_{\mu}) \frac{p_{\mu}}{z_{\mu}},$$
so \( Z_{\text{Par}} = \text{Frob}(\text{Par}) \). Because \( \text{Par} \) is the composition \( \mathbf{E} \circ \mathbf{E}_+ \) of the species of sets and nonempty sets, we also know that its cycle index series is given by plethystic substitution: \( Z_{\mathbf{E}\circ\mathbf{E}_+} = Z_{\mathbf{E}}[Z_{\mathbf{E}_+}] \). See Theorem 2 and (12) in [1, I.4]. Combining these two results will give the proof.

First, we are only interested in that piece of \( \text{Frob}(\text{Par}) \) coming from set partitions of shape \( \mu \). For this we need weighted combinatorial species. If a set partition has shape \( \mu \), give it the weight \( q_{a_1}^{a_1}q_{a_2}^{a_2} \cdots q_{a_k}^{a_k} \) in the cycle index series enumeration. The relevant identity is

\[
Z_{\mathbf{P}}(q) = \exp \sum_{k \geq 1} \frac{1}{k} \left( \exp \left( \sum_{j \geq 1} q_j^k \frac{p_{jk}}{j} \right) - 1 \right)
\]

(cf. Example 13(c) of Chapter 2.3 in [1]). Collecting the terms of weight \( q_\mu \) gives \( \text{Frob}(N_\mu) \).

We get

\[
\text{coeff}_{q_\mu} [Z_{\text{Par}}(q)] = \prod_{i=1}^{k} \left( \sum_{\lambda \vdash a_i} \frac{p_\lambda}{z_\lambda} \right) \left[ \sum_{\nu \vdash \mu} \frac{p_\nu}{z_\nu} \right].
\]

Standard identities [12, (2.14’) in I.2] between the \( h_i \)’s and \( p_j \)’s finish the proof.

As an example, we consider \( \mu = 222 = 2^3 \). Since

\[
h_2 = \frac{p_1^2}{2} + \frac{p_2}{2} \quad \text{and} \quad h_3 = \frac{p_1^3}{6} + \frac{p_1p_2}{2} + \frac{p_3}{3},
\]

a plethysm computation (and a change of basis) gives

\[
h_3[h_2] = \frac{p_1^3}{6} \left[ \frac{p_1^2}{2} + \frac{p_2}{2} \right] + \frac{p_1p_2}{2} \left[ \frac{p_1^2}{2} + \frac{p_2}{2} \right] + \frac{p_3}{3} \left[ \frac{p_1^2}{2} + \frac{p_2}{2} \right]
\]

\[
= \frac{1}{6} \left( \frac{p_1^2}{2} + \frac{p_2}{2} \right)^3 + \frac{1}{2} \left( \frac{p_1^2}{2} + \frac{p_2}{2} \right) \left( \frac{p_2^2}{2} + \frac{p_1}{2} \right) + \frac{1}{3} \left( \frac{p_2^3}{2} + \frac{p_6}{2} \right)
\]

\[
= s_6 + s_{42} + s_{222}.
\]

That is, \( N_{222} \) decomposes into three irreducible components, with the trivial representation \( s_6 \) being the span of \( m_{222} \) inside \( \Lambda \).

### 4.3 \( \Lambda \) meets \( S^\otimes \)

We begin by explaining the choice of normalizing coefficient in (16). Analyzing the abelianization map \( \text{ab} : T \to S \) (the map making the variables \( x \) commute), Rosas and Sagan [15, Thm. 2.1] show that \( \text{ab}|_{\Lambda} \) satisfies:

\[
\text{ab}(m_\mathbf{A}) = \lambda(\mathbf{A})^t m_{\lambda(\mathbf{A})}.
\]

In particular, \( \text{ab} \) maps onto \( S^\otimes \) and

\[
\text{ab}(m_\mu) = m_\mu.
\]
Note that $ab$ is also an algebra map. The reader may wish to use (19) to compare (8) and (13). Formula (20) suggests that a natural right-inverse to $ab|_{\mathcal{N}}$ is given by

$$\iota : S^\otimes \hookrightarrow \mathcal{N}, \quad \text{with} \quad \iota(m_\mu) := m_\mu \quad \text{and} \quad \iota(1) = 1. \quad (21)$$

This fact, combined with the observation that $\iota(S^\otimes) = \Lambda$, affords a quick proof of Theorem 1 when $|x| = \infty$. We explain this now.

