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Coordinate realizations of deformed Lie algebras with three generators

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Differential realizations in coordinate space for deformed Lie algebras with three generators are obtained using bosonic creation and annihilation operators satisfying Heisenberg commutation relations. The unified treatment presented here contains as special cases all previously given coordinate realizations of so(2,1), so(3), and their deformations. Applications to physical problems involving eigenvalue determination in nonrelativistic quantum mechanics are discussed. [S1050-2947(99)01711-4]

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I. INTRODUCTION

Lie groups and their associated algebras are extensively used in the analysis of the symmetry properties of physical systems. For example, realizations of so(2,1) have been used to obtain the eigenvalues of many quantum-mechanical problems. Recent studies show that coordinate realizations of nonlinear Lie algebras may also be interesting in determining eigenspectra of certain physical problems in an algebraic approach [1]. The main purpose of this paper is to set up a unified approach for obtaining differential realizations in one- and two-dimensional coordinate space for nonlinear Lie algebras with three generators.

The deformed Lie algebras, which we consider, are described by

$$[J_3,J_+]=J_+,\quad [J_3,J_-]=-J_-,\quad [J_+,J_-]=f(J_3). \tag{1}$$

$J_\pm \equiv J_1 \pm iJ_2$ are the well-known raising and lowering operators. $f(J_3)$ is an arbitrary analytic function of the operator J_3. Note that the special choice $f(J_3)=2J_3$ corresponds to so(3) and $f(J_3)=-2J_3$ corresponds to so(2,1). In terms of the Cartesian generators J_1, J_2, and J_3, the commutation relations are

$$[J_1,J_2]=\frac{i}{2}f(J_3),\quad [J_2,J_3]=iJ_1,\quad [J_3,J_1]=iJ_2. \tag{2}$$

The plan of this paper is as follows. In Sec. II, we review some simple general properties of Lie algebras. In Sec. III, we describe how to obtain realizations of Eq. (1) in terms of bosonic creation and annihilation operators ($a^\dagger$ and a) satisfying Heisenberg commutation relations $[a,a^\dagger]=1$. Although we are using the conventional notation a and $a^\dagger$ for these operators, they do not necessarily have to be Hermitian conjugates of each other. The Appendix contains a discussion of specific one-dimensional realizations of the Heisenberg algebra. In particular, it is shown that realizations involving derivatives higher than the first can all be reduced to first and zero order. Section IV contains a description of one-dimensional coordinate realizations of the Lie algebra given in Eq. (1). We show that our unified approach reproduces all previously known realizations in the literature [2–6]. Two-dimensional coordinate realizations are described in Sec V, along with some applications involving eigenvalue determination for some nonrelativistic quantum-mechanical potentials.

II. SOME PROPERTIES OF THE LIE ALGEBRA

For completeness and to establish notation, we describe some properties of Lie algebras. Some are well known, but others are new.

(i) The function $f(J_3)$ characterizes the Lie algebra given in Eq. (1). For subsequent work, it is convenient to define the function $g(J_3)$ as follows:

$$f(J_3)\equiv g(J_3)-g(J_3-1). \tag{3}$$

For example, for so(3), $f(J_3)=2J_3$ and one gets $g(J_3)=J_3(J_3+1)$. It is easy to check that the function $g(J_3)$ is not unique—any periodic function of unit period can be added while maintaining Eq. (3). Note that the Casimir operator for the Lie algebra of Eq. (1) is given by

$$C=J_-J_+ + g(J_3)=J_+J_- + g(J_3-1). \tag{4}$$

This observation is useful for many physical applications. For instance, we use it in Sec. V for eigenvalue determination.

(ii) The operators J_+ and J_- satisfy the important property

$$T(J_3)J_+=J_+T(J_3+1),\quad T(J_3)J_-=J_-T(J_3-1), \tag{5}$$

for any analytic function $T(J_3)$. This property is extensively used in obtaining realizations.

