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Reply to "Comment on Gravitational slingshot," by C. L. Cook [Am. J. Phys. 73 (4), 363 (2005)]

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Reply to "Comment on 'Gravitational slingshot,'" by C. L. Cook [Am. J. Phys. 73 (4), 363 (2005)]

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NOTES AND DISCUSSIONS

"Shortcut to the Slingshot Effect"

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The Lorentz transformation approach is a very elegant way to derive the gravitational slingshot effect.[1] An equally elegant shortcut starts with the classical Lagrangian

$$L=L(\mathrm{r},\mathrm{u},t)=\frac{1}{2}mu^2+\frac{mMG}{|\mathrm{r}-\mathrm{v}t|},\tag{1}$$

where r is the position vector of a space probe with velocity u≡dr/dt, vt the position vector of a planet moving with an approximately constant velocity v at time t, m the mass of the probe, M the mass of the planet, and G the gravitational constant.

Defining $\boldsymbol{\rho}\equiv\mathrm{r}-\mathrm{v}t$, $\mathrm{w}\equiv d\boldsymbol{\rho}/dt=\mathrm{u}-\mathrm{v}$, and $|\boldsymbol{\rho}|\equiv\rho$, Eq. (1) becomes

$$L=L(\rho,\mathrm{w})=\frac{1}{2}m|\mathrm{w}+\mathrm{v}|^2+\frac{mMG}{\rho}.\tag{2}$$

The canonical momentum derived from Eq. (2) is $\mathrm{p}=m(\mathrm{w}+\mathrm{v})$, giving the Hamiltonian

$$H=H(\mathrm{p},\rho)=\mathrm{w}\cdot\mathrm{p}-L\tag{3a}$$

$$=\frac{\mathrm{p}^2}{2m}-\mathrm{v}\cdot\mathrm{p}-\frac{mMG}{\rho}\tag{3b}$$

$$=\frac{1}{2}m\mathrm{w}^2-\frac{mMG}{\rho}-\frac{1}{2}m\mathrm{v}^2\tag{3c}$$

$$=\frac{1}{2}m\mathrm{u}^2-\frac{mMG}{|\mathrm{r}-\mathrm{v}t|}-m\mathrm{v}\cdot\mathrm{u}\tag{3d}$$

$$=E-m\mathrm{v}\cdot\mathrm{u},\tag{3e}$$

where $E=E(u,\mathrm{r},t)$ is the total energy of the probe and $H(\mathrm{p},\rho)$ is a constant of the motion. The Hamiltonian (3a) and (3b), though conserved, is not the energy E, which is not conserved.

Since the increment $\Delta H=0$ between any two positions of the probe, Eq. (3e) gives

$$\Delta E=m\mathrm{v}\cdot\Delta\mathrm{u}=m\mathrm{v}\cdot\Delta\mathrm{w}\tag{4}$$

as the energy increment. If the positions are chosen so that ρ (the distance between the probe and the planet) is the same at both positions, Eq. (3c) indicates that w (the speed of the probe relative to the planet) is also the same. Defining the unit vectors $\hat{\mathrm{v}}\equiv\mathrm{v}/v$ and $\hat{\mathrm{w}}\equiv\mathrm{w}/w$, Eq. (4) becomes

$$\Delta E=mvw\Delta(\hat{\mathrm{v}}\cdot\hat{\mathrm{w}})=mvw\Delta(\cos\theta),\tag{5}$$

where the angle θ is between vectors v and w.

The energy increment (5) is equivalent to Eq. (7) of Ref. 1. Both approaches depend on the assumption that the velocity v of the planet can be treated as constant during the time when the interaction between the planet and the probe is significant, but it is not necessary to assume that the interaction is insignificant in the initial and final states. It is only necessary to choose the initial and final positions symmetrically so that ρ is the same at both, i.e., so that $\Delta\rho=0$, for which Eq. (3c) gives $\Delta w=0$, a necessary condition for the validity of Eq. (5). The quantity ΔE is the change in the kinetic energy, because the potential energy is the same at these symmetrically located points.

The analysis here is performed relative to the "sun-centered frame" defined in Ref. 1, except that the term "relative" used here refers to Newtonian relativity based on Galilean transformations, rather than Einsteinian relativity based on Lorentz transformations. It is an approach which seems to eliminate G from the problem. Another approach which emphasizes the role of G is obtained by noting that the energy E is the Hamiltonian $H_1(p,\mathrm{r},t)$ obtained from Lagrangian (1), so that

$$E=H_1(p,\mathrm{r},t)=\frac{p^2}{2m}-\frac{mMG}{|\mathrm{r}-\mathrm{v}t|}.\tag{6}$$

It follows that

$$\frac{dE}{dt}=\frac{\partial H_1}{\partial t}=-\frac{mMG\mathrm{v}\cdot(\mathrm{r}-\mathrm{v}t)}{|\mathrm{r}-\mathrm{v}t|^3},\tag{7}$$

quantifying the relation between the strength of the gravitational interaction and the rate at which energy is exchanged between the planet and the probe. Equation (7) can be put in the form

$$-\frac{dE}{dt}=\mathrm{F}\cdot\mathrm{v},\tag{8}$$

where F is the force that the probe exerts on the planet, and v is the velocity of the planet, so F·v is the rate at which the probe does work on the planet. When F·v is negative, the planet does work on the probe, creating a slingshot effect.