5 The coinvariant space of $\mathcal{N}$ (Case: $|x| = \infty$)

5.1 Quick proof of main result

When $|x| = \infty$, the pair of maps $(ab, \iota)$ have further properties: the former is a Hopf algebra map and the latter is a coalgebra map [2, Props. 4.3 & 4.5]. Together with (20) and (21), these properties make $\iota$ a coalgebra splitting of $ab : \mathcal{N} \to S^\otimes \to 0$. A theorem of Blattner, Cohen, and Montgomery immediately gives our main result in this case.

**Theorem 4** ([5], Thm. 4.14). If $H \xrightarrow{\pi} \mathcal{P} \to 0$ is an exact sequence of Hopf algebras that is split as a coalgebra sequence, and the splitting map $\iota$ satisfies $\iota(\bar{1}) = 1$, then $H$ is isomorphic to a crossed product $A \# \mathcal{P}$, where $A$ is the left Hopf kernel of $\pi$. In particular, $H \simeq A \otimes \mathcal{P}$ as vector spaces.

For the technical definition of crossed products, we refer the reader to [5, §4]. We mention only that: (i) the crossed product $A \# \mathcal{P}$ is a certain algebra structure placed on the tensor product $A \otimes \mathcal{P}$; and (ii) the **left Hopf kernel** is the subalgebra

$$A := \{h \in H : (id \otimes \pi) \circ \Delta(h) = h \otimes \mathcal{T}\}.$$  

We take $H = \mathcal{N}$, $\mathcal{P} = S^\otimes$, and $\pi = ab$. Since our $\iota$ is a coalgebra splitting, the coinvariant space $\mathcal{C}$ we seek seems to be the left Hopf kernel of $ab$. Before setting off to describe $\mathcal{C}$ more explicitly, we point out that the left Hopf kernel is graded: the maps $\Delta$, $id$, and $ab$ are graded, as is the map $\mathcal{C} \# \Lambda \xrightarrow{\sim} \mathcal{N}$ used in the proof of Theorem 4 (which is simply $a \otimes h \mapsto a \cdot \iota(h)$). Theorem 1 follows immediately from this result.

5.2 Atomic set partitions.

Recall the main result of Wolf [20] that $\mathcal{N}$ is freely generated by some collection of functions. We announce our first choice for this collection now, following the terminology of [3]. Let $\Pi$ denote the set of all set partitions (of $[d]$, $\forall d \geq 0$). The **atomic set partitions** $\Pi$ are defined as follows. A set partition $A = \{A_1, A_2, \ldots, A_r\}$ of $[d]$ is atomic if there does not exist a pair $(s, c)$ ($1 \leq s < r, 1 \leq c < d$) such that $\{A_1, A_2, \ldots, A_s\}$ is a set partition of $[c]$. Conversely, $A$ is not atomic if there are set partitions $B$ of $[d']$ and $C$ of $[d'']$ splitting $A$ in two: $A = B \cup C + d'$, $C + d' = C + d''$. We write $A = B|C$ in this situation. A **maximal splitting** $A = A_1|A_2|A_3| \cdots |A_t$ of $A$ is one where each $A^{(i)}$ is atomic. For example, the partition 17.235.4.68 is atomic, while 12.346.57.8 is not. The maximal splitting of
the latter would be 12|124.35|1, but we abuse notation and write 12|346.57|8 to improve legibility.

It follows from [3, Corollary 9] that \( N \) is freely generated by the atomic monomial functions \( \{m_A : A \in \hat{\Pi}\} \). We now introduce an order on \( \Pi \) that will make this explicit. First we introduce the restricted growth function associated to a set partition (see Section 6.1): if \( A = \{A_1, A_2, \ldots, A_r\} \vdash [d] \), define \( w(A) \in \mathbb{N}^d \) by

\[
w(A) = w_1w_2 \cdots w_d, \quad \text{with} \quad w_i := k \iff i \in A_k. \tag{22}
\]

For example, \( w(13.24) = 1212 \) and \( w(17.235.4.68) = 12232414 \). Now, given two atomic set partitions \( A \vdash [c] \) and \( B \vdash [d] \), we put:

- \( A > B \) when \( c > d \); or
- \( A > B \) when \( c = d \) and \( w(A) > \text{lex} w(B) \).