(iii) If operators J_+, J_-, and J_3 satisfy the standard so(3) Lie algebra, so do operators $\tilde{J}_+$, $\tilde{J}_-$, and $\tilde{J}_3$ defined by $\tilde{J}_m=\Sigma_n M_{mn}J_n$ provided the matrix M satisfies $M^TM=1$

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and det $M=+1$. Note that the elements of the matrix M do not have to be real, but if they are, the matrix is orthogonal. This property is very useful in relating all the so(3) realizations currently available in the literature.

(iv) Given operators J_1, J_2, and J_3, which satisfy the so(3) Lie algebra, one can find operators K_1, K_2, and K_3, that satisfy a more general algebra

$$[K_1,K_2]=iq_3K_3, \quad [K_2,K_3]=iq_1K_1, \quad [K_3,K_1]=iq_2K_2, \tag{6}$$

by choosing $K_1=\sqrt{q_2q_3}J_1$, $K_2=\sqrt{q_3q_1}J_2$, and $K_3=\sqrt{q_1q_2}J_3$. In particular $K_1=iJ_1$, $K_2=iJ_2$, and $K_3=J_3$ is a realization of so(2,1).

(v) Given operators J_+, J_-, and J_3, which satisfy the standard so(3) Lie algebra, one can find operators $\tilde{J}_+$, $\tilde{J}_-$, and $\tilde{J}_3$, which satisfy the deformed algebra of Eq. (1) [7]. These operators are given by

$$\tilde{J}_+=J_+A(J_3,C), \quad \tilde{J}_-=B(J_3,C)J_-, \quad \tilde{J}_3=J_3, \tag{7}$$

where $C=J_-J_++J_3(J_3+1)$ is the Casimir operator of so(3). The form of the operators in Eq. (7) was chosen so that the two conditions $[\tilde{J}_3,\tilde{J}_\pm]=\pm\tilde{J}_\pm$ are trivially satisfied. In order to satisfy the third condition $[\tilde{J}_+,\tilde{J}_-]=f(\tilde{J}_3)$, one needs functions $A(J_3,C)$ and $B(J_3,C)$, which satisfy the following condition:

$$A(J_3-1,C)B(J_3-1,C)[C-J_3(J_3-1)]-B(J_3,C)A(J_3,C)\times[C-(J_3+1)J_3]=f(J_3). \tag{8}$$

If $A(J_3,C)$ and $B(J_3,C)$ commute, this condition reduces to

$$H(J_3,C)[C-J_3(J_3+1)]=-g(J_3)+p(J_3); \quad H(J_3,C)\equiv A(J_3,C)B(J_3,C), \tag{9}$$

where $p(J_3)$ is an arbitrary periodic function of period unity. It is important to realize that only the product $H(J_3,C)$ is fixed by the above constraint equation, but not the individual functions $A(J_3,C)$ and $B(J_3,C)$. Given Eq. (7), it is sufficient to restrict our attention to realizations of so(3) in order to obtain realizations of any deformed Lie algebra with three generators.

Note that for the special case of so(3) itself, the choice $p(J_3)=C$ gives $H(J_3,C)=1$. The simplest choice of factors $A(J_3,C)=B(J_3,C)=1$ reproduces the initial so(3) realization, whereas a more general choice $B(J_3,C)=A^{-1}(J_3,C)$ yields a new realization. Furthermore, other choices of $p(J_3)$ give additional new realizations of so(3). In particular, the choice $p(J_3)=0$ gives the realization

$$\tilde{J}_+=-J_+\frac{J_3(J_3+1)}{C-J_3(J_3+1)}, \quad \tilde{J}_-=J_-, \quad \tilde{J}_3=J_3,$$

which differs from the original one only in one generator J_+. This freedom in choosing the periodic function $p(J_3)$ is analogous to gauge fixing in field theories.

