Reply to “Comment on Gravitational slingshot,” by C. L. Cook [Am. J. Phys. 73 (4), 363 (2005)]

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Response to “Comments regarding recent articles on relativistically rotating frames” [Am J. Phys. 67 (2), 158 (1999)]
Am. J. Phys. 67, 159 (1999); 10.1119/1.19214
“Shortcut to the Slingshot Effect”

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A recent paper\(^1\) used the Lorentz transformation for energy-momentum four vectors to analyze the gravitational slingshot. It claimed that “the relativistic method is shorter and more compact than its nonrelativistic counterpart.” We will present a nonrelativistic treatment that is more compact and just as elegant and simple.

In mechanics, energy transfer occurs when forces do work. The kinetic energy of a spacecraft increases if it does negative work on a planet,\(^2\)

\[ 0 > \int \mathbf{F}_{\text{spacecraft}} \cdot d\mathbf{r}_{\text{planet}}, \]

or, in terms of the reaction force of the planet on the spacecraft,

\[ 0 < \int \mathbf{F}_{\text{planet}} \cdot d\mathbf{r}_{\text{planet}} = \int \mathbf{F}_{\text{planet}} \cdot \mathbf{V}_{\text{planet}} \, dt. \tag{2} \]

The presence of the planetary displacement vector \(d\mathbf{r}_{\text{planet}}\) or velocity vector \(\mathbf{V}_{\text{planet}}\) makes the work integral frame dependent.\(^3\)

Because the spacecraft-planet interaction occupies a time interval much less than the planet’s orbital period, \(\mathbf{V}_{\text{planet}}\) may be assumed to be constant.\(^4\) Following Ref. 1 we set

\[ \mathbf{V}_{\text{planet}} = \mathbf{V}_0 \tag{3} \]

in the Sun rest frame.

If we substitute Eq. (3) into Eq. (2) and discard the constant positive factor \(V\), the condition for an increase in the spacecraft’s kinetic energy in the Sun rest frame becomes

\[ 0 < \mathbf{V} \cdot \int \mathbf{F}_{\text{planet}} \cdot d\mathbf{r}_{\text{planet}}. \tag{4} \]

where \(\int \mathbf{F}_{\text{planet}} \cdot d\mathbf{r}_{\text{planet}}\) is the impulse, \(\Delta p\), delivered to the spacecraft by the planet. It has the same value in any reference frame because force and time are Galilean invariants.\(^5\)

We evaluate \(\Delta p\) in the planet center-of-mass frame,

\[ \Delta p = m \mathbf{u} \{(\cos \theta_2 - \cos \theta_1)\mathbf{x} + (\sin \theta_2 - \sin \theta_1)\mathbf{y}\}, \tag{5} \]

where the notation of Ref. 1 has been employed.\(^6\) Equation (4) becomes

\[ 0 < \mathbf{V} \cdot \Delta \mathbf{p} = m \mathbf{u} (\cos \theta_2 - \cos \theta_1). \tag{6} \]

That is, for an increase in the spacecraft’s kinetic energy, \(\cos \theta_1 < \cos \theta_2\) as derived using the Lorentz transformation in Ref. 1.

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\(^3\) The scalar product of the attractive force that the spacecraft exerts on the planet and the planet’s displacement is negative while the spacecraft passes behind the planet. The work done by the spacecraft is then negative and the slingshot “fires.”


\(^5\) The tiny change in the planet’s velocity due to the energy transfer is negligible.


\(^7\) The spacecraft of mass \(m\) initially travels in the \(xy\) plane at an angle \(\theta_1\) relative to the \(x\) axis; after the interaction it travels at an angle \(\theta_2\): its speed has the same initial and final values, \(u\).
kinetic energy in the Sun frame is 

\[ \frac{1}{2}m|\vec{V} + \vec{u}_i|^2 - \frac{1}{2}|\vec{V} + \vec{u}_f|^2. \]

Because \( \vec{u}_i \cdot \vec{V} = Vu \cos \theta_i \) (\( i = 1,2 \)), where the angles \( \theta_1 \) and \( \theta_2 \) are between the planet’s velocity \( \vec{V} \) and the craft’s velocities \( \vec{u}_i \) and \( \vec{u}_f \) in the planet frame, the desired result, \( m\nu V \cos (\theta_2 - \cos \theta_1) \), is immediate.

The relativistic derivation in Ref. 2 is more involved, but it gives further insight into the nature of the slingshot effect. As an example, we discuss what the gravitational slingshot effect would be for a photon. Of course, it cannot accelerate a photon, but it does change its frequency in accordance with a generalization of Compton scattering which allows for a moving mass. This result cannot be understood as a nonrelativistic slingshot effect even though the planet’s speed is nonrelativistic.

