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Managing Inventory in Global Supply Chains Facing Port-of-Entry Disruption Risks

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Ports-of-entry are critical components of the modern international supply chain infrastructure, particularly container seaports and airfreight hubs. The potential operational and economic impact resulting from their temporary closure is unknown, but is widely believed to be very significant. This paper investigates one aspect of this potential impact, focusing specifically on the use of supply chain inventory as a risk mitigation strategy for a one supplier, one customer system in which goods are transported through a port-of-entry subject to temporary closures. Closure likelihood and duration are modeled using a completely observed, exogenous Markov chain. Order lead times are dependent on the status of the port-of-entry, including potential congestion backlogs of unprocessed work. An infinite-horizon, periodic-review inventory control model is developed to determine the optimal average cost ordering policies under linear ordering costs with backlogged demand. When congestion is negligible, the optimal policy is state invariant. In the more complex case of non-negligible congestion, this result no longer holds. For studied scenarios, numerical results indicate that operating margins may decrease 10% for reasonable-length port-of-entry closures, that margins may be eliminated completely without contingency plans, and expected holding and penalty costs may increase 20% for anticipated increases in port-of-entry utilization.

Key words: supply chain risk, inventory management, seaport operations

1. Introduction

Modern global supply chain systems are increasingly vulnerable to disruption. Such systems are vitally important to the world economy; the value of export merchandise transported globally in 2008 was an astonishing $15.8 trillion [World Trade Organization (2009)]. Disruptions result in recovery costs, and can significantly impact the market value of a company moving its goods through the supply chain. Results of an empirical study in Hendricks and Singhal (2003) estimate a mean decrease in firm market value of 10.28% over the two-day period after the public announcement of a major supply chain disruption.

Among the most critical and vulnerable nodes are ports-of-entry, particularly container seaports and airfreight hubs. Maritime transportation is the dominant mode for international trade, and 80% of the goods (measured by value) moved by ocean are transported in intermodal containers. Temporary seaport closures could result from a number of events, including natural disasters, labor disruptions and shortages, and security incidents. In August 2005, Hurricane Katrina caused the port of New Orleans to close for over two weeks, and full capacity was not restored for many months thereafter. A lockout of stevedores in 2002 closed 29 Western U.S. seaports for 10 days, resulting in a backlog of cargo vessels and port congestion that again did not dissipate for months.

While security incidents have not yet resulted in a seaport closure in the U.S., they have severely disrupted freight transportation systems. In the aftermath of the September 11, 2001 terrorist attacks, all traffic to U.S. airports was suspended, including international cargo flights and truck delays at the U.S.-Canadian border increased from a few minutes to an extreme 12 hours [Bonner (2004)]. Due to the resulting parts shortages, Ford Motor Company intermittently idled production at five of its assembly plants [Rice and Caniato (2003)], and Toyota came within hours of halting production at one plant [Sheffi (2001)]. In the event of another terrorist event or "false alarm," seaport closures may be likely. Gerencser et al. (2002) present the results of a port security wargame simulating a terrorist attack using radiological ("dirty") bombs in intermodal containers. The participating business and government leaders decided to close every U.S. seaport for eight days, and the resulting import freight backlog required three months to clear. The forecasted total loss to the U.S. economy was $58 billion. In another study, Abt (2003) estimates that the impact on U.S. trade alone from a nuclear attack on a U.S. seaport is $100-$200 billion.

In this paper, a model for quantifying the potential supply chain cost impacts of temporary port closures is developed. To do so, focus is placed on a simple one supplier, one customer system in which goods are transported through a port. When closed, the port may develop a backlog of container processing work which is cleared over time after reopening. An assumption is made that the customer in this system uses inventory for mitigating the risks of transportation disruption, and deploys a minimum long-run average cost inventory management policy.

Initially, a probability mass function is developed for the expected leadtime under the threat of port closure. An expression for the long-run average cost of a stationary, state-dependent basestock policy is then derived using Markov reward theory. Constraining the model to the case where port congestion is negligible (all containers may be processed as long as the port is open), the optimal ordering policy is proven to be state-invariant and closed-form solutions for the optimal basestock level and associated long-run average cost are presented. Finally, the theoretical model is used to quantify the effects of port-of-entry closures for the more complex non-negligible congestion case through a numerical analysis. A range of probabilities related to likelihood and length of closure are evaluated, displaying the impact on costs and inventory levels. Further, the results are presented in the context of three important strategic issues for international supply chain systems facing port disruptions:

1. Effective contingency plans that reduce the duration of temporary port closures and prioritize speedy return to normal operation;
2. Inventory management strategies that explicitly plan for potential port closures; and
3. Management strategies for increasing the freight processing capability of highly utilized and congested ports during states of emergency.

The numerical results emphasize the need for development of these strategies by all segments of the supply chain. For example, increasing port closure duration from two to 20 days may lead to a 10% reduction in operating margin. Also, implementing an inventory plan that considers potential port closures results in a cost savings over plans that do not in virtually every scenario tested. Further, increasing port utilization from approximately 70% to 96% may increase holding and penalty costs by nearly 20%. These findings may be used to encourage supply chain managers and policy makers alike to create anticipatory policies that consider the likelihood of a port closure.

The rest of this paper is organized as follows. Section 2 presents related literature on supply chain systems facing disruptions and inventory management with uncertain supply. Section 3 introduces the problem setting, and a Markov decision model for inventory control in this setting. Section 4 then presents theoretical results for a best-case scenario in which port congestion is negligible. Since these results likely underestimate the true impact of port closures, Section 5 treats the case with port congestion in more detail, and presents the results of a numerical study which highlights the relative importance of the three strategic issues identified above. Finally, Section 6 provides concluding remarks and areas of future research.

2. Related Literature

Supply disruptions can be categorized into two types: disruptions in a supplier's ability to output product, and disruptions to the transportation of product from supplier to customer. Inventory control models that address supplier availability generally assume zero fulfillment lead time or that at most a single replenishment order is outstanding at any given time. Examples of papers in this category include Parlar and Berkin (1991), Weiss and Rosenthal (1992), Parlar and Perry (1995), and Snyder (2006), which consider EOQ-based inventory ordering policies, and Arreola-Risa and DeCroix (1998), Özekici and Parlar (1999), and Parlar et al. (1995), which consider (s, S) policies. Production disruptions in a manufacturing environment can be viewed as a form of internal supplier disruption, and is studied in Moinzadeh and Aggarwal (1997), Yang et al. (2004), and Yang et al. (2005). Qi et al. (2009) consider a continuous-time inventory control model where both the supplier and retailer may face disruptions. In the retailer case, a disruption destroys all on-hand inventory.

Some recent research also focuses on issues of supply chain strategy and design given supplier disruptions. Tomlin (2005) investigates dual-sourcing policies to mitigate risk, while Tomlin (2009) studies other potential strategies such as supplier diversification, contingent sourcing, and demand management for a single-period inventory model for short life-cycle products. Snyder and Shen (2006) discuss important differences in multi-echelon supply chain designs for dealing with demand uncertainty and supply uncertainty. Tomlin and Snyder (2009) show that an adaptive inventory ordering policy based on a threat advisory system for supplier availability can lead to substantial cost savings over a static policy. Lewis (2005) also considers an adaptive ordering policy based on a threat advisory system, but where the advisory system provides information about future port-of-entry closures, rather than supplier availability. Leadtime uncertainty is countered through the use of forecast updating by Wang and Tomlin (2009), who show that as forecast updating becomes more efficient, leadtime uncertainty becomes less of an issue.

Little work in the inventory control literature specifically focuses on disruptions to the transportation of product from supplier to customer. Minor disruptions that may cause delays are generally considered to contribute to regular lead time variability, and so are not explicitly modeled. Kaplan (1970) is the earliest work to prove the optimality of an (s, S) inventory policy for a finite-horizon, periodic-review inventory system with stochastic lead times and multiple outstanding orders. Order crossover is prohibited by assumption, so that an order placed at time t must arrive no later than the one placed at time $t+1$. Ehrhardt (1984) extends this result to the infinite horizon.

Song and Zipkin (1996) generalize these models by allowing the lead time distribution to depend on an exogenous system that is modeled as a discrete-time Markov chain (DTMC). When ordering costs have no fixed component, they show the optimality of a stationary basestock policy for both the total expected discounted cost and long-run average cost models, where the basestock (or order-up-to) levels depend on the state of the DTMC. While bounds on the optimal order-up-to levels are discussed, no explicit procedures are presented to determine optimal policy parameters, the long-run average cost of an arbitrary state-dependent basestock policy, or the optimal long-run average cost.

This paper provides new theoretical results for the generic model in Song and Zipkin (1996), specifically an expression for the long-run average cost of a stationary, state-dependent basestock policy as well as a sufficient condition for the optimality of a stationary, state-*invariant* basestock policy, a method for its calculation, and an expression for the associated long-run average cost. A specialization of the generic model is presented, representing a supply chain moving goods through an international port-of-entry that is subject to the risk of temporary closures and resulting congestion. The key steps in this specialization are the modeling of the port-of-entry's freight processing operations and the derivation of the probability distribution for order leadtimes. For a special case of negligible congestion, it is shown that the optimal policy is, in fact, a stationary, state-invariant basestock policy.

Chen and Song (2001) also develop a specialized algorithm for Markov-modulated demand models that can be used to determine the optimal order-up-to levels as well as the optimal long-run average cost for the Song and Zipkin (1996) model. Chen and Yu (2005) consider a specialization of Song and Zipkin (1996) to investigate the value of observability of the DTMC which models a simple lead time distribution.

3. Problem Definition and Solution Approach

To develop an understanding of the potential cost impacts of temporary port-of-entry closures on an international supply chain, consider a simplified system in which a domestic customer orders a single product from a foreign supplier with unlimited supply. Product is shipped from supplier to customer using a single fixed transport route passing through a domestic port-of-entry for importation; see Figure 1. Transit time $L > 0$ from supplier to port is assumed to be deterministic, and is measured in discrete periods. Orders then face a stochastic processing delay through the congested, and possibly closed, port-of-entry. After clearing the port, assume that orders arrive at the customer instantaneously (our results are extendable to the case of positive transit time from the port to the customer).

Suppose that the customer operates a periodic-review inventory control system. At the beginning of each period, the customer observes its inventory state and the state of the supply system (to be described below), and places an order if necessary. Ordering cost is immediately incurred. Next, the state of the supply system is updated, resulting in the arrival to on-hand inventory of some subset of the outstanding orders, and demand is realized. Demand is stochastic and is satisfied

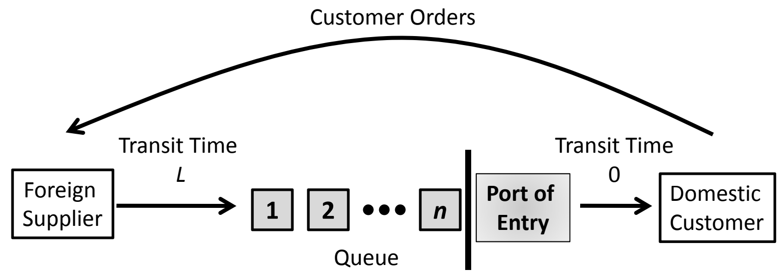

Figure 1 A supply system with possible port-of-entry closures and congestion.

from on-hand inventory if possible; otherwise, it is fully backlogged. Finally, the on-hand inventory holding cost or the backlog penalty cost is assessed. The problem faced by the customer is to determine order quantities with the objective to minimize per period average cost considering an infinite planning horizon.

