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Comment on "The Hidden Symmetry for a Quantum System with an Infinitely Deep Square-well Potential by Shi-Hai Dong and Zhong-Qi Ma [Am. J. Phys. 70 (5) 520-521 (2002)]

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Note on thermal heating efficiency

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Kelvin showed the maximum efficiency with which heat can be converted into work; but there is a dual theorem about the maximum efficiency with which heat at one temperature can be converted into heat at another temperature. It has some surprising implications, in particular that the efficiency with which we heat our buildings could in principle be improved by a large factor. This long known, but still little known, fact is of current pedagogical interest and practical importance.

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I. INTRODUCTION

For over 200 years the University of Glasgow has played a uniquely important role in the development of thermodynamics. There the distinction between temperature as a measure of *intensity* of something, and heat as a *quantity* of something, was first seen clearly by Joseph Black in about 1760. This knowledge contributed to the work of his colleague, James Watt, in the first practical means of converting heat into work. Then Carnot and others tried to find the maximum theoretical efficiency of this conversion, but the one who finally succeeded was Wm. Thomson (later Lord Kelvin) at the University of Glasgow.

Recently an addition to this was made, which is not only of theoretical interest as representing in a sense the completion of the logical structure of classical thermodynamics; it has immediate practical implications. Yet the principle is hardly new; it is such a simple and immediate consequence of Thomson's work that it must have been known to Thomson in 1870.[1] Today it cannot be really unknown to anyone familiar with the theory of heat pumps. But to the best of our knowledge it has not yet appeared in any physics textbook, stated in a form where it is seen as logically independent of Carnot engines, and forming the natural dual theorem to the one on the efficiency of Carnot engines.[2] It seems appropriate that this way of looking at the result was finally pointed out by Robert S. Silver,[3] the James Watt Professor (now emeritus) of the University of Glasgow.

In Sec. II we give the almost trivial derivation, and in Sec. III we point out its practical implications by numerical examples. Because a large part of the world's energy resources are actually used for heating rather than production of work, these implications are not trivial. Section IV points out another surprising application.

II. THEORETICAL DERIVATION

We have a source of heat Q_2 which is available at Kelvin temperature T_2. By this we mean, as was stressed long ago by Gibbs,[4] that the source is capable of delivering that heat to a heat reservoir which is at temperature T_2; T_2 is the highest temperature to which it can deliver that heat. If there is available a cold reservoir at temperature $T_1 < T_2$, then according to classical thermodynamics we may exploit this temperature difference to obtain work W. By applying the first and second laws, we obtain $W = Q_2 - Q_1$, $Q_1/T_1 \geqslant Q_2/T_2$; if we solve these for W and Q_1, we have

$$W \leqslant Q_2\left(1 - \frac{T_1}{T_2}\right), \quad Q_1 \geqslant Q_2 \frac{T_1}{T_2}, \tag{1}$$

with an equality if and only if the engine is reversible. In the latter case the "wasted energy"

$$Q_1(\text{Carnot}) = Q_2 \frac{T_1}{T_2} \tag{2}$$

is delivered as heat to the reservoir at temperature T_1. This is the standard result.

But now suppose that our objective is not to produce work, but to deliver the maximum possible heat to that lower temperature reservoir. This is the conversion problem faced in every home, where one has heat from a gas, oil, wood, or coal flame but wants heat at room temperature. At present, we simply allow the primary heat Q_2 to degrade itself directly to the lower temperature T_1 by passing through ducts, radiators, etc. In this way we obtain, at best (neglecting heat loss through chimneys) the amount of heat $Q_1(\text{direct}) = Q_2$. But this process is irreversible because there is a net entropy increase $\Delta S = Q_2/T_1 - Q_2/T_2 > 0$, indicating that something has been wasted, and we can do better. The first and second laws imply that, not only in the conversion of heat to work, but also in the conversion of heat to heat, the maximum efficiency will be attained if we can carry out the process reversibly.

Suppose we have an ambient heat reservoir (the outside world) at temperature $T_0 < T_1$, and we use a perfect Carnot engine to obtain the heat $Q_1(\text{Carnot})$. Then we still have the work W available, which we can use to drive a heat pump between T_0 and T_1, yielding the additional heat

$$Q_1(\text{pump}) = \frac{T_1 W}{T_1 - T_0}. \tag{3}$$

If we combine Eqs. (2) and (3), we have now obtained the total heat

$$Q_1 = Q_1(\text{Carnot}) + Q_1(\text{pump}) = Q_2 \frac{T_1 T_2 - T_0}{T_2 T_1 - T_0}, \tag{4}$$

and there is always a net gain, because Q_1 is always greater than Q_2 whenever $T_0 < T_1 < T_2$. But while we know that a reversible Carnot engine delivers the maximum attainable work, this argument does not make it obvious whether Eq. (4) is the maximum attainable heat.