Because our “craft” is a photon, we will first remove the craft’s mass \( m \) from the kinetic-energy equation ([Eq. 6) in Ref. 1] by looking at the fractional change in its kinetic energy. This change is

\[
\frac{\text{KE}_2}{\text{KE}_1} = \frac{1 + \frac{uV}{c^2} \cos \theta_2}{1 + \frac{uV}{c^2} \cos \theta_1},
\]

which holds for any mass that is negligible compared to the planet’s mass. In this instance, the speed of the craft is \( c \) in any frame, so \( u = c \), and we have

\[
\frac{\text{KE}_2}{\text{KE}_1} = \frac{1 + \beta \cos \theta_2}{1 + \beta \cos \theta_1},
\]

where \( \beta = V/c \). For a photon, \( E = h \nu \), and thus

\[
\nu_2 = \frac{1 + \beta \cos \theta_2}{1 + \beta \cos \theta_1} \nu_1.
\]

Equation (3) gives the relation between the initial and final frequencies of the photon, \( \nu_1 \) and \( \nu_2 \), in the Sun frame due to the gravitational slingshot effect.

To see that Eq. (3) is equivalent to the Doppler shift, we assume that a photon approaches the planet at the angle \( \theta_1 \) and leaves at the angle \( \theta_2 \) due to the gravitational pull of the planet (the angles \( \theta_1 \) and \( \theta_2 \) are in the planet frame). In this frame, the initial and final energies (frequencies) are the same. As before, we denote the photon’s initial and final frequencies in the Sun frame by \( \nu_1 \) and \( \nu_2 \), and by \( \nu' \) in the planet frame.

Due to the relativistic Doppler shift, the observed frequency \( \nu' \) of radiation that has frequency \( \nu \) in a source frame with velocity \( \vec{V} \) is

\[
\nu' = \frac{1 - \beta^2}{1 - \beta \cos \psi} \nu,
\]

where \( \psi \) is the angle in the observer frame between the planet’s velocity and the source velocity. From Eq. (4) the frequency of the radiation observed in the planet frame is, assuming a moving source with velocity \( -\vec{V} \)

\[
\nu' = \frac{1 - \beta^2}{1 - \beta \cos (\pi - \theta_1)} \nu = \frac{1 - \beta^2}{1 + \beta \cos \theta_1} \nu_1,
\]

where \( \psi = \pi - \theta_1 \).

After deflection by the planet’s gravity, the photon departs in the direction \( \theta_2 \) in the planet frame. If we switch back to the Sun frame, the frequency \( \nu_2 \) for the departing photon is again given by that for a moving source. This time the source has velocity \( \vec{V} \) and \( \psi = \phi_2 \), the angle the departing photon makes in the Sun frame with the planet’s velocity. From Eq. (4) we have

\[
\nu_2 = \frac{(1 - \beta^2)^{1/2}}{1 - \beta \cos \phi_2} \nu' = \frac{(1 - \beta^2)^{1/2}}{1 + \beta \cos \phi_2} \nu_1.
\]

We let \( \text{sgn} \) denote the sign of \( \cos \phi_2 \), and use Eq. (4) in Ref. 2 to obtain

\[
\cos \phi_2 &= \text{sgn}(1 + \tan^2 \phi_2) \nu_2^{1/2},
\]

\[
= \text{sgn}\left(1 + \beta \cos \phi_2 \right)^{1/2} - 1/2,
\]

\[
= \text{sgn} \left( 1 + \beta \cos \phi_2 \right)^{1/2},
\]

Because \( 1 + \beta \cos \phi_2 > 0 \), we have

\[
\cos \phi_2 = \frac{\beta + \cos \phi_2}{1 + \beta \cos \phi_2}.
\]

If we substitute Eq. (8) into Eq. (6), we find that the frequency in the Sun frame due to the gravitational bending is

\[
\nu_2 = \frac{1 + \beta \cos \phi_2}{1 + \beta \cos \phi_2} \nu_1.
\]

which agrees with Eq. (3).

This longer derivation based on the Doppler shift provides additional insight into the reason for this result, Eq. (9). The shorter derivation leading to the same result, Eq. (3), is based on a direct application of the Lorentz transformation to the energy-momentum four-vector, Eq. (3) of Ref. 2.