An interesting nonlinear example using the above formalism comes from the choice $g(J_3)=J_3^2(J_3+1)^2$ and $p(J_3)=C^2$. This choice gives the realization

$$\tilde{J}_+=J_+[C+J_3(J_3+1)], \quad \tilde{J}_-=J_-, \quad \tilde{J}_3=J_3, \tag{10}$$

for the deformed Lie algebra corresponding to $f(J_3)=4J_3^3$.

III. REALIZATIONS OF THE DEFORMED LIE ALGEBRA IN TERMS OF BOSONIC OPERATORS

In this section, we develop a procedure for obtaining realizations of the Lie algebra defined by Eq. (1) in terms of bosonic creation and annihilation operators $a^\dagger$ and a, which obey the Heisenberg algebra commutator $[a,a^\dagger]=1$. The number operator is defined by $N\equiv a^\dagger a$. It follows that $[N,a^\dagger]=a^\dagger$ and $[N,a]=-a$. More generally,

$$[N,a^{\dagger m}]=ma^{\dagger m}, \quad [N,a^m]=-ma^m \quad (m=0,\pm1,\pm2,\dots). \tag{11}$$

To generate realizations of a deformed Lie algebra using the operators $a^\dagger$, a, and N, we choose the following ansatz:

$$J_+=PF(N), \quad J_-=G(N)Q, \quad J_3=N+c, \tag{12}$$

where c is a constant. P and Q are functions of a and $a^\dagger$ chosen to satisfy the property

$$[N,P]=P, \quad [N,Q]=-Q. \tag{13}$$

Clearly, from Eqs. (11) and (13) it follows that two possible choices for $P(a,a^\dagger)$ are $a^\dagger$ and $1/a$ and two possible choices for $Q(a,a^\dagger)$ are a and $1/a^\dagger$. In fact, one can choose the linear combination

$$P=\alpha_1(N)a^\dagger+\alpha_2(N)\frac{1}{a}, \quad Q=\beta_1(N)a+\beta_2(N)\frac{1}{a^\dagger}. \tag{14}$$

Using Eq. (13), it is easy to show that $PN^m=(N-1)^mP$ and $N^mQ=Q(N-1)^m$, so that one has the property $PT(N)=T(N-1)P$ and $T(N)Q=QT(N-1)$ for any analytic function $T(N)$. Also, the dependence on a and $a^\dagger$ of the products PQ and QP clearly comes only through the combination $a^\dagger a=N$.

Our ansatz of Eq. (12) will satisfy the conditions of Eq. (1) provided

$$F(N-1)G(N-1)PQ-G(N)F(N)QP=f(N+c). \tag{15}$$

If $F(N)$ and $G(N)$ commute, the above condition becomes

$$H(N-1)PQ-H(N)QP=f(N+c), \quad H(N)\equiv F(N)G(N). \tag{16}$$

It only remains to determine $H(N)$ from Eq. (16). As in Sec. II, note again that the functions $F(N)$ and $G(N)$ do not appear separately but only appear as their product $H(N)$. Also, note that in Sec. V, we will discuss a situation where $F(N)$ and $G(N)$ do not commute.

IV. ONE-DIMENSIONAL COORDINATE REALIZATIONS

Here we consider one-dimensional coordinate realizations for $a,a^\dagger$ such that $[a,a^\dagger]=1$. Equations (12), (14), and (16) now immediately give a realization for the nonlinear algebra of Eq. (1). As an example, we consider the same deformed Lie algebra with $f(J_3)=4J_3^3$ as in Sec. II. We make the simple choice $P=a^{\dagger}=x$, $Q=a=d/dx$, and $c=0$, which gives $PQ=N$, $QP=N+1$, and $N=xd/dx$. Equation (16) now reads $H(N-1)N-H(N)(N+1)=4N^3$ whose solution is $H(N)=-N^2(N+1)$. Taking $G(N)=1$ our coordinate realization is

$$J_+=-x\left(x\frac{d}{dx}\right)^2\left(x\frac{d}{dx}+1\right),\quad J_-=\frac{d}{dx},\quad J_3=x\frac{d}{dx}. \tag{17}$$