[1] John J. Dykla, Robert Cacioppo, and Asim Gangopadhyaya, "Gravitational slingshot," Am. J. Phys. **72**(5), 619–621 (2004).

Comment on "Gravitational slingshot," by John J. Dykla, Robert Cacioppo, and Asim Gangopadhyaya [Am. J. Phys. 72 (5), 619–621 (2004)]

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A recent paper[1] used the Lorentz transformation for energy-momentum four vectors to analyze the gravitational slingshot. It claimed that "the relativistic method is shorter and more compact than its nonrelativistic counterpart." We will present a nonrelativistic treatment that is more compact and just as elegant and simple.

In mechanics, energy transfer occurs when forces do work. The kinetic energy of a spacecraft increases if it does negative work on a planet,[2]

$$0 > \int \mathbf{F}_{spacecraft\ on\ planet} \cdot \mathbf{dr}_{planet}, \tag{1}$$

or, in terms of the reaction force of the planet on the spacecraft,

$$0 < \int \mathbf{F}_{planet\ on\ spacecraft} \cdot \mathbf{dr}_{planet} = \int \mathbf{F}_{planet\ on\ spacecraft} \cdot \mathbf{V}_{planet} dt. \tag{2}$$

The presence of the planetary displacement vector $\mathbf{dr}_{planet}$ or velocity vector $\mathbf{V}_{planet}$ makes the work integral frame dependent.[3]

Because the spacecraft-planet interaction occupies a time interval much less than the planet's orbital period, $\mathbf{V}_{planet}$ may be assumed to be constant.[4] Following Ref. 1 we set

$$\mathbf{V}_{planet} = V\hat{\mathbf{x}} \tag{3}$$

in the Sun rest frame.

If we substitute Eq. (3) into Eq. (2) and discard the constant positive factor V, the condition for an increase in the spacecraft's kinetic energy in the Sun rest frame becomes

$$0 < \hat{\mathbf{x}} \cdot \int \mathbf{F}_{planet\ on\ spacecraft} dt, \tag{4}$$

where $\int \mathbf{F}_{planet\ on\ satellite} dt$ is the impulse, $\Delta\mathbf{p}$, delivered to the spacecraft by the planet. It has the same value in any reference frame because force and time are Galilean invariants.[5] We evaluate $\Delta\mathbf{p}$ in the planet center-of-mass frame,

$$\Delta\mathbf{p} = mu\{(\cos\theta_2 - \cos\theta_1)\hat{\mathbf{x}} + (\sin\theta_2 - \sin\theta_1)\hat{\mathbf{y}}\}, \tag{5}$$

where the notation of Ref. 1 has been employed.[6]

Equation (4) becomes

$$0 < \hat{\mathbf{x}} \cdot \Delta\mathbf{p} = mu(\cos\theta_2 - \cos\theta_1). \tag{6}$$

That is, for an increase in the spacecraft's kinetic energy, $\cos\theta_1 < \cos\theta_2$ or $\theta_1 > \theta_2$ as derived using the Lorentz transformation in Ref. 1.

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[1]John J. Dykla, Robert Cacioppo, and Asim Gangopadhyaya, "Gravitational slingshot," Am. J. Phys. **72**(5), 619–621 (2004).

[2]The scalar product of the attractive force that the spacecraft exerts on the planet and the planet's displacement is negative while the spacecraft passes behind the planet. The work done by the spacecraft is then negative and the slingshot "fires."

[3]This point is clearly made in James A. Van Allen's, "Gravitational assist in celestial mechanics: A tutorial," Am. J. Phys. **71**(5), 448–451 (2003).

[4]The tiny change in the planet's velocity due to the energy transfer is negligible.

[5]C. L. Cook, "Note on actually *using* impulse," Am. J. Phys. **58**(11), 1106 (1990).

[6]The spacecraft of mass m initially travels in the xy plane at an angle θ_1 relative to the x axis; after the interaction it travels at an angle θ_2; its speed has the same initial and final values, u.