The change in frequency due to the interaction includes the familiar result for Compton scattering in which the frequency of the outgoing photon is less than the frequency of the incoming photon. Equation (9) generalizes the usual decrease of frequency for scattering from a stationary mass to scattering from a moving mass as long as \( \theta_2 > \theta_1 \). (Note that for a stationary scatterer \( \theta_1 = 0 \), so that this condition is always satisfied.) Thus we have a simple way involving the angles of the photon propagation in the scatterer’s frame of reference to distinguish between the case in which the photon loses energy in the scattering event and what could be called “inverse Compton scattering.” The case of inverse scattering involves the photon gaining energy from the moving scatterer, if and only if \( \theta_2 < \theta_1 \).

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Due to a copyediting error, Tables I and II of this Reply [Am. J. Phys. 73 (2), 185–189 (2005)] were omitted. They are provided below:

Table I. Optimum control variables and maximum range for typical pitches. $C_{D_{min}} = 0.15$, $\rho = 1.205$ Kg/m$^3$, and $\mu = 1.8 \times 10^{-5}$ N-s/m$^2$.

<table>
<thead>
<tr>
<th>Pitch Type</th>
<th>$V_{b0}$ (m/s)</th>
<th>$V_{g0}$ (m/s)</th>
<th>$\omega_{b0}$ (rad/s)</th>
<th>$V_{bf}$ (m/s)</th>
<th>$\omega_{bf}$ (rad/s)</th>
<th>$\xi$ (rad)</th>
<th>$E_{opt}$ (m)</th>
<th>$\phi_{opt}$ (rad)</th>
<th>Optimal range (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>fast</td>
<td>42.00</td>
<td>30.00</td>
<td>$-$200.00</td>
<td>44.46</td>
<td>194.75</td>
<td>0.4921</td>
<td>0.0277</td>
<td>0.1944</td>
<td>135.108</td>
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<tr>
<td>knuckle</td>
<td>36.00</td>
<td>30.00</td>
<td>0.00</td>
<td>44.09</td>
<td>232.30</td>
<td>0.4712</td>
<td>0.0259</td>
<td>0.1723</td>
<td>135.922</td>
</tr>
<tr>
<td>curve</td>
<td>35.00</td>
<td>30.00</td>
<td>200.00</td>
<td>44.23</td>
<td>267.64</td>
<td>0.4385</td>
<td>0.0227</td>
<td>0.1475</td>
<td>139.047</td>
</tr>
</tbody>
</table>

Table II. Optimum control variables and maximum range for typical pitches; $C_{D_{min}} = 0.25$, $\rho = 1.205$ Kg/m$^3$, and $\mu = 1.8 \times 10^{-5}$ N-s/m$^2$.

<table>
<thead>
<tr>
<th>Pitch Type</th>
<th>$V_{b0}$ (m/s)</th>
<th>$V_{g0}$ (m/s)</th>
<th>$\omega_{b0}$ (rad/s)</th>
<th>$V_{bf}$ (m/s)</th>
<th>$\omega_{bf}$ (rad/s)</th>
<th>$\xi$ (rad)</th>
<th>$E_{opt}$ (m)</th>
<th>$\phi_{opt}$ (rad)</th>
<th>Optimal range (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>fast</td>
<td>42.00</td>
<td>30.00</td>
<td>$-$200.00</td>
<td>44.64</td>
<td>204.43</td>
<td>0.5380</td>
<td>0.0294</td>
<td>0.2363</td>
<td>124.362</td>
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<tr>
<td>knuckle</td>
<td>36.00</td>
<td>30.00</td>
<td>0.00</td>
<td>44.13</td>
<td>250.32</td>
<td>0.5153</td>
<td>0.0277</td>
<td>0.1972</td>
<td>124.929</td>
</tr>
<tr>
<td>curve</td>
<td>35.00</td>
<td>30.00</td>
<td>200.00</td>
<td>44.33</td>
<td>284.64</td>
<td>0.4880</td>
<td>0.0248</td>
<td>0.1807</td>
<td>127.517</td>
</tr>
</tbody>
</table>

In addition the sentence in the last full paragraph of the second column of p. 187 should read: “As an example, in the direct impact of a spinning baseball with a bat of normal incidence, a tangential impulse $p_t$ is required to create the angular impulse $\omega_b = \frac{2m_r \omega_c r_b^2}{5}$ necessary to stop the spin, where $m_r$, $r_b$, and $I$ are the ball mass, radius and moment of inertia, respectively.” In the first paragraph of the second column of page 188, the units of $\mu$ are “N-s/m$^2$”. The symbol $\mu$ in footnote 19 should be $\mu_c$. On January 27, 2005 the online version of the paper was changed to contain the two missing tables.