General coordinate realizations of $a,a^{\dagger}$ are discussed in the Appendix. Any of these can be used to generate different one-dimensional realizations of deformed Lie algebras. Our formalism is very flexible since there is freedom in choosing $a,a^{\dagger}$ (Appendix) and the operators P and Q in Eq. (14). Furthermore, once $H(N)$ has been determined from Eq. (16), one has various choices for factorization into the functions $F(N)$ and $G(N)$, which appear in the final realization given in Eq. (12). Our formalism contains as special cases all the coordinate realizations published in the literature. We shall now illustrate this statement for specific realizations discussed in [4] and [2].

Filho and Vaidya [4] have discussed physical applications based on the following representation of so(2,1):

$$J_+=2\frac{d^2}{dy^2}-\frac{2\alpha}{y^2},\quad J_-=\frac{y^2}{8},\quad J_3=-\frac{y}{2}\frac{d}{dy}-\frac{1}{4}, \tag{18}$$

where α is an arbitrary constant. In order to obtain this realization as a special case of our formalism, we choose $a,a^{\dagger}$ by taking $\theta=0$, $h(y)=1/y^2$, and $r(y)=-y^2/4$ in Eq. (A2) in the Appendix. This gives

$$a=-\frac{y^3}{2}\frac{d}{dy}-\frac{y^2}{4},\quad a^{\dagger}=\frac{1}{y^2},\quad N=-\frac{y}{2}\frac{d}{dy}-\frac{1}{4}.$$

Furthermore, choosing $P=a^{\dagger}$, and $Q=1/a^{\dagger}$ in Eq. (14) implies that constraint (16) on $H(N)$ reads

$$H(N-1)-H(N)=-2N.$$

The solution is $H(N)=N(N+1)+\beta$, where β is an arbitrary constant. Choosing the factorization $G(N)=1/8$ and $F(N)=8H(N)$, Eq. (12) with $c=0$ and $\beta=(3-4\alpha)/16$ after simplification gives the Filho-Vaidya realization of Eq. (18).

Another example of a differential realization of the so(2,1) algebra was given by Barut and Bornzin [2]. Their expressions for the generators are

$$T_1=\frac{1}{2}\left(\frac{y^{2-n}}{n^2}p_y^2+\frac{\xi}{y^n}-y^n\right),\quad T_2=\frac{1}{n}\left(yp_y-i\frac{n-1}{2}\right),$$
$$T_3=\frac{1}{2}\left(\frac{y^{2-n}}{n^2}p_y^2+\frac{\xi}{y^n}+y^n\right). \tag{19}$$

Here $p_y=-iy^{-1}(d/dy)y$, n is an arbitrary positive integer, and ξ is an arbitrary constant. To make contact with our formalism, using (iii) from Sec. II, we first rotate T_1, T_2, and T_3 to the new operators $J_1=iT_3$, $J_2=-T_1$, and $J_3=iT_2$. This gives

$$J_+=iy^n,\quad J_-=-i\left(\frac{y^{2-n}}{n^2}\frac{d^2}{dy^2}+2\frac{y^{1-n}}{n^2}-\frac{\xi}{y^n}\right),$$

$$J_3=\frac{y}{n}\frac{d}{dy}+\frac{n+1}{2n}.$$

Next, let us take $\theta=0$, $h(y)=y^n$, and $r(y)=-[n(2c-1)-1]/(ny^n)$ in Eq. (A2) of the Appendix. This implies

$$a=\frac{y^{1-n}}{n}\frac{d}{dy}-\frac{n(2c-1)-1}{2ny^n},\quad a^{\dagger}=y^n,$$