Finally, given two set partitions \( A \) and \( B \), put \( A > B \) if \( \lambda(A) < \text{lex} \lambda(B) \) in the usual lexicographic order on integer partitions. If \( \lambda(A) = \lambda(B) \), then determine maximal splittings of \( A \) and \( B \), view them as words in the atomic set partitions and use the lexicographic order induced by \( > \). The following chain of set partitions of shape 3221 illustrates our total ordering on \( \Pi \):

\[
1|23|45|678 < 13.2|45|6|78 < 13.24|568.7 < 13.24|578.6 < 17.235.4.68 < 17.236.4.58.
\]

In fact, \( 1|23|45|678 \) is the unique minimal element of \( \Pi \) of shape 3221.

Define the leading term of a sum \( \sum_C \alpha_C m_C \) to be the monomial \( m_{C_0} \) such that \( C_0 \) is greatest (according to \( > \) above) among all \( C \) with \( \alpha_C \neq 0 \). Combined with (14), our definition of \( > \) makes it clear that the leading term of \( m_A \cdot m_B \) is \( m_{A,B} \) and that \( N \) is freely generated by the atomic monomial functions. Moreover, it is clear that multiplication in \( N \) is shape-filtered. Since the left Hopf kernel \( C \) is a subalgebra, \( C \) is shape-filtered as well. Finally, the isomorphism \( \mathcal{C} \not\# \Lambda \rightarrow N \) constructed in the proof of Theorem 4 is also shape-filtered. These facts give Corollary 2 immediately.

### 5.3 Explicit description of the Hopf algebra structure of \( \mathcal{C} \)

We begin by partitioning \( \hat{\Pi} \) into two sets according to length,

\[
\hat{\Pi}_{(1)} := \{ A \in \hat{\Pi} : \ell(A) = 1 \} \quad \text{and} \quad \hat{\Pi}_{(>1)} := \{ A \in \hat{\Pi} : \ell(A) > 1 \}.
\]

It is easy to find elements of the left Hopf kernel \( C \). For instance, if \( A \) and \( B \) belong to \( \hat{\Pi}_{(1)} \), then the Lie bracket \( [m_A, m_B] \) belongs to \( C \). Indeed,

\[
\Delta ([m_A, m_B]) = \Delta (m_{A,B} - m_{B,A}) = m_{A,B} \otimes 1 + m_A \otimes m_B + m_B \otimes m_A + 1 \otimes m_{A,B} - m_{B,A} \otimes 1 - m_B \otimes m_A - m_A \otimes m_B - 1 \otimes m_{B,A} = (m_{A,B} - m_{B,A}) \otimes 1 + 1 \otimes (m_{A,B} - m_{B,A}).
\]
Since $ab(m_{A|B}) = ab(m_{B|A})$, we have

$$(\text{id} \otimes ab) \circ \Delta ([m_A, m_B]) = [m_A, m_B] \otimes 1$$

as desired. Similarly, the difference of monomial functions $m_{13,2} - m_{12,3}$ belongs to $C$. The leading term here is indexed by $13.2 \in \hat{\Pi}_{(>1)}$. These two simple examples essentially exhaust the different ways in which an element can belong to $C$. The following discussion makes this precise.

From [3, Theorem 15], we learn that $N$ is cofree cocommutative with minimal cogenenerating set indexed by the Lyndon words in $\hat{\Pi}$. (This result and the previously mentioned freeness result may also be deduced from the techniques developed in [9].) Since single letters are Lyndon words, we know there are primitive elements associated to each atomic set partition. Recall that an element $h$ in a Hopf algebra is primitive if $\Delta(h) = h \otimes 1 + 1 \otimes h$. Let $\text{Prim}(N)$ denote the set of primitive elements in $N$—a Lie algebra under the commutator bracket.

Bearing the free and cofree cocommutative results in mind, a classical theorem of Milnor and Moore [13] guarantees that $N$ is isomorphic to the universal enveloping algebra $U(\mathcal{L}(\hat{\Pi}))$ of the free Lie algebra $\mathcal{L}(\hat{\Pi})$ on the set $\hat{\Pi}$. In the isomorphism $\mathcal{L}(\hat{\Pi}) \mapsto \text{Prim}(N)$, one may map $A \in \hat{\Pi}_{(1)}$ to $m_A$ since these monomial functions are already primitive. The choice of where to send $A \in \hat{\Pi}_{(>1)}$ is the subject of the next proposition.