Orders are placed in discrete quantities (*e.g.* full containers) at a cost of c per unit. Holding costs are h per unit per period for any inventory held. Penalty costs are b per unit per period for any backlogged demand. Given an on-hand inventory level of $\hat{x}_t$ at the beginning of period t, the holding/penalty cost assessed for period t is given by

$$\hat{C}(\hat{x}_{t+1}) = \begin{cases} -b\hat{x}_{t+1} & \text{if } \hat{x}_{t+1} < 0 \\ h\hat{x}_{t+1} & \text{if } \hat{x}_{t+1} \geq 0. \end{cases} \tag{1}$$

Note that the cost for period t is evaluated in period $t+1$ as it is dependent on the inventory level carried into the following period.

Let D_t be a non-negative, integer random variable representing the demand in period t, where demands in different periods are identically and independently distributed. Demand is bounded such that $D_t \in \{d_1, d_2, ..., d_J\}$ where $J < \infty$ and $0 \leq d_1 < d_2 < ... < d_J < \infty$. Let $D^{(l)}$ represent cumulative demand over l periods.

The stochastic processing delay at the port-of-entry (and thus the uncertain lead time for receipt of orders) is modeled as follows. First, assume that during any time period t, the port may be *open* or *closed*. Port status is modeled using a discrete-time Markov chain $\mathbf{I} = \{i_t, t \geq 0\}$ with state space $S_I = \{O, X\}$, where $i_t = O$ indicates the port is open, and $i_t = X$ indicates the port is closed. Assume that the transition probabilities of $\mathbf{I}$ are exogenous to the decisions of the customer, time-homogeneous, and known. Define $p_{ij} = P(i_{t+1} = j | i_t = i)$ for all $t \geq 0$, and let $P_I = [p_{ij}]$ be the resulting one-step transition probability matrix. Since the chain has finite state space, let π^I be the unique stationary distribution. Also define $p_{ij}^{(l)} = P(i_{t+l} = j | i_t = i)$ for all $t \geq 0$ and $l \geq 0$, and let $P_I^l = [p_{ij}^{(l)}]$ be the l-step transition probability matrix. Lewis et al. (2006) briefly discusses how the customer might determine the transition probabilities.

Second, the processing operation at the port-of-entry is assumed to be capacitated such that a queue of unprocessed freight does not necessarily immediately dissipate after periods of closure. The port-of-entry simultaneously services many customers and a simple deterministic queueing approach is used to model port freight processing. For the purposes of this research, it is not necessary to model the operations of the port-of-entry in great detail, for example the loading, unloading, and stacking operations of individual containers at a seaport. Instead, a *unit* is defined as an aggregate quantity of work to be completed by the port-of-entry, such that the port has the capability to process multiple units daily when open; for example, for a seaport that processes an average of $1,000$ import containers daily, a unit may be defined as a group of 100 containers.

Assume that unprocessed units build up in a queue at the port, and that the queue is processed in first in, first out (FIFO) order. Define u_t^a and u_t^p respectively to be the number of new work units that arrive to the port, and the maximum number of units that are processed through the port, in period t. Assume for simplicity that, for all t,

$$u_t^a = r_0, \tag{2}$$

and

$$u_t^p = \begin{cases} r_1 & \text{if } i_t = O \\ 0 & \text{if } i_t = X, \end{cases} \tag{3}$$

where r_0 and r_1 are finite, positive integer constants, and $r_1 \geq r_0$. Thus, the same amount of work arrives at the port each period (independent of the customer's ordering decision and the port's open/closed status), and the amount of work that may be processed is dependent on the state i_t. The queue length, n, at the beginning of period $t+1$ is

$$\begin{aligned} n_{t+1} &= (n_t + u_t^a - u_t^p)^+ \\ &= \begin{cases} (n_t + r_0 - r_1)^+ & \text{if } i_t = O \\ n_t + r_0 & \text{if } i_t = X, \end{cases} \end{aligned} \tag{4}$$

where $(x)^+ = max\{x, 0\}$. Define the utilization of the port-of-entry to be

$$\rho = \lim_{t\to\infty} \frac{E[u_t^a]}{E[u_t^p]} = \frac{r_0}{\pi_O^I r_1}, \tag{5}$$

where E is the expectation operator conditioned on i_0, π_O^I is the unique stationary distribution for an open port, and the limit is the Cesaro limit. To ensure queue length stability, assume $\rho < 1$.

Finally, the movement of customer orders through the port processing system may be described. At each time t, r_0 new units of work arrive at the port-of-entry representing freight for many customers. Assume that the customer's order arriving at the port at time t (the order placed in

Symbol	Definition
i_t	port status at beginning of period t, $O =$ port open status, $X =$ port closed status
$\hat{x}_t$	on-hand inventory level at the beginning of period t
x_t	inventory position prior to ordering in period t
n_t	queue length at beginning of period t
z_{kt}	order quantity in position k at time t
γ	dummy position with all orders that have arrived at customer
y	order-up-to level
$\Lambda(i_t, n_t)$	leadtime random variable
$\hat{C}(\hat{x}_t)$	total holding/penalty cost in period $t-1$
u_t^a	number of units arriving at the port in period t, set equal to r_0
u_t^p	maximum number of units that may be processed by the port in period t, set equal to r_1 when port open
D_t	demand in period t
L	transit time
c	ordering cost per unit
h	holding cost per unit per period
b	backlog penalty per unit per period

Table 1 **Summary of key terminology**

period $t-L$) becomes part of the *last* arriving unit of work; note also then that we implicitly assume that the per period customer order quantity is always smaller than one unit of port work. Given the definition of a unit of work, this is a reasonable assumption. Orders move through the port processing system with their assigned unit, and arrive at the customer during the time period when the unit is processed. Note that by assigning the customer order to the last arriving unit of work, a worst-case processing scenario is modeled. Also note that the minimum lead time for an order is $L+1$ periods. The symbols used throughout the paper are summarized in Table 1.

3.1. Modeling the Supply System Using an Order Movement Function

Given this problem setting, an optimal inventory control policy may be determined. To be consistent with existing literature, we adopt the notation for the *order position* and *order movement function* presented in Song and Zipkin (1996). To track outstanding orders through the supply system, each order is given a *position* attribute. Let z_{kt} represent the order quantity in position k at time t, where $k \in \{-(L-1), ..., -1, 0, 1, ..., n\}$. Position $k \in \{-(L-1), ..., -1, 0\}$ is used to hold an order quantity in transit (*e.g.* z_{kt} corresponds to the total order quantity that is $-k$ periods away from arriving to the port). Position $k \in \{1, 2, 3, ..., n\}$ is used to hold order quantities that have arrived at the port and are assigned to work unit k in the port processing queue. Since the queue is processed FIFO, quantities assigned to higher-numbered work units will be processed first.

Finally, dummy position $k=-L$ holds the order placed in the current period and dummy position $k=\gamma$ represents all orders that have arrived at the customer.

For every port status and queue length, (i_t,n_t), the *order movement function* $M(k|i_t,n_t)$ determines the position to which the order currently in position k will move in the next period, such that $z_{k,t+1}=\sum_{j:M(j|i_t,n_t)=k} z_{jt}$ for all k. The specification of $M(k|i_t,n_t)$ depends on u_t^p, the number of work units to be processed in period t, as follows. If $u_t^p \geq n_t+u_t^a$, the entire queue is processed:

$$M(k|i_t,n_t)=\begin{cases}\gamma & \text{if } k\geq 0\\ k+1 & \text{if } k<0.\end{cases} \tag{6}$$

If $n_t \leq u_t^p < n_t+u_t^a$, the existing queue is processed but some of the new work arriving in period $t+1$ is not:

$$M(k|i_t,n_t)=\begin{cases}\gamma & \text{if } k> 0\\ k+1 & \text{if } k\leq 0.\end{cases} \tag{7}$$

Finally if $u_t^p<n_t$, only part of the existing queue and none of the new work is processed:

$$M(k|i_t,n_t)=\begin{cases}\gamma & \text{if } k>n_t-u_t^p\\ k+r_0 & \text{if } 0<k\leq n_t-u_t^p\\ k+1 & \text{if } k\leq 0.\end{cases} \tag{8}$$

Note that this order movement function prevents order crossover.

Using this representation, the on-hand inventory $\hat{x}_{t+1}$ at the beginning of period $t+1$ is

$$\hat{x}_{t+1}=\hat{x}_t+\sum_{\{k:M(k|i_t,n_t)=\gamma\}} z_{kt}-D_t, \tag{9}$$

while the inventory position x_t (the sum of all outstanding orders plus remaining on-hand inventory) prior to ordering in period t is

$$x_t=\hat{x}_t+\sum_{-(L-1)\leq k\leq n} z_{kt}. \tag{10}$$

The leadtime random variable is represented by $\Lambda(i_t,n_t)$, where leadtime is the amount of time from when the customer places the order until the order is received, composed of both transit time, L, and the time required to process the order at the port. The order movement function can be used to specify $\Lambda(i_t,n_t)$ for the order placed in period t given (i_t,n_t):

$$\Lambda(i_t,n_t)=\min\left\{l\geq L \mid M^{l+1}(-L|i_t,n_t)=\gamma\right\}. \tag{11}$$

where $M^l(k|i_t,n_t)$ is the random variable representing the position to which the order in position k at time t will move by time $t+l$. Note that given that port utilization is less than 100%, $\Lambda(i_t,n_t)$ is finite with probability one.

3.2. Probability Mass Function for Order Leadtime

Just as the length of the leadtime in a real supply chain is key in determining levels of inventory, the leadtime in this model drives many of the theoretical and analytical results regarding optimal order-up-to policies. Specifically, the probability distribution for the order leadtime is used to derive the optimal basestock policy for the closure model with negligible congestion in Section 4.2 and to conduct the numerical analyses in Sections 4.3 and 5.2. Developing this distribution is not a simple task for this problem. If the port-of-entry is open at time t, all orders arriving to, or waiting at, the border are not necessarily processed in period t. Whether an order is processed depends on both the port status *and* on the specific position of the order in the port work queue. A careful accounting of how the state of the port changes over time and how the order moves through the port's queue is key to the derivation.

For the case when the minimum transit time $L > 1$, the expression for the leadtime distribution may be described as follows.