Now from a theoretical standpoint it is more general and more elegant to apply the first and second laws directly to this process, as we did in Eq. (1). Because some heat Q_0 is removed from the outside reservoir, we must have

$$Q_1 = Q_0 + Q_2, \quad \frac{Q_1}{T_1} \geqslant \frac{Q_0}{T_0} + \frac{Q_2}{T_2}. \tag{5}$$

By solving these equations for Q_1 and Q_0, we have

$$Q_1 \leqslant Q_2 \frac{T_1}{T_2} \frac{T_2 - T_0}{T_1 - T_0}, \quad Q_0 \leqslant Q_2 \frac{T_0}{T_2} \frac{T_2 - T_1}{T_1 - T_0}, \tag{6}$$

where the equality holds if and only if the process is reversible. Thus we obtain automatically the same result Eq. (4), plus the statement that it is the *maximum attainable* heating, without invoking Carnot engines at all. It is in this simple argument that the main theoretical and pedagogical interest of this discussion lies.

III. PRACTICAL IMPLICATIONS

Consider heating from a primary temperature $T_2 = 1000$ K to room temperature, $T_1 = 25\,°\text{C} = 298$ K, with an outside temperature $T_0 = 0\,°\text{C} = 273$ K. Comparing our ideal Q_1 with the present maximum Q_2, we have from Eq. (6), the gain factor

$$G \equiv \frac{Q_1}{Q_2} = \frac{1 - 0.273}{1 - 0.916} = 8.66. \tag{7}$$

This seems at first glance quite startling; if we take into account that we are at present far from getting even Q_2 because of heat loss up chimneys, the conclusion is that it is in principle possible to heat our homes with an order of magnitude less fuel than we are now consuming.

A better idea of the numerical improvement allowed by the second law is given in Fig. 1, where we give contours of constant gain $G \equiv Q_1/Q_2$ in the (T_0, T_2) plane for $T_1 = 25\,°\text{C}$, room temperature. Even in cold climates, average gains of the order of 5 are indicated. The reason for this high efficiency is that T_0 and T_1 are not very different on the Kelvin scale. With the values of inside and outside temperature assumed in Eq. (7), one Joule of work will pump

$$T_0/(T_1 - T_0) = 10.9 \tag{8}$$

Joules of heat from the outside world, and deliver 11.9 Joules to the inside. Unfortunately, presently available heat pumps are far from realizing this theoretical efficiency. Silver[3] notes that if present engines realize only half of the theoretical efficiency, then the heat pump component of Q_1 will be only a quarter of our calculated value.

Evidently, the development of heat pumps that approach the theoretical efficiency for small temperature differences would be of very great economic importance, and no physical law stands in the way of realizing them. It is only a matter of the ingenuity of inventors, and the one who succeeds will be one of the world's great benefactors. We suspect that the successful technology will avoid the crude mechanical pumps of our present realizations, perhaps depending on thermoelectric or electrochemical means that avoid all mechanical moving parts, although perhaps with circulating fluids.

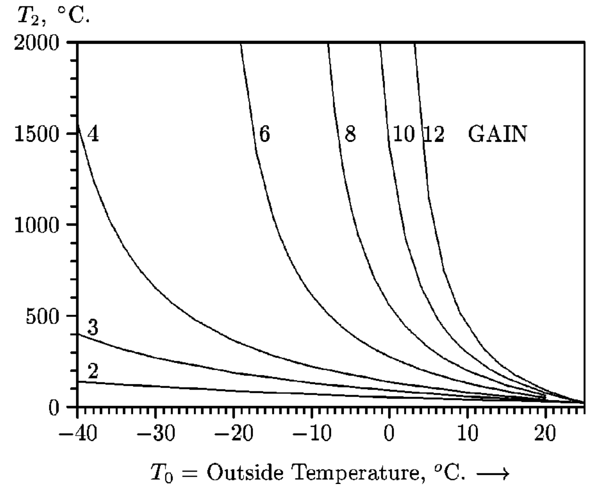

Fig. 1. Contours of constant gain in the (T_0, T_2) plane, for $T_1 = 25\,°\text{C}$.