**Proposition 5.** For each $A \in \hat{\Pi}_{(>1)}$, there is a primitive element $\tilde{m}_A$ of $N$,

$$\tilde{m}_A = m_A - \sum_{B \in \Pi} \alpha_B m_B,$$

satisfying: (i) if $B \in \hat{\Pi}$ or $\lambda(B) \neq \lambda(A)$, then $\alpha_B = 0$; and (ii) $\sum_B \alpha_B = 1$.

**Proof.** Suppose $A \in \hat{\Pi}_{(>1)}$. A primitive $\tilde{m}_A$ exists by the Milnor–Moore theorem, as explained above.

(i). Since $N = \bigoplus_{\mu} N_{\mu}$ is a coalgebra grading by shape, we may assume $\lambda(B) = \lambda(A)$ for any nonzero coefficients $\alpha_B$. Now, since there are linearly independent primitive elements in $N$ associated to every atomic set partition, we may use Gaussian elimination and our ordering on $\Pi$ to ensure that $\alpha_B = 0$ for any $B \in \hat{\Pi}$.

(ii). Define linear maps $\Delta_+^j : N_+ \rightarrow N \otimes N$ recursively by

$$\Delta_+^1(h) := \Delta(h) - h \otimes 1 - 1 \otimes h,$$

$$\Delta_+^{j+1}(h) := (\Delta_+ \otimes \text{id}^j) \circ \Delta_+^j(h) \quad \text{for} \quad j > 0.$$ 

Assume that (i) is satisfied for $\tilde{m}_A$ and that $A = \{A_1, A_2, \ldots, A_r\}$. Since $\Delta_+(\tilde{m}_A) = 0$, we have $\Delta_+^j(\tilde{m}_A) = \Delta_+^j(\sum_B \alpha_B m_B)$ for all $j > 1$. Now,

$$\Delta_+^r(m_A) = \sum_{\sigma \in \mathfrak{S}_r} m_{A_{\sigma_1}} \otimes m_{A_{\sigma_2}} \otimes \cdots \otimes m_{A_{\sigma_r}}.$$
Indeed, the same holds for any $B$ with $\lambda(B) = \lambda(A)$:

$$\Delta^r_+ \left( \sum_B \alpha_B m_B \right) = \left( \sum_B \alpha_B \right) \sum_{\sigma \in S_r} m_{A_{\sigma_1}} \otimes m_{A_{\sigma_2}} \otimes \cdots \otimes m_{A_{\sigma_r}}.$$ 

Conclude that $\sum_B \alpha_B = 1$. \hfill $\square$

We say an element $h \in N_\mu$ has the “zero-sum” property if it satisfies (ii) from the proposition. Put $\tilde{m}_A := m_A$ for $A \in \hat{\Pi}_{(1)}$. We next describe the coinvariant space $\mathcal{C}$.

**Corollary 6.** Let $\mathcal{C}$ be the Lie ideal in $\mathfrak{L}(\hat{\Pi})$ given by $\mathcal{C} = [\mathfrak{L}(\hat{\Pi}), \mathfrak{L}(\hat{\Pi})] \oplus \hat{\Pi}_{(>1)}$. If $\varphi : \mathfrak{U}(\mathfrak{L}(\hat{\Pi})) \to N$ is the Milnor–Moore isomorphism given by putting $\varphi(A) := \tilde{m}_A$ for all $A \in \hat{\Pi}$ and extending multiplicatively, then the left Hopf kernel $\mathcal{C}$ is the Hopf subalgebra $\varphi(\mathfrak{U}(\mathcal{C}))$.

**Proof.** We first show that $\varphi(\mathfrak{U}(\mathcal{C})) \subseteq \mathcal{C}$. We certainly have $\tilde{m}_A \in \mathcal{C}$ for all $A \in \hat{\Pi}_{(1)}$, since the zero-sum property means $ab(\tilde{m}_A) = 0$. Next suppose $f \in [\mathfrak{L}(\hat{\Pi}), \mathfrak{L}(\hat{\Pi})]$ is a sum of Lie brackets $[A] = [[\ldots [A', A''], \ldots], A^{(0)}]$. In this case, $\varphi(f) \in \mathcal{C}$ because each $\varphi([A])$ is primitive and $ab$ is an algebra map. Indeed, $ab(\tilde{m}_{A'}, \tilde{m}_{A''}) = 0$. The inclusion follows, since $\mathfrak{U}(\mathcal{C})$ is generated by elements of these two types.