Proposition 1. *For all (i,n), the probability mass function of $\Lambda(i,n)$ is*
$P(\Lambda(i,n)=l|i_t=i,n_t=n)=$

$$\begin{cases} 0 & \text{if } l<L, \\ \sum_{0\le m\le r_1-r_0} f_{(i,n)}(O,m) & \text{if } l=L, \\ \sum_{0\le m\le r_1-r_0} f_{(i,n)}(X,m)p_{XO}+\sum_{r_1-r_0<m\le 2r_1-r_0} f_{(i,n)}(O,m)p_{OO} & \text{if } l=L+1, \\ \sum_{j\in\{O,X\}}\sum_{m\ge 0}\sum_{k\in\{O,X\}} P(N_{jO}(t+L,l-L-1)=\beta(m), & \\ \qquad i_{t+l-1}=k|i_{t+L}=j,n_{t+L}=m,i_t=i,n_t=n)p_{kO}f_{(i,n)}(j,m) & \text{if } l>L+1, \end{cases} \tag{12}$$

where

$$\beta(m)=\begin{cases} \lfloor\frac{m+r_0}{r_1}\rfloor & \text{if } \frac{m+r_0}{r_1}\notin Z, \\ \frac{m+r_0}{r_1}-1 & \text{otherwise} \end{cases} \tag{13}$$

$$f_{(i,n)}(j,m)=P(i_{t+L}=j,n_{t+L}=m|i_t=i,n_t=n) \tag{14}$$

and $N_{jO}(t+L,l-L-1)$ represents the number of times the port status is open from period $t+L$ through the next $l-L-1$ periods given that $i_{t+L}=j$.

The full proof of this proposition may be found in the Appendix. It is essentially an accounting exercise of the port queue dynamics. Given $(i_t,n_t)=(i,n)$, an order placed at time t arrives to the border at time $t+L$ where the length of the queue is n_{t+L}. The customer order is part of r_0 units of work to arrive at time $t+L$, such that there are $n_{t+L}+r_0$ units of work in the updated queue, and the order is assigned as a part of the last arriving unit. If the order is to be processed in period $t+l\ge t+L$, then all work units must be completely processed by the end of period $t+l$, and moreover, the last of which must be processed *in* period $t+l$. The proof models these constraints and all allowable intermediate states.

3.3. Determination of Average-Cost Optimal Ordering Decisions

Recall that the problem of the customer is to determine an ordering decision $z_{-L,t}$ each period such that long-run average costs are minimized. To do so, a dynamic programming optimality equation may be defined.

Let $(i,n,\hat{x},\mathbf{z})$ be the complete model state space for each time period $t \geq 0$, where $\mathbf{z} = (z_{-L+1}, z_{-L+2}, ...)$. A decision rule at time t is a function $\delta_t : (i,n,\hat{x},\mathbf{z}) \to Z^+$ such that $z_{-L,t} = \delta_t(i,n,\hat{x},\mathbf{z})$, and a policy is the set of decision rules $\Delta = \{\delta_t, t \geq 0\}$.

Since the average cost model does not discount future costs, and since orders cannot cross each other in time, all costs associated with the order placed in period t can be assessed to period t. A discounted cost criterion model may be used under these conditions to take into account future costs. However, the discounted cost model is considerably more complex and the results would be virtually equivalent given that a discount modeled on realistic rates is essentially inconsequential.

As shown in Song and Zipkin (1996) for the generalized model, the total cost V assessed to period t under Δ is

$$V_\Delta(i,n,\hat{x},\mathbf{z}) = c\delta(i,n,\hat{x},\mathbf{z}) + C(i,n,x+\delta(i,n,\hat{x},\mathbf{z})), \tag{15}$$

where

$$C(i,n,x+\delta(i,n,\hat{x},\mathbf{z})) = \sum_{l\geq 0} P(\Lambda(i,n) \leq l \leq \Lambda(i_+,n_+)) E\left[\hat{C}\left(x+\delta(i,n,\hat{x},\mathbf{z}) - D^{(l+1)}\right)\right], \tag{16}$$

and (i_+,n_+) represents that state of the supply system in period $t+1$. Note that time subscripts and superscripts are suppressed in this expression for simplicity, for example writing simply δ rather than δ_t. Expression $C(i,n,x+\delta(i,n,\hat{x},\mathbf{z}))$ is the expected cumulative holding and penalty costs incurred from the time the current order arrives at the customer until the time period prior to the arrival of the order placed in the next period.

For each starting state $(i_0,n_0,\hat{x}_0,\mathbf{z}_0)$, the total expected cost incurred from period 0 through some arbitrary time horizon ending in period $T-1$ under policy Δ is then

$$v_T^\Delta(i_0,n_0,\hat{x}_0,\mathbf{z}_0) = E_{(i_0,n_0,\hat{x}_0,\mathbf{z}_0)}\left\{\sum_{t=0}^{T-1} V_\Delta(i_t,n_t,\hat{x}_t,\mathbf{z}_t)\right\}, \tag{17}$$

and the average expected cost or gain is

$$g^\Delta(i_0,n_0,\hat{x}_0,\mathbf{z}_0) = \lim_{T\to\infty}\frac{1}{T}v_T^\Delta(i_0,n_0,\hat{x}_0,\mathbf{z}_0) = \lim_{T\to\infty}\frac{1}{T}E_{(i_0,n_0,\hat{x}_0,\mathbf{z}_0)}\left\{\sum_{t=1}^{T-1} V_\Delta(i_t,n_t,\hat{x}_t,\mathbf{z}_t)\right\}. \tag{18}$$

Under linear ordering costs, Song and Zipkin (1996) show for the generalized model that average cost optimality for such systems is achieved by some stationary state-dependent basestock policy that depends only on the inventory position x_t, and not on $\hat{x}_t$ and $\mathbf{z}_t$. The basestock decision rule at time t for such systems can be written as

$$\delta_t(i_t, n_t, x_t) = z_{-L,t} = \begin{cases} 0 & \text{if } x_t \geq y(i_t, n_t) \\ y(i_t, n_t) - x_t & \text{if } x_t < y(i_t, n_t) \end{cases}, \tag{19}$$

where $y(i,n)$ is the state-dependent *order-up-to* level. The resulting average-cost optimality equation is

$$g(i,n,x) + B(i,n,x) = \min_{y(i,n) \geq x} \{c(y(i,n) - x) + C(i,n,y(i,n)) + E[B(i_+, n_+, y(i,n) - D)]\}, \tag{20}$$

where $B(i,n,x)$ is the bias, and time subscripts and superscripts are again suppressed.

Theorem 1 may now be presented, providing an expression for the long-run average cost of a stationary, state-dependent basestock policy that is derived using Markov reward theory. The theorem states a new, general cost formula for the Song and Zipkin (1996) model. Let $\mathbf{W} = \{W_t : t \geq 0\}$ be a Markov chain with countable or finite state space S and transition probability matrix P. Let $V : S \to \Re$ be the cost function such that a cost of $V(s)$ is incurred at time t when $W_t = s$. The bivariate stochastic process $\{(W_t, V(W_t)) : t \geq 0\}$ is known as a Markov reward process (MRP). It is well known that in Markov decision processes (MDP), every stationary policy Δ produces an MRP (denoted $\mathbf{W}_\Delta$) with transition probability matrix P_Δ and cost V_Δ [Puterman (1994)]. This concept is central to the analysis. Subscripts and superscripts are again suppressed when appropriate, for example writing P for P_Δ.

Since there exists an optimal stationary, state-dependent basestock policy $\mathbf{y} = \{y(i,n)\}$, this analysis is confined to $\Delta = \mathbf{y}$. The resulting MRP, $\mathbf{W}_\mathbf{y}$, has finite state space S_y. Since demand is bounded and due to the structure of the policy, the state space for the inventory position is finite. The probability transition matrix is $P_\mathbf{y}$ and the cost assessed to period t is

$$V_\mathbf{y}(i,n,x) = c(y(i,n) - x)^+ + C(i,n,x + (y(i,n) - x)^+). \tag{21}$$

Then for each $(i,n,x) \in S_y$, the average expected cost or gain of policy $\mathbf{y}$ is

$$\begin{aligned} g^\mathbf{y}(i,n,x) &= \lim_{T\to\infty} \frac{1}{T} v_T^\mathbf{y}(i,n,x) = \lim_{T\to\infty} \frac{1}{T} E_{(i,n,x)} \left\{ \sum_{t=0}^{T-1} V_\mathbf{y}(W_t) \right\} \\ &= \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} P_\mathbf{y}^t V_\mathbf{y}(i,n,x) = [P_\mathbf{y}^* V_\mathbf{y}](i,n,x), \end{aligned} \tag{22}$$

where the limit exists since S_y is a finite set and where P^* is defined to be the *limiting matrix* of $\mathbf{W}$. The limiting matrix is defined by the Cesaro limit (see Appendix A.4 in Puterman (1994)) to be

$$P^* = \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} P^t. \tag{23}$$

Regardless of the periodicity characteristics of $\mathbf{W}$, the Cesaro limit exists for both countable- and finite-state Markov chains (and is equivalent to the regular limit if the chain is aperiodic). Furthermore if the Markov chain is irreducible and positive recurrent, then a unique stationary distribution π solves the system of equations $\pi = \pi P$ subject to $\sum_{(i,n,x)\in S_y} \pi_{(i,n,x)} = 1$ and $\pi_{(i,n,x)} \geq 0$ for all $(i,n,x) \in S_y$. A property of the limiting matrix is that $P^*P = P^*$. Therefore, since the stationary distribution is unique, $P^* = \pi^T e^T$ where e is a column vector of ones. That is, the rows of P^* are identical and are each equivalent to the stationary distribution π. Finally, since $\mathbf{W}$ has finite state space and is irreducible, the gain is constant for all $(i,n,x) \in S_y$ and is

$$g^{\mathbf{y}} = [P^*_{\mathbf{y}} V_{\mathbf{y}}] = \pi_{\mathbf{y}} V_{\mathbf{y}} = \sum_{(i,n,x)\in S_y} \pi^y_{(i,n,x)} \left[c(y(i,n) - x)^+ + C(i,n,x + (y(i,n) - x)^+) \right]. \tag{24}$$

The long-run average cost of a stationary, state-dependent basestock policy may be defined as follows.