$$N=\frac{y}{n}\frac{d}{dy}+\frac{n+1}{2n}-c.$$

Further, choosing $\alpha_1=i$, $\beta_2=1$, and $\alpha_2=\beta_1=0$ in Eq. (14), we find a solution of Eq. (16) of the form $H(N)=b_2N^2+b_1N+b_0$ with $b_2=-i/\beta_2$, $b_1=-i(2c+1)/\beta_2$, and $b_0=-(i/\beta_2)[(2c+1)^2/4-\xi-1/(4n^2)]$. Finally, the factorization $H=FG$ with $F=1$, concludes the proof that Eqs. (19) are a particular case of our formalism. Note that the initial rotation of generators seems to be essential in getting the realizations of [2].

Similarly, our formalism also gives the one-dimensional realizations described in Refs. [3] and [6].

V. TWO-DIMENSIONAL COORDINATE REALIZATIONS

In this section we will introduce realizations of so(2,1) using two coordinates. In contrast to the one-coordinate realizations, we now allow the functions F and G appearing in Eq. (12) to be functions of N as well as an internal coordinate x and its derivative d/dx. It is important to observe that due to this generalization, the functions F and G no longer commute with each other, and as a result, Eq. (15) must be used.

To construct explicit realizations of so(2,1), we choose $P=a^{\dagger}=\exp(i\phi)$ and $Q=1/a^{\dagger}=\exp(-i\phi)$, i.e., $\alpha_2=\beta_1=0$ in Eq. (14). The simplest choice of the operator a, which satisfies $[a,a^{\dagger}]$, is $a=-i\,\exp(i\phi)\partial/\partial\phi$. This gives $N=a^{\dagger}a=-i\,\partial/\partial\phi$. As a simple example, we consider

$$F(N)=\left[-\frac{\partial}{\partial x}+W\left(x,-i\frac{\partial}{\partial\phi}\right)\right],$$
$$G(N)=\left[\frac{\partial}{\partial x}+W\left(x,-i\frac{\partial}{\partial\phi}\right)\right], \tag{20}$$

where W is a function to be determined. Substitution in Eq. (15) yields

$$\left[W^2\left(x,-i\frac{\partial}{\partial\phi}-1\right)-\frac{dW\left(x,-i\frac{\partial}{\partial\phi}-1\right)}{dx}\right]-\left[W^2\left(x,-i\frac{\partial}{\partial\phi}\right)+\frac{dW\left(x,-i\frac{\partial}{\partial\phi}\right)}{dx}\right]=f\left(-i\frac{\partial}{\partial\phi}+c\right). \tag{21}$$

The left-hand side of this equation depends on x while the right-hand side does not. In order to get a two-dimensional realization one needs a solution of Eq. (21). In supersymmetric quantum mechanics, this equation is well known to be the shape-invariance condition. Its solutions are shape-invariant superpotentials [8]. One solution is

$$W=-i\frac{\partial}{\partial\phi}\tanh x+B\,\mathrm{sech}\,x. \tag{22}$$

In this case, an explicit calculation yields $f(-i\partial/\partial\phi+c)=-2\,(-i\partial/\partial\phi)+1$. This implies that we are dealing with a deformed Lie algebra with $f(J_3)=-2J_3+2c+1$. For the choice $c=-1/2$ this is the so(2,1) algebra and its realization is

$$J_+=e^{i\phi}\left[-\frac{\partial}{\partial x}-i\frac{\partial}{\partial\phi}\tanh x+B\,\mathrm{sech}\,x\right],$$

$$J_-=\left[\frac{\partial}{\partial x}-i\frac{\partial}{\partial\phi}\tanh x+B\,\mathrm{sech}\,x\right]e^{-i\phi},\quad J_3=-i\frac{\partial}{\partial\phi}+\frac{1}{2}.$$

There are several other solutions possible [9] and they can be derived analytically using a point canonical transformation described in Ref. [10].