It remains to show that $\mathcal{C} \subseteq \varphi(\mathfrak{U}(\mathcal{C}))$. To begin, note that $\mathfrak{L}(\hat{\Pi})/\mathcal{C}$ is isomorphic to the abelian Lie algebra generated by $\hat{\Pi}_{(1)}$. The universal enveloping algebra of this latter object is evidently isomorphic to $S^\mathfrak{g}$. (Send $A = \{[d]\}$ to $m_d$.) The Poincaré–Birkhoff–Witt theorem guarantees that the map $\varphi(\mathfrak{U}(\mathcal{C})) \otimes S^\mathfrak{g} \to N$ given by $a \otimes b \mapsto a \cdot \iota(b)$ is onto $N$. Conclude that $\mathcal{C} \subseteq \varphi(\mathfrak{U}(\mathcal{C}))$, as needed. \hfill $\square$

Before turning to the case $|x| < \infty$, we remark that we have left unanswered the question of finding a systematic procedure (e.g., a closed formula in the spirit of Möbius inversion) that constructs a primitive element $\tilde{m}_A$ for each $A \in \hat{\Pi}_{(>1)}$. This is accomplished in [11].

### 6 The coinvariant space of $N$ (Case: $|x| \leq \infty$)

#### 6.1 Restricted growth functions

We repeat our example of Section 3.3 in the case $n = 3$. The leading term with respect to our previous order would be $m_{13245}$, except that this term does not appear because $13245$ has more than $n = 3$ parts:

$$m_{132} \cdot m_{12} = 0 + m_{13425} + m_{13524} + m_{13245} + m_{13254} + m_{13524} + m_{13425}.$$ 

Fortunately, the map $w$ from set partitions to words on the alphabet $\mathbb{N}_{>0}$ reveals a more useful leading term, underlined below:

$$m_{121} \cdot m_{12} = 0 + m_{12113} + m_{12131} + m_{12123} + m_{12132} + m_{12121} + \underline{m_{12112}}.$$ (23)
Notice that the words appearing on the right in (23) all begin by 121 and that the concatenation 12112 is the lexicographically smallest word appearing there. This is generally true and easy to see: if \( w(A) = u \) and \( w(B) = v \), then \( uv \) is the lexicographically smallest element of \( w(A \cup B) \).

The map \( w \) maps set partitions to restricted growth functions, i.e., the words \( w = w_1 w_2 \cdots w_d \) satisfying \( w_1 = 1 \) and \( w_i \leq 1 + \max\{w_1, w_2, \ldots, w_{i-1}\} \) for all \( 2 \leq i \leq d \). We call them restricted growth words here. See [16, 17, 19] and [6, 8] for some of their combinatorial properties and applications. These words are also known as “rhyme scheme words” in the literature; see [14] and [18, A000110]. Before looking for a coinvariant space \( \mathcal{C} \) within \( \mathcal{N} \), we first fix the representatives of \( \Lambda \). Consider the partition \( \mu = 3221 \). Of course, \( m_\mu \) is the sum of all set partitions of shape \( \mu \), but it will be nice to have a single one in mind when we speak of \( m_\mu \). A convenient choice turns out to be 123.45.67.8: if we use the length plus lexicographic order on \( w(\Pi) \), then it is easy to see that \( w(123.45.67.8) = 11122334 \) is the minimal element of \( \Pi \) of shape 3221. We are led to introduce the words

\[
w(\mu) := 1^{\mu_1}2^{\mu_2}\cdots k^{\mu_k}
\]

associated to partitions \( \mu = (\mu_1, \mu_2, \ldots, \mu_k) \); we call such restricted growth words convex words since \( \mu_1 \geq \mu_2 \geq \cdots \geq \mu_k \).

6.2 Proof of main theorem

We say that a restricted growth word is non-splittable if \( w_i \cdots w_{n-1}w_n \) is not a restricted growth word for any \( i > 1 \). The maximal splitting of a restricted growth word \( w \) is the maximal deconcatenation \( w = w'|w''|\cdots|w^{(r)} \) of \( w \) into non-splittable words \( w^{(i)} \). For example, 12314 is non-splittable while 11232411 is a string of four non-splittable words 11232411.