IV. FREE OVENS FOR ESKIMOS

Note that the derivation of Eq. (6) is general in that it holds for any exchange of heat between three reservoirs whatever the relative temperatures and the signs of the Q_i, although the arrangement of Carnot engines envisaged in our derivation of Eq. (4) would no longer apply. But this seems to contradict a common statement of the second law attributed to Kelvin that "It is impossible for heat to flow of itself from a cold reservoir to a hotter one." The statement actually made by Kelvin is that it is impossible to do this *without leaving changes in external bodies*. Equation (6) demonstrates the need for this qualification for it is quite possible for heat to flow spontaneously from room temperature T_1 to a higher temperature T_2, if there is at the same time a compensating flow to a lower temperature T_0.

Suppose then that we want to heat an oven at the standard cooking temperature of $T_2 = 400\,°\text{F} = 204\,°\text{C} = 477$ K, using heat extracted from the air of a kitchen at room temperature $T_1 = 25\,°\text{C} = 298$ K. Our equations use the sign convention that Q_1 is the heat *delivered to* the reservoir at T_1, while Q_0 and Q_2 represent heat *extracted from* those at T_0, T_2. Therefore Q_0, Q_1, and Q_2 are now all negative, so $(-Q_1)$ is the heat extracted from the room and $(-Q_2)$ is the resulting heat delivered to the oven; but Eq. (6) still holds. If we write the first as

$$(-Q_2) \leqslant (-Q_1) \frac{1 - T_0/T_1}{1 - T_0/T_2}, \tag{9}$$

we see that the maximum heat that can be delivered to the oven is less than that extracted from the room, but if the outside temperature T_0 is low enough, the efficiency can be quite high; unlike room heating, oven heating becomes more efficient as the outside temperature is lowered.

Indeed, we have only to run a Carnot engine between T_1 and T_0 extracting the work $W = (-Q_1)(1 - T_0/T_1)$, then use that to run a heat pump between T_0 and T_2, which delivers the heat $(-Q_2)=W/(1-T_0/T_2)$, in agreement with Eq. (9). If the outside temperature T_0 is $-40\,°\mathrm{F}=-40\,°\mathrm{C}=233$ K then according to Eq. (9), 1000 calories of heat removed from the room can deliver 426 calories to the oven. If this leaks back eventually to reheat the room, it might appear that the "cost" of running the oven was not the 1000 calories removed from the room, but only the 574 calories lost to the outside.

But this leaking back is again an irreversible process in which something is wasted, and we can do better. If the oven is well insulated, then when we are done with it the heat $(-Q_2)$ is still in it, so we have only to run those Carnot engines backward, obtaining the work $W=426(1-T_0/T_2)$ from which the heat pump can return the heat $W/(1-T_0/T_1)=1000$ calories to the room, completely restoring the *status quo*. The second law allows us to operate an oven, at whatever temperature we please, at zero cost, the outside reservoir T_0 serving only as a temporary repository for the entropy that must be disposed of in heating the oven.[5]

Unfortunately, the second law will not allow us to supply our cooling needs as easily; it offers free (that is, zero operating cost) ovens to eskimos, but not free air-conditioning to hottentots because they have no lower temperature reservoir to take up that entropy.

[a]Professor Edwin Jaynes died on 28 April 1998 and this paper was found among his unpublished works; its message remains timely. Correspondence concerning this paper can be addressed to W. T. Grandy, Jr. at wtg@uwyo.edu.

[1]WTG—Actually, Kelvin had realized it much earlier: W. Thomson (Lord Kelvin), "On the economy of the heating or cooling of buildings by means of currents of air," Proc. R. Philos. Soc. (Glasgow) **3**, 269–272 (1852).

[2]WTG—There is, however, at least one textbook that presents the principle in terms of Carnot engines: F. H. Crawford, *Heat, Thermodynamics, and Statistical Physics* (Harcourt, Brace & World, New York, 1963), pp. 217–219.

[3]R. S. Silver, "Reflexions sur la puissance chaleurique du feu," J. Heat Recovery Syst. **1**, 205–207 (1981).