Reply to "Comment on 'Gravitational slingshot,'" by C. L. Cook [Am. J. Phys. 73 (4), 363 (2005)]

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Cook[1] makes the valid point that a nonrelativistic explanation of the slingshot effect is shorter than the relativistic derivation given in Ref. 2. Because gravity is a conservative force, the initial and final speeds of the craft are $v_1 = v_2 = u$ in the planet frame. In the Sun frame the initial and final velocities are $\vec{V} + \vec{v}_1$ and $\vec{V} + \vec{v}_2$, respectively. The change in kinetic energy in the Sun frame is $\frac{1}{2}m|\vec{V}+\vec{v}_2|^2-\frac{1}{2}|\vec{V}+\vec{v}_1|^2$. Because $\vec{v}_i\cdot\vec{V}=Vu\cos\theta_i$ $(i=1,2)$, where the angles θ_1 and θ_2 are between the planet's velocity $\vec{V}$ and the craft's velocities $\vec{v}_1$ and $\vec{v}_2$ in the planet frame, the desired result, $mVu(\cos\theta_2-\cos\theta_1)$, is immediate.

The relativistic derivation in Ref. 2 is more involved, but it gives further insight into the nature of the slingshot effect. As an example, we discuss what the gravitational slingshot effect would be for a photon. Of course, it cannot accelerate a photon, but it does change its frequency in accordance with a generalization of Compton scattering which allows for a moving mass. This result cannot be understood as a nonrelativistic slingshot effect even though the planet's speed is nonrelativistic.

Because our "craft" is a photon, we will first remove the craft's mass m from the kinetic-energy equation [Eq. (6) in Ref. 1] by looking at the fractional change in its kinetic energy. This change is

$$\frac{\mathrm{KE}_2}{\mathrm{KE}_1}=\frac{1+\dfrac{uV}{c^2}\cos\theta_2}{1+\dfrac{uV}{c^2}\cos\theta_1}, \tag{1}$$

which holds for any mass that is negligible compared to the planet's mass. In this instance, the speed of the craft is c in any frame, so $u=c$, and we have

$$\frac{\mathrm{KE}_2}{\mathrm{KE}_1}=\frac{1+\beta\cos\theta_2}{1+\beta\cos\theta_1}, \tag{2}$$

where $\beta=V/c$. For a photon, $E=h\nu$, and thus

$$\nu_2=\frac{1+\beta\cos\theta_2}{1+\beta\cos\theta_1}\nu_1. \tag{3}$$

Equation (3) gives the relation between the initial and final frequencies of the photon, ν_1 and ν_2, in the Sun frame due to the gravitational slingshot effect.

To see that Eq. (3) is equivalent to the Doppler shift, we assume that a photon approaches the planet at the angle θ_1 and leaves at the angle θ_2 due to the gravitational pull of the planet (the angles θ_1 and θ_2 are in the planet frame). In this frame, the initial and final energies (frequencies) are the same. As before, we denote the photon's initial and final frequencies in the Sun frame by ν_1 and ν_2, and by ν' in the planet frame.

Due to the relativistic Doppler shift, the observed frequency ν_0 of radiation that has frequency ν in a source frame with velocity $\vec{V}$ is

$$\nu_0=\frac{(1-\beta^2)^{1/2}}{1-\beta\cos\psi}\nu, \tag{4}$$

where ψ is the angle in the observer frame between the photon's velocity and the source velocity.[3] From Eq. (4) the frequency of the radiation observed in the planet frame is, assuming a moving source with velocity $-\vec{V}$ is

$$\nu'=\frac{(1-\beta^2)^{1/2}}{1-\beta\cos(\pi-\theta_1)}\nu_1=\frac{(1-\beta^2)^{1/2}}{1+\beta\cos\theta_1}\nu_1, \tag{5}$$

where $\psi=\pi-\theta_1$.

After deflection by the planet's gravity, the photon departs in the direction θ_2 in the planet frame. If we switch back to the Sun frame, the frequency ν_2 for the departing photon is again given by that for a moving source. This time the source has velocity $\vec{V}$ and $\psi=\phi_2$, the angle the departing photon makes in the Sun frame with the planet's velocity. From Eq. (4) we have

$$\begin{aligned}\nu_2&=\frac{(1-\beta^2)^{1/2}}{(1-\beta\cos\phi_2)}\nu'\\&=\frac{(1-\beta^2)}{(1+\beta\cos\theta_1)(1-\beta\cos\phi_2)}\nu_1.\end{aligned} \tag{6}$$

We let sgn denote the sign of $\cos\phi_2$, and use Eq. (4) in Ref. 2 to obtain

$$\cos\phi_2=\mathrm{sgn}(1+\tan^2\phi_2)^{-1/2}, \tag{7a}$$

$$=\mathrm{sgn}\left(\frac{(1+\beta\cos\theta_2)^2}{(\beta+\cos\theta_2)^2}\right)^{-1/2}, \tag{7b}$$

$$=\frac{\mathrm{sgn}|\beta+\cos\theta_2|}{1+\beta\cos\theta_2}. \tag{7c}$$

Because $1+\beta\cos\theta_2>0$, we have

$$\cos\phi_2=\frac{\beta+\cos\theta_2}{1+\beta\cos\theta_2}. \tag{8}$$

If we substitute Eq. (8) into Eq. (6), we find that the frequency in the Sun frame due to the Doppler shift caused by the gravitational bending is

$$\nu_2=\frac{1+\beta\cos\theta_2}{1+\beta\cos\theta_1}\nu_1, \tag{9}$$

which agrees with Eq. (3).