It is easy to see that if \( a, b, c, \) and \( d \) are non-splittable, then \( ac = bd \) if and only if \( a = b \) and \( c = d \). Together with the remarks on \( A \cup B \) following (23), this implies that if \( \{u_1, u_2, \ldots, u_r\} \) and \( \{v_1, v_2, \ldots, v_s\} \) are two sets of non-splittable words, then

\[m_{u_1}m_{u_2}\cdots m_{u_r} \quad \text{and} \quad m_{v_1}m_{v_2}\cdots m_{v_s}\]

share the same leading term (namely, \( m_{u_1[u_2|\cdots|u_r]} \) if and only if \( r = s \) and \( u_i = v_i \) for all \( i \). In other words, our algebra \( \mathcal{N} \) is non-splittable word-filtered and freely generated by the monomial functions \( \{m_{w(A)} : w(A) \text{ is non-splittable}\} \). This is one of the collections of monomial functions originally chosen by Wolf [20].

We aim to index \( \mathcal{C} \) by the restricted growth words that don’t end in a convex word. Toward that end, we introduce the notion of bimodal words. These are words with a maximal (but possibly empty) convex prefix, followed by one non-splittable word. The bimodal decomposition of a restricted growth word \( w \) is the expression of \( w \) as a product \( w = w'|w''|\cdots|w^{(r)}|w^{(r+1)} \), where \( w', w'', \ldots, w^{(r)} \) are bimodal and \( w^{(r+1)} \) is a possibly empty convex word (which we call a tail). For a given word \( w \), this decomposition is accomplished by first splitting \( w \) into non-splittable words, then recombining, from
left to right, consecutive non-splittable words to form bimodal words. For instance, the maximal splitting of 1122212 into non-splittable words is 1|1222|12. The first two factors combine to make one bimodal word; the last factor is a convex tail: 1122212 \rightarrow \hat{1}1222\hat{1}2.

Similarly, 

1231231411231231411122311 \rightarrow 123|12314|1|1|1223|1|1 \rightarrow \hat{1}23\hat{1}23\hat{1}4\hat{1}\hat{1}\hat{1}\hat{2}2\hat{3}\hat{1}\hat{1}\hat{1}\hat{2}23\hat{1}1\hat{1}.

Suppose now that \( u \) and \( v \) are restricted growth words and that the bimodal decomposition of \( u \) is tail-free. Then by construction, the bimodal decomposition of \( uv \) is the concatenation of the respective bimodal decompositions of \( u \) and \( v \). We are ready to identify \( \mathcal{C} \) as a subalgebra of \( \mathbb{N} \).

**Theorem 7.** Let \( \mathcal{C} \) be the subalgebra of \( \mathbb{N} \) generated by \( \{m_v : v \text{ is bimodal}\} \). Then \( \mathcal{C} \) has a basis indexed by restricted growth words \( w \) whose bimodal decompositions are tail-free. Moreover, the map \( \varphi : \mathcal{C} \otimes \Lambda \rightarrow \mathbb{N} \) given by \( m_{w_1}m_{w_2} \cdots m_{w_s} \otimes m_\mu \mapsto m_{w_1w_2\cdots w_s}\otimes m_\mu \) is a vector space isomorphism.

**Proof.** The advertised map is certainly onto, since \( \{m_w : w \in \mathcal{w}(\Pi)\} \) is a basis for \( \mathbb{N} \) and every restricted growth word has a bimodal decomposition \( w_1|w_2|\cdots|w_s|w(\mu) \). It remains to show that the map is one-to-one.

Note that the monomial functions \( \{m_v : v \text{ is bimodal}\} \) are algebraically independent: certainly, the leading term in a product \( m_{v_1}m_{v_2}\cdots m_{v_s} \) (with \( v_i \) bimodal) is \( m_{v_1v_2\cdots v_s} \); now, since every word has a unique bimodal decomposition, no (nontrivial) linear combination of products of this form can be zero. Finally, the leading term in the simple tensor \( m_{w_1}m_{w_2}\cdots m_{w_s}\otimes m_\mu \) is the basis vector \( m_{w_1w_2\cdots w_s}\otimes m_\mu \), so no (nontrivial) linear combination of these will vanish under the map \( \varphi \).

\[\square\]

**References**