THEOREM 1. *Let* $\boldsymbol{y}$ *be any stationary, state-dependent basestock policy whose resulting MRP has state-space* S_y *and stationary distribution* π^y*. Then*

$$g^y = cE[D] + \sum_{(i,n,x)\in S_y} \pi^y_{(i,n,x)} C(i,n,x + (y(i,n) - x)^+). \tag{25}$$

Proof: Note that for all $t \geq 0$, $x_{t+1} = x_t + (y(i_t,n_t) - x_t)^+ - D_t$. It follows that

$$\lim_{t\to\infty} E\left[(y(i_t,n_t) - x_t)^+ \right] = \lim_{t\to\infty} E[x_{t+1}] - \lim_{t\to\infty} E[x_t] + \lim_{t\to\infty} E[D_t], \tag{26}$$

where E is the expectation operator conditioned on (i_0,n_0,x_0) and the limit is the Cesaro limit. For any bounded or non-negative function ϕ,

$$\lim_{t\to\infty} E[\phi(i_t,n_t,x_t)] = \sum_{(i,n,x)\in S_y} \phi(i,n,x)\pi^y_{(i,n,x)}. \tag{27}$$

Applying equation (27) to the terms in equation (26),

$$\sum_{(i,n,x)\in S_y} \pi^y_{(i,n,x)} (y(i,n) - x)^+ = E[D] \tag{28}$$

From equations (24) and (28),

$$\begin{aligned} g^y &= \sum_{(i,n,x)\in S_y} \pi^y_{(i,n,x)} \left[c(y(i,n) - x)^+ + C(i,n,x + (y(i,n) - x)^+) \right] \\ &= cE[D] + \sum_{(i,n,x)\in S_y} \pi^y_{(i,n,x)} C(i,n,x + (y(i,n) - x)^+). \end{aligned}$$

□

The result also holds for a generic state component instead of the specific (i,n) that we have presented here, provided that the associated state space component is finite.

4. Systems with Negligible Port Congestion

To develop insight into the potential impact of port-of-entry closures on global supply chains, we initially consider a simplified system where the workload queue is assumed to dissipate immediately after a port is reopened (i.e., $r_1 = \infty$). This case would result if two of the three strategic areas for improvement were successfully implemented: contingency planning to reduce closure duration and congestion and investment in strategies to increase processing capabilities at ports-of-entry during states of emergency. These findings are important because they indicate the potential benefits in inventory management that may be gained through the implementation of these strategies.

If the border is open when outstanding orders arrive at the port-of-entry, the order crosses and arrives to the domestic customer without delay. Otherwise, the order is held at the port-of-entry until the border reopens. When the border reopens, all orders arriving to, or currently waiting at, the border cross and arrive at the domestic customer. In this case there is no border queue and the only relevant state information is the port-of-entry status, solely represented by the DTMC $\mathbf{I}=\{i_t, t \geq 0\}$.

Given a system where the queue length n may be ignored, several observations may be made regarding the closure model. This section will provide conditions for the optimality of a state-invariant basestock policy and show that the optimal policy for the model with negligible congestion is a stationary, state-invariant policy.

4.1. The Special Class of State-Invariant Basestock Policies

Note that the definition of a state-dependent basestock policy simply permits, but does not require, the order-up-to levels to vary from state to state. Policies in which the optimal order-up-to levels do not vary from state to state are known as a special class called state-*invariant* basestock policies. In this section, a corollary is introduced providing a sufficient condition for the optimality of a state-invariant basestock policy, a method for its calculation, and an expression for the associated long-run average cost. These results are critical to prove that the optimal policy for a system with negligible port congestion is state-invariant, which is presented in Section 4.2.

In order to provide a sufficient condition for the optimality of a state-invariant basestock policy, we first describe additional characteristics of $C(i,n,y)$, the expected cumulative holding and penalty costs incurred from the time the current order arrives until just before the order placed in the next period arrives. In the case of negligible congestion, the queue length has no impact on the leadtime probability distribution and therefore can be eliminated from the definition of $C(i,n,y)$, such that Equation (16) is reduced to the following:

$$C(i, x+\delta(i,\hat{x},\mathbf{z})) = \sum_{l \geq 0} P(\Lambda(i) \leq l \leq \Lambda(i_+)) E\left[\hat{C}\left(x+\delta(i,\hat{x},\mathbf{z}) - D^{(l+1)}\right)\right]. \tag{29}$$

Let the demand D be identically and independently distributed with probability mass function q and cumulative distribution function Q. Recall that d_1 and d_J represent the lower and upper bounds on the demand distribution, respectively, and that $D^{(l)}$ represents the cumulative demand over l periods. Then, let the demand $D^{(l)}$ be identically and independently distributed with probability mass function q_l and cumulative distribution function Q_l. Define $\xi_i = \sum_{l\geq 0} P(\Lambda(i) \leq l \leq \Lambda(i_+))$. Recall that $S_I = \{O, X\}$. Now, $C(i,y)$ may be described by the following two lemmas.

LEMMA 1. *For all $i \in S_I$, $C(i,y)$ is convex in y and $\lim_{|y|\to+\infty} C(i,y) = +\infty$.*

Proof: The convexity of $C(i,y)$ follows from the convexity of $\hat{C}(x)$ and the definition of $C(i,y)$. Since $\hat{C}(x) = \max\{-bx, hx\}$,

$$\begin{aligned}
C(i,y) &= \sum_{l\geq 0} P(\Lambda(i) \leq l \leq \Lambda(i_+)) E\left[\hat{C}(y - D^{(l+1)})\right] \\
&= \sum_{l\geq 0} P(\Lambda(i) \leq l \leq \Lambda(i_+)) E\left[\max\left\{-b(y - D^{(l+1)}), h(y - D^{(l+1)})\right\}\right] \\
&\geq \sum_{l\geq 0} P(\Lambda(i) \leq l \leq \Lambda(i_+))) \max\left\{E[-b(y - D^{(l+1)})], E[h(y - D^{(l+1)})]\right\} \\
&= \sum_{l\geq 0} P(\Lambda(i) \leq l \leq \Lambda(i_+)) \max\left\{-by + bE[D^{(l+1)}], hy - hE[D^{(l+1)}]\right\},
\end{aligned}$$

which completes the proof. The third step is valid by Jensen's Inequality. $\square$

LEMMA 2. *For all $i \in S_I$ and y,*

$$\partial C(i,y) \equiv C(i,y+1) - C(i,y) = (b+h) \sum_{l\geq 0} P(\Lambda(i) \leq l \leq \Lambda(i_+)) Q_{l+1}(y) - b\xi_i.$$

Proof: It follows from equation (1) that

$$\partial \hat{C}(y-d) \equiv \hat{C}(y+1-d) - \hat{C}(y-d) = \begin{cases} -b & \text{if } d > y \\ h & \text{if } d \leq y. \end{cases}$$

Then from Lemma 1,

$$\begin{aligned}
\partial C(i,y) &= \sum_{l\geq 0} P(\Lambda(i) \leq l \leq \Lambda(i_+)) E\left[\partial \hat{C}\left(y - D^{(l+1)}\right)\right] \\
&= \sum_{l\geq 0} P(\Lambda(i) \leq l \leq \Lambda(i_+)) \sum_{d=(l+1)d_1}^{(l+1)d_J} q_{l+1}(d) \partial \hat{C}\left(y - D^{(l+1)}\right) \\
&= \sum_{l\geq 0} P(\Lambda(i) \leq l \leq \Lambda(i_+)) \left(\sum_{d=(l+1)d_1}^{y} q_{l+1}(d) h + \sum_{d=y+1}^{(l+1)d_J} q_{l+1}(d)(-b) \right) \\
&= \sum_{l\geq 0} P(\Lambda(i) \leq l \leq \Lambda(i_+)) \left(h \sum_{d=(l+1)d_1}^{y} q_{l+1}(d) - b \left(1 - \sum_{d=(l+1)d_1}^{y} q_{l+1}(d) \right) \right)
\end{aligned}$$

$$= \sum_{l \geq 0} P(\Lambda(i) \leq l \leq \Lambda(i_+)) \left((b+h) \left(\sum_{d=(l+1)d_1}^{y} q_{l+1}(d) \right) - b \right)$$
$$= \sum_{l \geq 0} P(\Lambda(i) \leq l \leq \Lambda(i_+)) \left((b+h) Q_{l+1}(y) - b \right)$$
$$= (b+h) \sum_{l \geq 0} P(\Lambda(i) \leq l \leq \Lambda(i_+)) Q_{l+1}(y) - b\xi_i,$$

where the expectation operator in the first equation is with respect to $D^{(l+1)}$. □

Considering all stationary, state-dependent basestock policies, let $y^*(i)$ denote the smallest among all optimal order-up-to levels for state i. Considering all stationary, state-invariant basestock policies (where $y^*(i){=}y^*(j)$ for all i and j in S_I), let $\hat{y}^*(i)$ denote the smallest among all optimal order-up-to levels for state i. Define the myopic cost function as $H(i,y) = cE[D] + C(i,y)$ and let $y^+(i)$ denote the smallest among all minimizers of $H(i,y)$. We will refer to $y^+(i)$ as the myopic order-up-to level for state i.

Corollary 1 provides a sufficient condition for the optimality of a state-invariant basestock policy, a method for its calculation, and an expression for the associated long-run average cost.

Corollary 1.

(i) Let $\tilde{i} = min\{argmin_i\{y^+(i)\}\}$. *Then for all* $i \in S_I$, $y^+(\tilde{i}) \leq y^*(i) \leq y^+(i)$.

(ii) For each $i \in S_I$, *if*

$$\tilde{y} = min \left\{ (l+1)d_1 \leq y < \infty : \ y \in Z, \ \sum_{l \geq 0} Q_{l+1}(y) \frac{P(\Lambda(i) \leq l \leq \Lambda(i_+))}{\xi_i} \geq \frac{b}{b+h} \right\},$$

then $\tilde{y} = y^+(i)$.

(iii) If $y^+(0) = y^+(1) = ... = y^+(N) \equiv y^+$, *then* $y^+ = y^*(0) = y^*(1) = ... = y^*(N) = \hat{y}^*$ *and*

$$g^{\hat{y}^*} = cE[D] + \sum_{i \in S_I} \pi_i^I C(i, \hat{y}^*) = g^*,$$

where g^* *is the minimal gain over* all *stationary, state-dependent basestock policies.*

Proof: Part (i) follows from Theorem 3(a) and 3(b) in Song and Zipkin (1996). For part (ii), note that by definition, $y^+(i) = \min\{\operatorname{argmin}_{y:y\in Z}\{H(i,y)\}\}$. Removing $cE[D]$, which has no affect on $y^+(i)$, we have $y^+(i) = \min\{\operatorname{argmin}_{y:y\in Z}\{C(i,y)\}\}$. Since $\Lambda(i)$ is finite with probability 1 and order crossover is prohibited, $\xi_i > 0$ for all $i \in S_I$. From Lemma 2, for all $i \in S_I$, if $y < 0$, then $\partial C(i,y) = -b\xi_i < 0$ since $Q_{l+1}(y) = 0$ for $y < (l+1)d_1$ and $\xi_i > 0$. It follows from Lemma 1 that $y^+(i)$ is finite. The variable $y^+(i)$ may be redefined with the new bounds and two necessary conditions for optimality as $y^+(i)$ =min$\{(l+1)d_1 \leq y < \infty : y \in Z, \partial C(i,y) \geq 0\}$. The result in part (ii) then follows from Lemma 2. In part (iii), the optimality of y^+ follows directly from part (i). The left equality in the expression for the gain in part (iii) holds by Theorem 1 and the right equality holds by definition of g^* and the optimality of $\hat{y}^*$. $\square$

Lemmas 1 and 2 and Corollary 1 all hold for a generic state component instead of the specific i that we have presented here, provided that the associated state space component is finite.