This longer derivation based on the Doppler shift provides additional insight into the reason for this result, Eq. (9). The shorter derivation leading to the same result, Eq. (3), is based on a direct application of the Lorentz transformation to the energy-momentum four-vector, Eq. (3) of Ref. 2.

The change in frequency due to the interaction includes the familiar result for Compton scattering in which the frequency of the outgoing photon is less than the frequency of the incoming photon. Equation (9) generalizes the usual decrease of frequency for scattering from a stationary mass to scattering from a moving mass as long as $\theta_2>\theta_1$. (Note that for a stationary scatterer $\theta_1=0$, so that this condition is always satisfied.) Thus we have a simple way involving the angles of the photon propagation in the scatterer's frame of reference to distinguish between the case in which the photon loses energy in the scattering event and what could be called "inverse Compton scattering." The case of inverse scattering involves the photon gaining energy from the moving scatterer, if and only if $\theta_2<\theta_1$.

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[1]C. L. Cook, "Comment on 'Gravitational slingshot'," Am. J. Phys. **73**, 363 (2005).
[2]John J. Dykla, Robert Cacioppo, and Asim Gangopadhyaya, "Gravitational slingshot," Am. J. Phys. **72**(5), 619–621 (2004).
[3]A. P. French, *Special Relativity* (MIT, Cambridge, MA, 1968).

Erratum: Reply to Comment on "How to hit home runs: Optimum baseball swing parameters for maximum range trajectories," by Gregory S. Sawicki, Mont Hubbard, and William J. Stronge [Am. J. Phys. 71 (11), 1152–1162 (2003)]

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Due to a copyediting error, Tables I and II of this Reply [Am. J. Phys. **73** (2), 185–189 (2005)] were omitted. They are provided below:

Table I. Optimum control variables and maximum range for typical pitches. $C_{Dmin}=0.15$, $\rho=1.205$ Kg/m^3, and $\mu=1.8\times10^{-5}$ N-s/m^2.

Pitch Type	V_{b0} (m/s)	V_{B0} (m/s)	ω_{b0} (rad/s)	V_{bf} (m/s)	ω_{bf} (rad/s)	ζ (rad)	$E_{\rm opt}$ (m)	$\psi_{\rm opt}$ (rad)	Optimal range (m)
fast	42.00	30.00	−200.00	44.46	194.75	0.4921	0.0277	0.1944	135.108
knuckle	36.00	30.00	0.00	44.09	232.30	0.4712	0.0259	0.1723	135.922
curve	35.00	30.00	200.00	44.23	267.64	0.4385	0.0227	0.1475	139.047

Table II. Optimum control variables and maximum range for typical pitches; $C_{Dmin}=0.25$, $\rho=1.205$ Kg/m^3, and $\mu=1.8\times10^{-5}$ N-s/m^2.

Pitch Type	V_{b0} (m/s)	V_{B0} (m/s)	ω_{b0} (rad/s)	V_{bf} (m/s)	ω_{bf} (rad/s)	ζ (rad)	$E_{\rm opt}$ (m)	$\psi_{\rm opt}$ (rad)	Optimal range (m)
fast	42.00	30.00	−200.00	44.64	204.43	0.5380	0.0294	0.2363	124.362
knuckle	36.00	30.00	0.00	44.13	250.32	0.5153	0.0277	0.1972	124.929
curve	35.00	30.00	200.00	44.33	284.64	0.4880	0.0248	0.1807	127.517

In addition the sentence in the last full paragraph of the second column of p. 187 should read: "As an example, in the direct impact of a spinning baseball with a bat of normal incidence, a tangential impulse p_t is required to create the angular impulse $r_b p_t = I\omega_o = 2m_b\omega_o r_b^2/5$ necessary to stop the spin, where m_b, r_b and I are the ball mass, radius and moment of inertia, respectively." In the first paragraph of the second column of page 188, the units of μ are "N-s/m^2". The symbol μ in footnote 19 should be $\mu_{\rm f}$. On January 27, 2005 the online version of the paper was changed to contain the two missing tables.