The above realizations have interesting applications. The operator J_+J_- is given by

$$J_+J_-=\left[-\frac{d^2}{dx^2}+W^2\left(x,-i\frac{\partial}{\partial\phi}-1\right)-\frac{dW\left(x,-i\frac{\partial}{\partial\phi}-1\right)}{dx}\right].$$

When acting on factorized basis functions $e^{im\phi}\psi(x)$, one gets

$$J_+J_-=\left[-\frac{d^2}{dx^2}+W^2(x,m-1)-\frac{dW(x,m-1)}{dx}\right],$$

which is recognized to be the standard Hamiltonian of supersymmetric quatum mechanics. For the choice of Eq. (22) the result is

$$J_+J_-=\left[-\frac{d^2}{dx^2}+(m-1)^2[B^2-(m-1)^2-(m-1)]\mathrm{sech}^2\,x+B(2(m-1)+1)\mathrm{sech}\,x\tanh x\right],$$

which is just the Hamiltonian for the Scarf potential[1] with $m-1$ being one of the parameters. The Scarf potential is well known to be shape invariant, hence exactly solvable [11]. We can also determine these eigenvalues using familiar algebraic methods of so(2,1). The Casimir is $C\equiv J^2$ and Eq. (4) gives $J_+J_-=J_3^2-J_3-J^2$. Since the eigenvalues of J^2 and J_3 are $j(j+1)$ and $m-1/2$, respectively, we find

$$E=\left(m-\frac{1}{2}\right)^2-\left(m-\frac{1}{2}\right)-j(j+1).$$

Now substituting $j=n-m+\frac{1}{2}$ [3], one gets

$$E_n=(m-1)^2-(m-n-1)^2,\quad n=0,1,2,\ldots \tag{23}$$

(Note that $E_0=0$ as expected from unbroken supersymmetric quantum mechanics.)

With a change of variable and appropriate similarity transformations of $F(N)$ and $G(N)$ [10], we can relate all solvable potentials of Ref. [8] to J_+J_- of this algebra and hence derive information about their spectrum algebraically.

In this paper, differential realizations in coordinate space for nonlinearly deformed Lie algebras with three generators were obtained using bosonic creation and annihilation operators. We have presented a unified formalism that contains as special cases all previously given coordinate realizations of $\mathrm{so}(2,1),\mathrm{so}(3)$, and their deformations. Although we have focused on deformations of the type specified by Eq. (1), coordinate realizations for other types of deformations have also been recently studied [12].

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APPENDIX: DIFFERENTIAL REALIZATIONS OF *a* AND $a^\dagger$

In this appendix, we discuss differential coordinate realizations of operators a and $a^\dagger$, which satisfy the Heisenberg commutation relation $[a,a^\dagger]=1$. The simplest choice is

$$a=\frac{d}{dx},\quad a^\dagger=x. \tag{A1}$$

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[1]The Scarf Hamiltonian is described by a potential $V_-(x,a_0,B)=[B^2-a_0(a_0+1)]\mathrm{sech}^2\,x+B(2a_0+1)\mathrm{sech}\,x\tanh x+a_0^2$. The eigenvalues of this system are [8] $E_n=a_0^2-(a_0-n)^2$.

As we shall see shortly, these operators are the basic building blocks for all other realizations, including those with higher-order derivatives. Note that although the notations a and $a^\dagger$ are being used, we are not requiring the two operators to be Hermitian conjugates of each other.