4.2. The Optimality of a State-Invariant Basetock Policy

In the model with negligible congestion, the potential position space, k, is redefined as $k \in \{0, 1, 2, ...L-1\}$. The component z_{kt} of the order vector represents the order that has been outstanding for k time periods at period t. Since orders may accumulate at the border when it is closed, z_{Lt} represents the sum of all orders that have been outstanding for *at least* L periods. The order movement function describing this system is

$$M(k|O) = \begin{cases} k+1 & \text{if } 0 \le k < L, \\ \gamma & \text{if } k = L, \end{cases}$$

$$M(k|X) = \begin{cases} k+1 & \text{if } 0 \le k < L, \\ L & \text{if } k = L. \end{cases}$$

This order movement function prevents crossover.

Let $\mathbf{W}=\{W_t \equiv (i_t, x_t) : t \ge 0\}$ be the Markov chain that arises under the stationary, state-dependent basestock policy $\mathbf{y}$. Suppose that $\mathbf{W}$ has transition probability matrix P. Assuming that $y^*(O) = y^*(X) \equiv \hat{y}$, the one-step transition probability of $\mathbf{W}$ is $[P]_{(i,x),(j,x')} = p_{ij} P(D = \hat{y} - x')$ for all (i, x) and (j, x').

The probability distribution for the order leadtime $\Lambda(i)$ is now developed. An order placed at time t when $i_t = i$ will arrive at time $t + \Lambda(i)$. From the order movement function, $P(\Lambda(i) = m) = 0$ for $0 \le m \le L-1$. The leadtime is exactly L if and only if $i_{t+L} = O$ and so $P(\Lambda(i) = L) = p_{iO}^{(L)}$. Similarly, the leadtime is exactly $L+1$ if and only if $i_{t+L} = X$ and $i_{t+L+1} = O$. Therefore $P(\Lambda(i) = L+1) = p_{iX}^{(L)} p_{XO}$. Note that $P(\Lambda(i) = L+1) \ne p_{iO}^{(L+1)}$ since i_{t+L} cannot be O. Similarly for $m \ge 2$, $P(\Lambda(i) = L+m) = p_{iX}^{(L)} p_{XX}^{m-1} p_{XO}$. Because the queue at the port-of-entry does not have to be accounted for in the case with negligible congestion, Proposition 1 reduces to:

$$P(\Lambda(i) = l) = \begin{cases} 0 & \text{if } l < L, \\ p_{iO}^{(L)} & \text{if } l = L, \\ p_{iX}^{(L)} p_{XX}^{l-L-1} p_{XO} & \text{if } l > L. \end{cases} \tag{30}$$

It can then be shown that

$$P(\Lambda(i) \le l) = \begin{cases} 0 & \text{if } l < L, \\ p_{iO}^{(L)} & \text{if } l = L, \\ 1 - p_{iX}^{(L)} p_{XX}^{l-L} & \text{if } l > L, \end{cases} \tag{31}$$

$$P(\Lambda(i) \le l \le \Lambda(i_+)) = P(\Lambda(i) \le l) - \sum_{j \in \{O,X\}} p_{ij} P(\Lambda(j) \le l-1) = \begin{cases} 0 & \text{if } l < L, \\ p_{iO}^{(L)} & \text{if } l = L, \\ p_{iO}^{(L)} p_{OX} p_{XX}^{l-L-1} & \text{if } l > L, \end{cases} \tag{32}$$

and

$$\xi_i = p_{iO}^{(L)} \left(1 + \frac{p_{OX}}{p_{XO}}\right). \tag{33}$$

The following corollary states that the optimal stationary, state-dependent basestock policy for the closure model with negligible congestion is actually a stationary, state-invariant basestock policy. Note that even though the optimal order-up-to levels are independent of the exogenous supply system state, it is not valid to claim that the model then reduces to one with a single-state exogenous border system. The system clearly affects the leadtime probability distribution, the order-up-to levels and the resultant long-run average cost.

Corollary 2. *For the closure model with negligible congestion, $y^*(O) = y^*(X)$.*

Proof: Since it is assumed that $0 < p_{OX} < 1$ and $0 < p_{XO} < 1$, it can be shown that $\xi_i > 0$ for all $i \in \{O, X\}$. Consider the left-hand side of the inequality in the second necessary condition within the minimization in Corollary 1(ii). From equations (32) and (33), this expression can be written as

$$\sum_{l \ge 0} Q_{l+1}(y) \frac{P(\Lambda(i) \le l \le \Lambda(i_+))}{\xi_i} = \frac{Q_{L+1}(y)}{1 + \frac{p_{OX}}{p_{XO}}} + \sum_{l > L} \left(\frac{p_{OX} p_{XX}^{l-L-1}}{1 + \frac{p_{OX}}{p_{XO}}} \right) Q_{l+1}(y),$$

which is independent of i. Thus the same $\tilde{y}$ will be found in Corollary 1(ii) for both border states and is therefore the optimal state-invariant order-up-to level by Corollary 1(iii). $\square$

It may appear to be counter-intuitive that the order-up-to level is the same whether the port-of-entry is open or closed. While the order-up-to level will change depending on the probability or expected duration of a closure, the optimal policy is not impacted by the current state of the port. Without congestion the leadtime distribution is independent of queue length and a port-of-entry closure is a temporary pause on flow through the system that does not create any lasting effects after reopening. If a port-of-entry is closed, demand at the customer arrives (and accumulates) with the same distribution that it did when the port-of-entry was open, while orders continue to be placed that will satisfy all accumulated demand and bring inventory levels back to normal when the closure ends. The orders arrive at the port-of-entry and wait for reopening. With negligible congestion all units are immediately passed to the customer as soon as the port-of-entry reopens, fulfilling the accumulated demand and restoring the inventory level. Altering the order-up-to level during a port-of-entry closure would simply change the resultant inventory level when the port-of-entry is reopened.

Alternatively, when congestion is a factor, the accumulated inventory must wait in a queue to be processed when the port-of-entry reopens after a closure, impacting the leadtime. Therefore, the leadtime used to calculate the order-up-to level varies dependent on the state of the port-of-entry and these results do not necessarily hold for the case with non-negligible congestion. Further, this result will only hold if the assumption of a finite state space holds.

This finding highlights the importance of efficient port-of-entry operations. A state-invariant policy makes inventory planning considerably simpler. Under conditions of negligible congestion, a supply chain manager may safely apply the same policies whether the port is open or closed. While the elimination of seaport congestion is already a vital, yet hard to achieve, goal that is facing many port operators, this provides further incentive to reach this goal.

4.3. Numerical Analysis

For reasonable ranges of transition probabilities, the effects of worsening closure conditions on the optimal long-run average inventory costs per day were limited for the case of negligible congestion. For example, the optimal long-run average cost per day did not increase by more than 6.16% from the best to the worst case instances within these ranges. Furthermore, for a case that represents well a modern Asia-US supply chain, the increase was only 0.60%. See Lewis (2005) for the complete numerical analysis results. Given that cost is not severely impacted for the case with negligible congestion, it is of interest to determine if the case with non-negligible congestion behaves in a similar manner.

5. Systems with Non-Negligible Congestion

While Section 4 provides a theoretical basis for studying the problem of port congestion, in all likelihood, most systems will experience at least some form of congestion when recovering from a closure. Developing similar closed form expressions for the case with non-negligible congestion is not necessarily possible due to increased complexity. Therefore, the impacts of port-of-entry closures and the resulting congestion on global supply chains are studied numerically using the model developed in Section 3. The results in this section highlight three important strategic issues that can improve the productivity and reliability of international supply chain systems:

1. The development of effective contingency plans that reduce the duration of port-of-entry closures and prioritize the return to a normal state of operation;

2. The development of inventory management strategies by supply chain firms that plan for the possibility of port-of-entry closures and congestion; and

3. The investment in strategies to increase the processing capabilities of highly utilized and congested ports-of-entry during states of emergency.

The model may be used to evaluate these issues by adjusting the appropriate parameters. Port-of-entry closure duration is analyzed through the modification of the probability of a port transitioning from closed to open, p_{XO}. As this probability value decreases and the port is closed for a longer period, the cost of the closure event rises. The scale of this cost increase reflects the relative importance of limiting closure duration.

The importance of planning for the possibility of closures and congestion is evaluated by determining the costs that a firm may incur when there is no contingency planning. Using a probability of port closure equal to zero, such that $p_{OX} = 0$, results in an inventory policy that does not consider potential port-of-entry closures. The cost will generally increase when this policy is adopted under conditions with nonzero closure probabilities. Quantifying this cost provides a metric to evaluate the need for port-of-entry closure contingency planning.

Finally, highly utilized ports-of-entry are evaluated by decreasing the processing parameter, r_1, mimicing a port with increasing utilization and, potentially, congestion. Similarly to the analysis of closure duration, the increase in costs as utilization rises reflects the need to combat congestion at the busiest of ports.

5.1. Determination of Optimal Policies

In order to evaluate these various scenarios, the model is tested using an appropriate algorithm. The numerical results were determined using a standard implementation of the value iteration algorithm (VIA) for Markov decision problems. A detailed description of the algorithm can be found in Puterman (1994). To enable efficient computation, the allowable state space and order quantities are constrained to be finite without substantially affecting the optimal solution.

The VIA terminates in a finite number of iterations with an ϵ-optimal policy. Its long-run average cost at the termination of the VIA (denoted g_ϵ) satisfies the following inequality: $g_\epsilon - g^* < \epsilon$. Furthermore, the true optimal long-run average cost is approximated as in Theorem 8.5.6(b) in Puterman (1994) (denoted g') and so $|g' - g^*| < \epsilon/2$. For any positive ϵ, no matter how small, the policy obtained at the termination of the VIA may be sub-optimal. However in this paper, as is common, the policy obtained by the VIA algorithm and the approximation of the optimal long-run average cost is referred to as the optimal policy and the optimal long-run average cost. For the numerical study that follows, ϵ is set at 0.01 and therefore the maximum difference between the approximate and true optimal long-run average cost is equivalent to less than \$0.005.

5.2. Experimental Design

Consider a supply chain where the domestic customer reviews its inventory and places daily re-supply orders in container-loads. Suppose that the deterministic leadtime from the supplier to the domestic seaport is L=15 days, corresponding to typical ocean carrier service from Asia to the Western United States. Assume a value of c=\$150,000 for the contents of a forty-foot container. This is a reasonable average value for goods shipped via ocean carrier, but may be low for some commodities such as electronics or pharmaceuticals. From Theorem 1, note that the purchase cost parameter does not directly affect the optimal inventory policy and that it contributes to the long-run average cost per day as an additive term, $cE[D]$. Therefore, only results for the optimal long-run average daily holding and penalty costs are presented. Given the value of the container contents, a daily holding cost of h=\$100 per day represents a 24.33% annual holding cost rate, a reasonable rate for many supply chains. The daily penalty cost for unfilled demand is b=\$1,000, representing an annual penalty cost of 2.4 times the container contents value.