[4]J. W. Gibbs, "Electrochemical thermodynamics," Report Brit. Assoc. Adv. Sci. 388 (1886); reprinted in *The Scientific Papers of J. Willard Gibbs* (Longmans, Green & Co., New York, 1906 and Dover Publications, Inc., New York, 1961), pp. 406–412.

[5]WTG—There is still some loss in the form of the energy required to change the state of whatever was cooked, but the *operating cost* is zero.

Comment on "The hidden symmetry for a quantum system with an infinitely deep square-well potential," by Shi-Hai Dong and Zhong-Qi Ma [Am. J. Phys. 70 (5), 520–521 (2002)]

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Dong and Ma have used the ladder-operator technique to solve the infinitely deep square well problem and have examined the group theoretical properties of their solutions, concluding that the eigenstates belong to representations of a spectrum generating SU(1,1) algebra.[1] We demonstrate here that their technique is an example of the method of supersymmetric quantum mechanics (SUSY-QM).[2] SUSY-QM provides an elegant and useful prescription for obtaining closed analytical expressions for both the energy eigenvalues and the eigenfunctions of a large class of one-dimensional problems. SUSY-QM extends Dirac's raising and lowering operators $a^\dagger$ and a, first developed for obtaining the energy eigenvalues of the one-dimensional harmonic oscillator, to a similar pair A and $A^\dagger$ which connect *different* potentials that share the *same* energy eigenvalues (except for the ground state). It also naturally imposes an algebraic structure on all analytically solvable problems of nonrelativistic quantum mechanics. In fact the infinite square well is a special case of the Eckart potential, one of the class of shape invariant potentials described earlier in this journal,[3] whose group theoretical properties have been extensively studied.[4] In particular, we have reviewed the characteristics of the SO(2,1) $\sim$SU(1,1) potential algebra.[5]

In SUSY-QM, each superpotential $W(x,a)$ produces two "partner potentials"

$$V_{\pm}=W^2(x,a)\pm\frac{dW(x,a)}{dx}, \tag{1}$$

and ladder operators

$$A(x,a)=\frac{d}{dx}+W(x,a), \quad A^\dagger(x,a)=-\frac{d}{dx}+W(x,a). \tag{2}$$

A subset of all possible superpotentials $W(x,a)$ has the property known as shape invariance.[6] Examples of two such shape invariant partner potentials are the infinite well and the $\mathrm{cosec}^2\, x$ potential (something one would hardly guess from the name "shape invariant"). The entire spectrum of these potentials can be determined by algebraic means,[2,3] analogous to the way that the one-dimensional harmonic oscillator is solved by Dirac's method.

In addition and independently, it was discovered that each of these exactly solvable systems possesses a SO(2,1) algebra,[7,8] as Dong and Ma have deduced for the infinite well. The connection between the SUSY-QM method of solution and the group theoretical potential algebra method was then established.[4,9]

In the following we will use units such that $\hbar$ and $2m=1$. In SUSY-QM, the partner potential V_- is adjusted to make the ground state energy $E_0=0$. Each of the excited state energies is thus shifted from the traditional Schrödinger value by $-E_0$. The superpotential W produces a ground state eigenfunction $\psi_0^{(-)}(x,a) \sim \exp(-\int_{x_0}^{x} W(x,a)dx)$. For $W(x,a) = -a \cot x$, we find $\psi_0^{(-)}(x,a) \sim \sin^a x$. We have considered the infinite well, for which $a=1$ for the ground state $\psi_0^{(-)}(x,1) \sim \sin x$.[9] If we operate successively with $A^\dagger$, we produce the excited states $\sin nx$. The primary difference between the SUSY-QM method[3] and that of Dong and Ma is that A^+ operates on a ground state with a shifted parameter a: $\psi(x,a+1) \sim \sin^{a+1} x$, while their P_+ operates on the customary ground state $\sin x$. The techniques are equivalent. The corresponding eigenvalues $E_n^{\pm}$ are obtained from the shape invariance condition, which represents them as a simple sum of algebraic remainders from the difference of the values of the two partner potentials.[4] They are the shifted eigenvalues $E_n^- = n^2\pi^2/L^2 - E_0$; $E_n^+ = E_{n+1}^-$. (As an added bonus, E_n^+ are the energy levels of the $\operatorname{cosec}^2 x$ potential.)

The connection between shape invariant potentials and SO(2,1) or its extension algebra has been obtained.[4] We have shown that they are special cases of the generalized Natanzon potential.[10] We have also shown that the set of already known potentials constitutes the full set.[4,9] Our approach therefore links the group theoretic (potential algebra) approach and the supersymmetric quantum mechanics approach for treating shape invariant potentials.