Given any two operators $a(x,d/dx)$ and $a^\dagger(x,d/dx)$ such that $[a,a^\dagger]=1$, several simple transformations can be used to generate new operators $\tilde{a}$ and $\tilde{a}^\dagger$, which satisfy $[\tilde{a},\tilde{a}]^\dagger=1$. These transformations are:

(i) rotations in the $(a,a^\dagger)$ plane,

$$\tilde{a}=a\cos\theta+a^\dagger\sin\theta, \quad \tilde{a}^\dagger=-a\sin\theta+a^\dagger\cos\theta;$$

(ii) change of variables $x=h(y)$,

$$\tilde{a}\left(y,\frac{d}{dy}\right)=a\left(h(y),\frac{1}{h'(y)}\frac{d}{dy}\right),$$

$$\tilde{a}^\dagger\left(y,\frac{d}{dy}\right)=a^\dagger\left(h(y),\frac{1}{h'(y)}\frac{d}{dy}\right),$$

where prime denotes the derivative with respect to y;

(iii) similarity transformations,

$$\tilde{a}=\phi^{-1}(x)a\phi(x), \quad \tilde{a}^\dagger=\phi^{-1}(x)a^\dagger\phi(x);$$

(iv) additions of arbitrary functions of the other operator,

$$\tilde{a}=a+\lambda(a^\dagger), \quad \tilde{a}^\dagger=a^\dagger; \quad \tilde{a}=a, \quad \tilde{a}^\dagger=a^\dagger+\mu(a).$$

Successive use of the first three transformations applied to Eq. (A1) yield

$$a=\frac{\cos\theta}{h'(y)}\frac{d}{dy}+(h(y)\sin\theta+r(y)\cos\theta),$$
$$a^\dagger=\frac{-\sin\theta}{h'(y)}\frac{d}{dy}+[h(y)\cos\theta-r(y)\sin\theta], \tag{A2}$$

where $h(y)$ and $r(y)$ are arbitrary analytic functions of coordinate y. It is easy to check that these are the most general operators linear in d/dy which satisfy $[a,a^\dagger]=1$.

A natural question to ask is whether one can construct differential coordinate realizations with second- and higher-order derivatives. This is, in fact, possible by starting with any first-order realization [say, Eq. (A1) or Eq. (A2)] and using transformation (iv) to generate higher-order derivatives. For example, using Eq. (A1) and taking $\mu(a)=a^2$ in transformation (iv) gives the realization

$$\tilde{a}=\frac{d}{dx}, \quad \tilde{a}^\dagger=x+\frac{d^2}{dx^2}.$$

Although this procedure can be readily extended to get realizations of the Heisenberg algebra involving derivatives of any desired order, it must be kept in mind that only the realizations involving first-order derivatives are fundamental.

[1] G. Junker and P. Roy, Phys. Lett. A **232**, 155 (1997).
[2] A.O. Barut and G. L. Bornzin, J. Math. Phys. **12**, 841 (1971); A. O. Barut, H. Beker, and A. J. Bracken, in *Proceedings of the XIIIth International Colloquium on Group Theoretical Methods in Physics*, edited by W. W. Zachary (World Scientific, Singapore, 1984), p. 269.
[3] B. G. Adams, J. Cizek, and J. Paldus, in *Advances in Quantum Chemistry*, edited by P. Loẅdin (Academic, New York, 1987), Vol. 19.
[4] H. B. Filho and A. N. Vaidya, Phys. Lett. A **145**, 69 (1990).
[5] D. M. Fradkin, J. Phys. A **27**, 1261 (1994).
[6] J. Beckers, Y. Brihaye, and N. Debergh, e-print hep-th/9803253.
[7] A. Polychronakos, Mod. Phys. Lett. A **5**, 2325 (1990).
[8] F. Cooper, A. Khare, and U. Sukhatme, Phys. Rep. **251**, 268 (1995), and references therein.
[9] A. Gangopadhyaya, J. V. Mallow, and U. P. Sukhatme, Phys. Rev. A **58**, 4287 (1998).
[10] A. Gangopadhyaya, P. Panigrahi, and U. Sukhatme, Helv. Phys. Acta **67**, 363 (1994).
[11] L. Gendenshtein, Zh. Eksp. Teor. Fiz., Pis'ma Red. **38**, 299 (1983) [JETP Lett. **38**, 356 (1983)].
[12] A. Ballesteros and F. J. Herranz, J. Phys. A **29**, L311 (1996).