Demand has a truncated Poisson distribution with mean demand of 0.5 containers per day, and a maximum realizable demand in any period of $d_J = 10$. A truncated Poisson distribution assigns Poisson probabilities to all demand realizations up through d_{J-1} and a probability of $1-Q(d_{J-1})$ to d_J, where d_J is chosen such that $1-Q(d_{J-1}) < \epsilon$ for some $\epsilon > 0$. In this problem instance, $P(D=10) = 1-Q(9) = 1.63\times 10^{-10}$.

For the freight processing model of a seaport, assume r_0=10 units of work per day and r_1=11 units of work per day. Therefore a port closure of 10 days results in congestion that will last for at least 100 days. The values of r_0 and r_1 were selected to provide a realistic model of potential seaport congestion. For example in the port security wargame described earlier, eight days of seaport closure resulted in 92 days of port congestion, and the 2002 10-day closure of Western U.S. seaports resulted in months of congestion.

Finally, several sets of port-of-entry closure and reopening probabilities that represent a wide-range of plausible real-world scenarios are considered: $p_{OX} \in \{0.001, 0.003, 0.01, 0.02\}$ and $p_{XO} \in \{0.05, 0.1, 0.2, 0.3, 0.4, 0.5\}$. These sets correspond to an expected inter-closure time $(1/p_{OX})$ ranging from approximately three years to 50 days, and an expected closure duration $(1/p_{XO})$ ranging from 20 days to two days. Note that the shorter inter-closure time and closure duration may replicate delays at an airport due to inclement weather. The probabilities can not allow for an expected port utilization that is equal to or greater than one; therefore, the following pairs are excluded: $(p_{OX}, p_{XO}) \in \{(0.01, 0.1), (0.01, 0.05), (0.02, 0.2), (0.02, 0.1), (0.02, 0.05)\}$. The remaining pairs translate to expected utilizations greater than 90%, which is representative of the situation at many major seaports. For example, many seaports in the U.S. operate at close to capacity throughout the year.

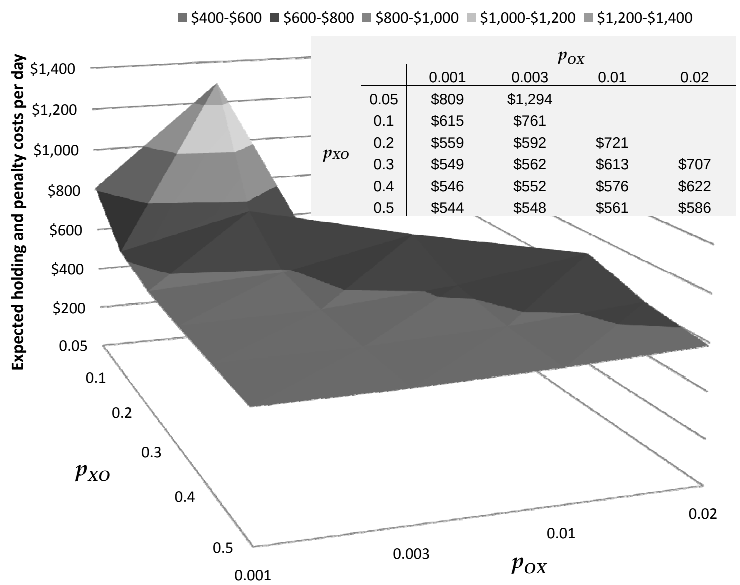

Figure 2 **Expected holding and penalty costs per day for the range of transition probabilities.**

5.3. Effect of Closure Duration on Costs and Order-up-to Levels

The most critical information resulting from this analysis is the effect that port closure duration has on holding and penalty costs, as well as order-up-to levels. The long-run average holding and penalty costs per day using the optimal action sequence are presented in Figure 2 and selected optimal order-up-to levels for a range of port status transition probabilities are shown in Table 2. When the port is open, the optimal order-up-to levels exhibit little variation over the range of the transition probabilities, so these are not presented graphically. However, the optimal order-up-to levels do vary with the transition probabilities when the port is closed; Table 2 depicts the cases when the port is closed and there are zero and 100 units of work in the port work queue, respectively.

		$n = 0$				$n = 100$			
		p_{ox}				p_{ox}			
		0.001	0.003	0.01	0.02	0.001	0.003	0.01	0.02
p_{xo}	0.05	25	28			30	33		
	0.1	18	18			23	24		
	0.2	14	14	15		20	20	21	
	0.3	13	13	14	14	19	19	19	20
	0.4	13	13	13	13	18	18	19	19
	0.5	12	12	13	13	18	18	18	18

Table 2 **Optimal order-up-to level for the range of transition values when the port is closed with queue lengths of 0 (no congestion) and 100 (significant congestion).**

For the studied ranges, observe that the optimal long-run average holding and penalty cost and order-up-to levels are non-decreasing as p_{XO} decreases or p_{OX} increases. As the likelihood of a closure (p_{OX}) or the expected duration of a closure ($1/p_{XO}$) increases, a firm should carry more inventory. Furthermore, the optimal long-run average holding and penalty cost and order-up-to levels are more sensitive to p_{XO} than to p_{OX}. Therefore, a firm will be impacted more by the length of a closure than the probability of closure. This result underlines the importance of reopening the port as soon as possible in the event of a closure.

As Figure 2 and Figure 3 show, the optimal long-run average holding and penalty costs per day increase superlinearly both as p_{XO} decreases and as $1/p_{XO}$ increases. Note that this will not always occur. For example, costs that simply decrease linearly with $1/\chi$ may increase superlinearly with χ (where χ is a dummy variable).

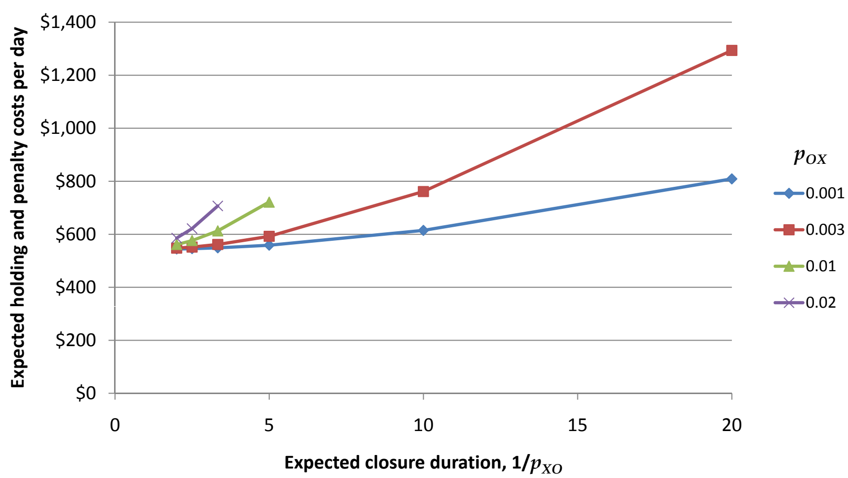

Figure 3 Expected holding and penalty costs per day based on expected closure duration).

As discussed in Lewis et al. (2006), longer port-of-entry closures and increased congestion can quickly erode profit margins. Not unexpectedly, the greatest increases in the optimal long-run average holding and penalty cost and order-up-to levels occur when p_{XO} is small, corresponding to long expected closures. For example, when p_{OX}=0.003 and the expected closure duration increases from two to 20 days, the expected holding and penalty cost per day increases by 136%, from \$548 to \$1,294 per day. Such cost increases represent significant reductions in operating margin. Assuming that this firm earns annual revenue equal to average yearly demand multiplied by the average value of a container's contents and operates with a 10% margin, its annual operating income would be about \$2.7 million. Thus, an increase in expected closure duration from two to 20 days results in a 10% reduction in operating margin.

These results have important implications for supply chain stakeholders, both private firms and public agencies. While prevention of closures is critically important (especially due to security-related incidents), economic impacts should also be mitigated by the deployment of effective contingency plans that reduce the duration of potential seaport closures and quickly return the freight transportation systems to normal operation. Such contingency plans may include the re-routing of in-transit freight to other ports-of-entry (when feasible) or, in the case of extreme residual congestion, specially allocated investment to enable temporary increases in processing capacity at the disrupted seaport or appropriate alternate ports-of-entry. The development and implementation of effective contingency plans will require the strategic engagement and cooperation of private firms and public agencies.

5.4. Performance of Inventory Policy without Consideration for Port-of-Entry Closure

In the previous section, it was assumed that the domestic customer makes optimal decisions given complete knowledge of the dynamics of port-of-entry closures and reopenings and the workload at the port-of-entry (e.g. queue length). Thus, the results of the analytical study indicate the best-expected performance that the domestic customer could achieve. Suppose that a domestic customer optimizes its inventory policy without regard to seaport closures and congestion. The policy may be sub-optimal when implemented and could be a costly managerial error. This policy is referred to as the implemented policy. The difference in cost between the optimal inventory policy and the implemented policy quantifies the savings from contingency planning for port-of-entry closures and congestion.

The implemented policy is determined by evaluating the optimal inventory policy when the probability of seaport closure is assumed to be zero, e.g. p_{OX}=0. The cost of this policy may then be compared to the long-run average cost given the true closure probability for each case. Figure 4 displays the amount that the cost decreases from that of the implemented policy when the potential for port closure is considered. Note that the implemented policy is independent of the system state, such that $y^*(i,n)$=12 containers regardless of the port status and border queue length.