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[1]S.-H. Dong and Z.-Q. Ma, "The hidden symmetry for a quantum system with an infinitely deep square-well potential," Am. J. Phys. **70**, 520–521 (2002).

[2]For a review of SUSY-QM, see F. Cooper, A. Khare, and U. Sukhatme, "Supersymmetry and quantum-mechanics," Phys. Rep. **251**, 268–385 (1995), and references therein.

[3]R. Dutt, A. Khare, and U. Sukhatme, "Supersymmetry, shape invariance and exactly solvable potentials," Am. J. Phys. **56**, 163–168 (1988).

[4]A. Gangopadhyaya, J. V. Mallow, and U. P. Sukhatme, "Shape invariance and its connection to potential algebra," in *Proceedings of Workshop on Supersymmetry and Integrable Models*, edited by H. Aratyn, T. Imbo, W.-Y. Keung, and U. Sukhatme (Springer-Verlag, Berlin, 1998); A. B. Balantekin, "Algebraic approach to shape invariance," Phys. Rev. A **57**, 4188–4191 (1998); S. Chaturvedi, R. Dutt, A. Gangopadhyaya, P. Panigrahi, C. Rasinariu, and U. Sukhatme, "Algebraic shape invariant models," Phys. Lett. A **248**, 109–113 (1998); A. Gangopadhyaya, J. V. Mallow, and U. P. Sukhatme, "Translational shape invariance and the inherent potential algebra," Phys. Rev. A **58**, 4287–4292 (1998); A. B. Balantekin, M. A. C. Ribeiro, and A. N. F. Aleixo, "Algebraic nature of shape-invariant and self-similar potentials," J. Phys. A **32**, 2785–2790 (1999).

[5]An algebra is a system of operators with commutation relations, in which one, called the Casimir operator, commutes with all the others. A common example is the angular momentum L^2, which commutes with its components L_i $(i=x,y,z)$, and which has the same eigenvalue $l(l+1)\hbar^2$ for all the eigenstates defined by it and one of its components (customarily L_z). Their eigenstates $|l,m\rangle$ constitute the representation for this example. A potential algebra is one for which the Hamiltonian is the Casimir operator. Because the Casimir operator has the same eigenvalue for each eigenstate, in this case the representation consists of a set of states with a common value for energy. But here, each state is an eigenstate of a distinct potential. Thus, reversing the usual paradigm for quantum mechanics—a single potential generating a set of energy eigenvalues and eigenstates—this supersymmetric algebra pulls together states of the same energy from an infinite set of different potentials. Hence, we are led to the name potential algebra. For more details, see Ref. 7 for the Eckart potential, as well as several other shape invariant potentials (see Ref. 9).

[6]L. Infeld and T. E. Hull, "The factorization method," Rev. Mod. Phys. **23**, 21–68 (1951); see Ref. 3 for more references.

[7]J. Wu, Y. Alhassid, and F. Gürsey, "Group theory approach to scattering. IV. Solvable potentials associated with SO(2,2)," Ann. Phys. (Leipzig) **196**, 163–181 (1989).

[8]A. O. Barut, A. Inomata, and R. Wilson, "A new realization of dynamical groups and factorization method," J. Phys. A **20**, 4075–4082 (1987); *ibid*. "Algebraic treatment of second Pöschl-Teller, Morse–Rosen, Eckart equations," *ibid*. **20**, 4083–4096 (1987); M. J. Englefield and C. Quesne, "Dynamical potential algebras for Gendenshtein and Morse potentials," *ibid*. **24**, 3557–3574 (1991).

[9]A. Gangopadhyaya, J. V. Mallow, C. Rasinariu, and U. P. Sukhatme, "Exactly solvable models in supersymmetric quantum mechanics and connection with spectrum-generating algebras," Theor. Math. Phys. **118**, 285–294 (1999); "Exact solutions of the Schrödinger equation: Connection between supersymmetric quantum mechanics and spectrum generating algebras," Chin. J. Phys. (Taipei) **39**, 101–121 (2001).

[10]G. A. Natanzon, Vestn. Leningr. Univ. [Biol.] **10**, 22–28 (1971) (in Russian). An English translation by H. Rosu is available at http://arxiv.org/PS_cache/physics/pdf/9907/9907032.pdf.