Port-of-entry closures are low probability events and some firms may substantially underestimate them or even choose to ignore them. Sub-optimal decision making that does not consider low probability events may be acceptable in many cases. However, there are clearly scenarios for which contingency planning for port closures and congestion is critically important. Total cost savings of only 1-2% correspond to annual dollar savings in the hundreds of thousands. The most impressive data point corresponds to the case when p_{OX}=0.003 and p_{XO}=0.05. In this case, the annual

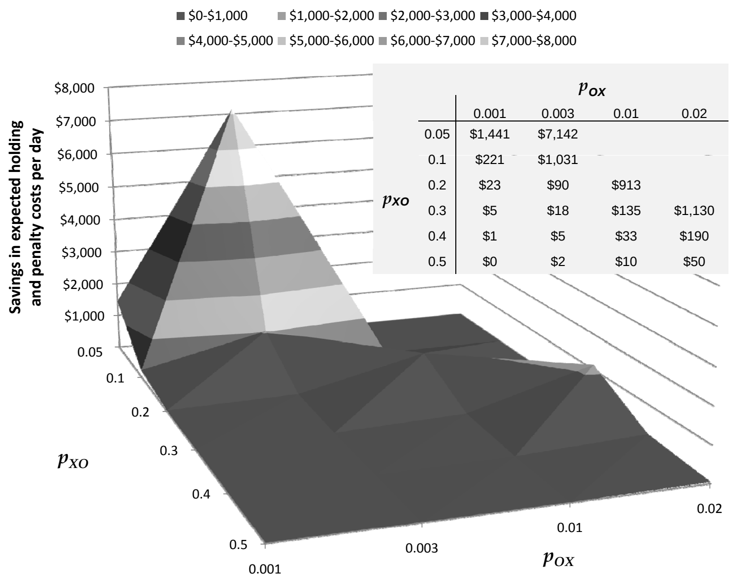

Figure 4 **Cost savings per day due to contingency planning that considers port-of-entry closures.**

expected cost penalty is approximately \$2.6 million; such a cost nearly consumes all operating income in this case. Supply chain firms operating under suboptimal inventory policies face exacerbated cost impacts of seaport closures and congestion when compared to firms operating optimally. Of course, firms must also be careful not to overestimate expected port closure times when developing inventory policies. Such over-estimations will lead to unnecessarily larger order-up-to levels and consequently larger inventory holding costs.

5.5. Impact of Port-of-Entry Utilization on Cost and Order-up-to Levels

Investment in strategies that increase the processing capabilities of highly utilized and congested ports-of-entry during states of emergency is critical to global supply chains. Utilization provides a means for measuring a seaport's excess processing capacity. Utilization and excess processing capacity are inversely proportional; as utilization increases, a seaport's ability to process the freight backlog after a disruption diminishes. Therefore, closures more negatively impact supply chains that rely on highly utilized seaports. In this section, the impact of seaport utilization on the optimal long-run average holding and penalty cost and order-up-to levels is investigated.

Recall that port utilization is given by $\rho = r_0/(\pi_O r_1)$ and is therefore affected by the processing parameters (r_0 and r_1) as well as the port status transition probabilities (through the stationary distribution). In this section, the arrival parameter (r_0=10) and the probability of transitioning from open to closed (p_{OX}=0.003) are fixed. The processing parameter (r_1) and the expected closure duration ($1/p_{XO}$) are then varied. Let Ω be the optimal long-run average holding and penalty cost. Table 3 displays the optimal expected holding and penalty costs per day and the order-up-to levels

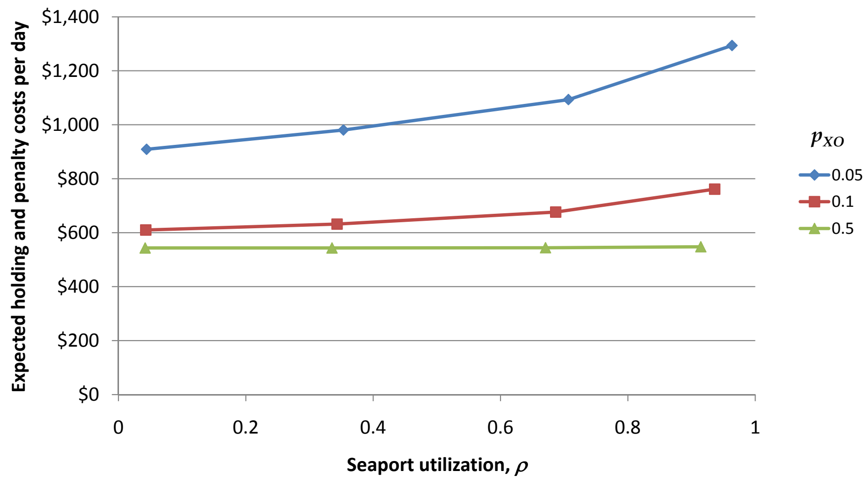

Figure 5 Change in long-run average holding and penalty costs per day as seaport utilization increases.

when there is significant congestion. The table also shows the differences in utilization values as the transition probabilities and arrival and processing parameters are varied. Figure 5 displays the optimal long-run average holding and penalty costs per day versus port utilization when r_0=10 and r_1=11.

p_{XO}	r_0	r_1	ρ	Ω	$y^*(O,100)$	$y^*(X,100)$
0.05	1	24	0.044	$909	13	13
0.05	10	30	0.353	$981	13	13
0.05	10	15	0.707	$1,093	14	18
0.05	10	11	0.964	$1,294	19	33
0.1	1	24	0.043	$610	12	12
0.1	10	30	0.343	$632	12	13
0.1	10	15	0.687	$676	13	16
0.1	10	11	0.936	$761	18	24
0.5	1	24	0.042	$543	12	12
0.5	10	30	0.335	$543	12	12
0.5	10	15	0.671	$544	12	13
0.5	10	11	0.915	$548	17	18

Table 3 Impact of seaport utilization (ρ) on expected costs and optimal order-up-to levels

As utilization increases and potential congestion becomes more severe, optimal expected holding and penalty costs per day and the optimal order-up-to levels increase. As r_1 decreases relative to a fixed value of r_0, fewer units of work can be completed in any open period, which means that queues will require a greater number of periods to be reduced. Figure 5 indicates that the holding and penalty costs per day (and the optimal order-up-to levels) increase more than linearly with seaport utilization. When p_{XO}=0.05 and the port utilization increases from 70.7% to 96.4%, the expected holding and penalty costs per day increase by nearly 20%. These results again assume optimal decision-making, and the expected costs per day would be even greater if the domestic customer implemented a sub-optimal inventory policy. Therefore, investments that increase the processing capabilities of highly-utilized seaports are crucial to reducing the impacts on supply chain productivity by seaport closures and congestion and need to be prioritized. Since additional processing capability is only required when congestion develops following a temporary closure, such investments should again be directed to strategies that enable such highly-utilized seaports to temporarily increase capacity immediately following a closure.

6. Conclusions

Ports-of-entry are among the most critical components of the modern international supply chain infrastructure, particularly container seaports. As global reliance on containerized cargo and ocean-shipping continues to increase, research about the potential impacts of disruptions such as unexpected port-of-entry closures and the resulting congestion is critical to both users and operators of international supply chain systems.

This research highlights three strategic activities involving supply chain users, operators, and government agencies that can improve the productivity and reliability of international supply chain systems. First, implementing a plan that considers potential port closures results in a cost savings over plans that do not in virtually every scenario tested, indicating that supply chain users should develop inventory management strategies that plan for the possibility of port-of-entry closures and resultant congestion. Firms that fail to do so may expect annual inventory-related cost increases that completely consume operating income. Second, it is shown that increasing port closure duration leads to a considerable reduction in operating margin. While the prevention of disruptions is critically important, the supply chain entities must also engage and cooperate with each other to design effective contingency plans that reduce the duration of port-of-entry closures and quickly return supply chain systems to normal states of operation. Finally, increasing port utilization may increase holding and penalty costs significantly. Investment by infrastructure operators and government agencies should be made to increase the processing capabilities of highly utilized ports-of-entry, especially during the states of emergency that immediately follow a closure when congestion is worst.

While the first issue relates to a potential strategic choice for supply chain firms, the latter two issues relate to choices to be made by government authorities and port terminal operators. However as supply chain firms are clearly impacted by the decisions of the other stakeholders, supply chain firms must be engaged in the latter two issues to ensure more favorable supply chain systems. One example of the interaction between supply chain firms and government authorities is the Customs-Trade Partnership Against Terrorism (C-TPAT), a program created by U.S. Customs and Border Protection through which supply chain firms voluntarily improve their supply chain security practices in exchange for more favorable treatment in clearing U.S. customs. Bakshi and Gans (2010) provide an analysis of C-TPAT, showing how this program can help alleviate seaport congestion.

Future research of micro-level operations of supply chain systems, as considered in this paper, is still necessary to fully understand the impacts of port-of-entry closures and the potential management strategies that firms can employ to deal with them. However, the results here indicate the importance of future research of macro-level operations of supply chain systems as a whole, involving multiple supply chain users, trade lanes, and ports-of-entry. While these types of centralized optimization and management policies may provide an overall benefit to a supply chain system, it is expected that individual supply chain users and operators may be negatively impacted by them. Compensation mechanisms should be considered in this research to account for cost-benefit imbalances. Some governments are beginning to research these types of systems-level policies (such as the U.S. Department of Transportation). A necessity for such policies is what Lee and Wolfe (2003) refer to as “total supply network visibility,” which is discussed as a strategy to help mitigate the impacts of a supply chain security breach. Continuing with the concept of visibility, another area of future research is the partial-observability of port-of-entry congestion. Domestic customers may have an indication of the level of congestion, but good metrics for port-of-entry congestion (and other components of supply chain system visibility) may be difficult to determine, monitor, and disseminate.

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Appendix. Proof of Proposition 1

Proof: Given $(i_t, n_t) = (i, n)$, $\Lambda(i, n) = l$ for $l \geq L$ if and only if the following two events occur:

(i) $i_{t+l} = O$, and

(ii) *If* $l = L$, *then*

$$n_{t+L} + r_0 \leq r_1. \tag{34}$$

If $l > L$ *and* $n_{t+L} = m$, *then*

$$N_{i_{t+L}O}(t+L, l-L-1) = \beta(m). \tag{35}$$

The first condition holds as $i_{t+l} = X$ implies that $u^p_{t+l} = 0$ and the order cannot be processed in that period. Thus i_{t+l} must be O. The second condition accounts for the dynamics of the port work queue. Given $(i_t, n_t) = (i, n)$, an order placed at time t arrives to the border at time $t+L$ where the length of the queue is n_{t+L}. Then r_0 units of work arrive and the customer order is assigned to the last arriving unit. After the arrival of new work this queue is referred to as the *full queue*. There are $n_{t+L} + r_0$ units of work in the full queue and the order placed at time t is at the very end. If the order is to be processed in period $t+l$, then all customers that have arrived to the queue by the end of period $t+L$ must be completely processed by the end of period $t+l$, and moreover, the last of which must be processed in period $t+l$.

If the order is to be processed in period $t+L$, then the number of units of work processed in period $t+L$ must be *at least* the number of units of work in the full queue. That is,

$$n_{t+L} + r_0 \leq u^p_{t+L} = r_1. \tag{36}$$

The equality holds since $i_{t+L} = O$ from part (i).

If the order is to be processed in period $t+l$ for $l > L$, then the number of units of work processed during the time interval $[t+L, t+l]$ must be at least the number of units of work in the full queue. This condition is clearly necessary but not sufficient, since it allows the full queue to be completely processed in a period prior to period $t+l$. Therefore a second condition is required to ensure that by the end of period $t+l-1$, there are still a positive number of units of work remaining in the full queue. This means that the customer order has not yet been processed. These two conditions are respectively represented by the following two inequalities:

$$N_{i_{t+L}O}(t+L, l-L)r_1 \geq n_{t+L} + r_0, \tag{37}$$

and

$$N_{i_{t+L}O}(t+L, l-L-1)r_1 < n_{t+L} + r_0. \tag{38}$$

Note that by definition, $N_{ij}(t, l)$ equals either $N_{ij}(t, l-1)$ or $N_{ij}(t, l-1) + 1$ with probability one. However given that the condition in part (i) holds, $N_{i_{t+L}O}(t+L, l-L) = N_{i_{t+L}O}(t+L, l-L-1) + 1$ with probability one. Substituting this into equation (37) and rearranging terms,

$$N_{i_{t+L}O}(t+L, l-L-1)r_1 \geq n_{t+L} + r_0 - r_1. \tag{39}$$

Combining equations (39) and (38) and dividing by r_1,

$$\frac{n_{t+L}+r_0}{r_1}-1\leq N_{i_{t+L}O}(t+L,l-L-1)<\frac{n_{t+L}+r_0}{r_1}$$

$$\Longleftrightarrow$$

$$\alpha-1\leq N_{i_{t+L}O}(t+L,l-L-1)<\alpha, \tag{40}$$

where $\alpha=\frac{n_{t+L}+r_0}{r_1}$.

Assume α is integer. Since $N_{i_{t+L}O}(t+L,l-L-1)$ is integer-valued, equation (40) holds if and only if $N_{i_{t+L}O}(t+L,l-L-1)=\alpha-1$. Now assume that α is not integer. Then equation (40) holds if and only if $N_{i_{t+L}O}(t+L,l-L-1)=\lfloor\alpha\rfloor$. The conditions in parts (i) and (ii) are both necessary and sufficient.

To determine $P(\Lambda(i,n)=l|i_t=i,n_t=n)$ for all $l\geq 0$, four cases for the value of l are considered. Leadtime can not be shorter than transit time, L, such that $P(\Lambda(i,n)=L|i_t=i,n_t=n)=0$ for $l<L$. If leadtime is equal to transit time, $l=L$, the port must be open when the order arrives and the port must be able to process the entire queue. Therefore,

$$\begin{aligned}P(\Lambda(i,n)=L|i_t=i,n_t=n) &= P(i_{t+L}=O,n_{t+L}+r_0\leq r_1|i_t=i,n_t=n)\\ &= P(i_{t+L}=O,n_{t+L}\leq r_1-r_0|i_t=i,n_t=n)\\ &= \sum_{0\leq m\leq r_1-r_0} P(i_{t+L}=O,n_{t+L}=m|i_t=i,n_t=n)\\ &= \sum_{0\leq m\leq r_1-r_0} f_{(i,n)}(O,m).\end{aligned} \tag{41}$$

In the case where $l=L+1$, the port must be open for period $t+L+1$ and the queue must be long enough such that the processing of units arriving during period $t+L$ cannot be completed until period $t+L+1$. Therefore,

$$\begin{aligned}&P(\Lambda(i,n)=L+1|i_t=i,n_t=n)\\ &= P(i_{t+L+1}=O,N_{i_{t+L}O}(t+L,0)=\beta(n_{t+L})|i_t=i,n_t=n)\\ &= \sum_{j\in\{O,X\}}\sum_{m\geq 0} P(i_{t+L}=j,n_{t+L}=m,i_{t+L+1}=O,N_{i_{t+L}O}(t+L,0)=\beta(n_{t+L})|i_t=i,n_t=n)\\ &= \sum_{j\in\{O,X\}}\sum_{m\geq 0} P(i_{t+L+1}=O,N_{jO}(t+L,0)=\beta(m)|i_{t+L}=j,n_{t+L}=m,i_t=i,n_t=n)\\ &\qquad *P(i_{t+L}=j,n_{t+L}=m|i_t=i,n_t=n)\\ &= \sum_{j\in\{O,X\}}\sum_{m\geq 0} P(i_{t+L+1}=O|N_{jO}(t+L,0)=\beta(m),i_{t+L}=j,n_{t+L}=m,i_t=i,n_t=n)\\ &\qquad *P(N_{jO}(t+L,0)=\beta(m)|i_{t+L}=j,n_{t+L}=m,i_t=i,n_t=n)f_{(i,n)}(j,m)\\ &= \sum_{j\in\{O,X\}}\sum_{m\geq 0} p_{jO}P(N_{jO}(t+L,0)=\beta(m)|i_{t+L}=j,n_{t+L}=m,i_t=i,n_t=n)f_{(i,n)}(j,m)\\ &= \sum_{m\geq 0:\ \beta(m)=0} f_{(i,n)}(X,m)p_{XO}+\sum_{m\geq 0:\ \beta(m)=1} f_{(i,n)}(O,m)p_{OO}\\ &= \sum_{0\leq m\leq r_1-r_0} f_{(i,n)}(X,m)p_{XO}+\sum_{r_1-r_0<m\leq 2r_1-r_0} f_{(i,n)}(O,m)p_{OO}.\end{aligned}$$

The second to last equation follows since $N_{jO}(t+L,0)$ represents the number of visits to state O only in period $t+L$ and can therefore only take on values of 0 (if $j=X$) or 1 (if $j=O$). The final equation follows from (13). For example $\beta(m)=0$ for values of $m \geq 0$ such that $\frac{m+r_0}{r_1} \leq 1$, which implies that $m \leq r_1 - r_0$. Since the customer queue length is non-negative, $m \geq 0$.

Similarly, when $l > L+1$ the port must be open for period $t+l$ and the queue must be long enough such that the processing of units arriving during period $t+l-1$ cannot be completed until period $t+l$. Therefore,

$$
\begin{aligned}
&P(\Lambda(i,n)=l|i_t=i,n_t=n)\\
&= P(i_{t+l}=O, N_{i_{t+L}O}(t+L,l-L-1)=\beta(n_{t+L})|i_t=i,n_t=n)\\
&= \sum_{j\in\{O,X\}}\sum_{m\geq 0} P(i_{t+L}=j,n_{t+L}=m,i_{t+l}=O,N_{i_{t+L}O}(t+L,l-L-1)=\beta(n_{t+L})|i_t=i,n_t=n)\\
&= \sum_{j\in\{O,X\}}\sum_{m\geq 0} P(i_{t+l}=O,N_{jO}(t+L,l-L-1)=\beta(m)|i_{t+L}=j,n_{t+L}=m,i_t=i,n_t=n)\\
&\qquad *P(i_{t+L}=j,n_{t+L}=m|i_t=i,n_t=n)\\
&= \sum_{j\in\{O,X\}}\sum_{m\geq 0} P(i_{t+l}=O,N_{jO}(t+L,l-L-1)=\beta(m)|i_{t+L}=j,n_{t+L}=m,i_t=i,n_t=n)f_{(i,n)}(j,m)\\
&= \sum_{j\in\{O,X\}}\sum_{m\geq 0} P(i_{t+l}=O|N_{jO}(t+L,l-L-1)=\beta(m),i_{t+L}=j,n_{t+L}=m,i_t=i,n_t=n)\\
&\qquad *P(N_{jO}(t+L,l-L-1)=\beta(m)|i_{t+L}=j,n_{t+L}=m,i_t=i,n_t=n)f_{(i,n)}(j,m)\\
&= \sum_{j\in\{O,X\}}\sum_{m\geq 0}\sum_{k\in\{O,X\}} P(i_{t+l-1}=k,i_{t+l}=O|N_{jO}(t+L,l-L-1)=\beta(m),i_{t+L}=j,n_{t+L}=m,i_t=i,n_t=n)\\
&\qquad *P(N_{jO}(t+L,l-L-1)=\beta(m)|i_{t+L}=j,n_{t+L}=m,i_t=i,n_t=n)f_{(i,n)}(j,m)\\
&= \sum_{j\in\{O,X\}}\sum_{m\geq 0}\sum_{k\in\{O,X\}} P(i_{t+l}=O|i_{t+l-1}=k,N_{jO}(t+L,l-L-1)=\beta(m),i_{t+L}=j,n_{t+L}=m,i_t=i,n_t=n)\\
&\qquad *P(N_{jO}(t+L,l-L-1)=\beta(m),i_{t+l-1}=k|i_{t+L}=j,n_{t+L}=m,i_t=i,n_t=n)f_{(i,n)}(j,m)\\
&= \sum_{j\in\{O,X\}}\sum_{m\geq 0}\sum_{k\in\{O,X\}} P(N_{jO}(t+L,l-L-1)=\beta(m),i_{t+l-1}=k|i_{t+L}=j,n_{t+L}=m,i_t=i,n_t=n)\\
&\qquad *p_{kO}f_{(i,n)}(j,m).
\end{aligned}
$$

□

It is not easy in general to develop a closed-form expression for probability distribution $f_{(i,n)}$, due to the max operator that governs queue processing. However, $f_{(i,n)}$ can of course be determined via explicit enumeration of all possible port-of-entry status sample paths for $L+1$ periods, given each possible initial state (i,n). In practice, such enumeration is reasonable computationally as long as L and n are relatively small.

The probability distribution for $N_{ij}(t,l)$ can be derived using methods developed first in Gabriel (1959). For all i, $j=O$, $t\geq 0$ and $l\geq 1$,

$$
P(N_{OO}(t,l)=1+\nu)=\begin{cases} p_{OO}^{\nu}p_{XX}^{l-\nu}\sum_{w=1}^{\tau_O}\binom{\nu}{\eta}\binom{l-\nu-1}{\theta-1}\left(\frac{p_{OX}}{p_{XX}}\right)^{\theta}\left(\frac{p_{XO}}{p_{OO}}\right)^{\eta} & \text{if } 0\leq\nu<l,\\ p_{OO}^{l} & \text{if } \nu=l,\\ 0 & \text{otherwise,}\end{cases} \tag{42}
$$

$$
P(N_{XO}(t,l)=\nu)=\begin{cases} p_{XX}^{l} & \text{if } \nu=0,\\ p_{OO}^{\nu}p_{XX}^{l-\nu}\sum_{w=1}^{\tau_X}\binom{\nu-1}{\theta-1}\binom{l-\nu}{\eta}\left(\frac{p_{OX}}{p_{XX}}\right)^{\eta}\left(\frac{p_{XO}}{p_{OO}}\right)^{\theta} & \text{if } 0<\nu\leq l,\\ 0 & \text{otherwise,}\end{cases} \tag{43}
$$

where $\tau_O = l + 0.5 - |2\nu + 0.5 - l|$ (note that this expression corrects a minor error in the corresponding equation in Gabriel (1959)), $\tau_X = l + 0.5 - |2\nu - 0.5 - l|$, and θ and η are functions of w such that if w is even, then $\theta = \eta = 0.5w$, and if w is odd, then $\theta = \lceil 0.5w - 1 \rceil$ and $\eta = \lceil 0.5w \rceil$. Due to the Markov property, $N_{ij}(t, l)$ is identically distributed for all t.