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Quantum Mechanics of Multi-Prong Potentials

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Abstract

We describe the bound state and scattering properties of a quantum mechanical particle in a scalar N-prong potential. Such a study is of special interest since these situations are intermediate between one and two dimensions. The energy levels for the special case of N identical prongs exhibit an alternating pattern of non-degeneracy and $(N - 1)$ fold degeneracy. It is shown that the techniques of supersymmetric quantum mechanics can be used to generate new solutions. Solutions for prongs of arbitrary lengths are developed. Discussions of tunneling in N-well potentials and of scattering for piecewise constant potentials are given. Since our treatment is for general values of N, the results can be studied in the large N limit. A somewhat surprising result is that a free particle incident on an N-prong vertex undergoes continuously increased backscattering as the number of prongs is increased.

1 Introduction

Various aspects of the solutions of the Schrödinger equation for both scalar and vector potentials on a wide variety of networks have been discussed by several authors [1-8]. Recent investigations have been motivated in part by the considerable interest in mesoscopic systems and the experimental observation of persistent currents[9]. Most of the results correspond to piecewise constant scalar potentials $V = V_0$ [3, 4, 8], vector potentials $\mathbf{A}$ associated with uniform magnetic fields [5], or $\delta-$function potentials at the network vertices [2, 6, 7]. In this paper we discuss the bound states and the scattering properties of a single particle moving in an arbitrary scalar N-prong potential. The treatment is kept simple in order to clearly show the generalization from the two-prong case, which is just the familiar one-dimensional (particle on a line) problem. The properties of the eigenstates of N-prong potentials are not a priori evident, since this system is in some sense intermediate between one and two dimensions.

The plan of this paper is as follows. In Sec. 2, we define N-prong potentials and the boundary conditions which the wave function must satisfy. The normalization of wave functions and their orthogonality properties are also discussed. Sec. 3 contains a discussion of bound states for potentials with N identical prongs. The general solution is quite easy to obtain and is in fact closely related to the solution of a symmetric one-dimensional potential. Sec. 4 contains a discussion of bound states in N-prong potentials with non-identical prongs. Several analytical and numerical solutions are given to illustrate eigenfunction properties. In particular, we show that a generalized version of the usual rule about one extra node for each higher eigenstate is probably true but the theorem about non-degeneracy in one-dimensional problems is not. Since the concept of supersymmetry has yielded many interesting results for one-dimensional quantum mechanics[10, 11], it is natural to investigate what new results come from using supersymmetry for N-prong potentials. This is discussed in Sec. 5. Sec. 6 illustrates the use of lowest order perturbation theory to obtain bound state energies. Scattering is discussed in Sec. 7. For the special case of no potential in any of the N prongs, very interesting properties are obtained which are quite different from naive expectations. Finally, in Sec. 8, we extend the usual discussion of tunneling for a double well potential to the case of N-well potential systems.

2 Multi-Prong Potentials

We are addressing the problem of formulating and solving the Schrödinger equation for a single particle constrained to remain in a space made up of N lines meeting at a vertex point (Fig. 1). The prongs are labeled by the indices i ($i = 1, 2, \cdots, N$) and the position on any prong is given by the positive coordinate x_i, with $x_i = 0$ being the vertex point. The potential is fully specified by giving the scalar potentials $V_i(x_i)$ on each prong i. The overall

wave function $\psi(\vec{x})$ is composed of the individual wave functions $\psi_i(x_i)$ on each prong i:

$$\psi(\vec{x}) \equiv \{\psi_1(x_1), \cdots, \psi_N(x_N)\}. \tag{1}$$

We are using the arrow notation to indicate the appropriate N-tuple. The physical requirement of single-valuedness of the wave function at the vertex implies for the component wave functions

$$\psi_1(0) = \psi_2(0) = \cdots = \psi_N(0). \tag{2}$$

In this paper, we will not be considering δ-function potentials at the vertex. Consequently, the second vertex condition requires the sum of all derivatives to add up to zero [1, 6]:

$$\sum_{i=1}^{N} \frac{d\psi_i}{dx_i}\bigg|_{x_i=0} = 0. \tag{3}$$

This can be readily derived as follows. The time independent Schrödinger equation, reads $\nabla^2 \psi = (V - E)\psi$, taking units in which $\hbar = 2m = 1$. We assume that the potential is finite in the neighborhood of the vertex, and integrate over a small sphere of radius ϵ with the vertex as the center. Using the divergence theorem and letting $\epsilon \to 0$ yields $\lim_{\epsilon \to 0} \int_A \nabla \psi \cdot dA = 0$, which reduces to Eq. (3) for an N-prong potential. Note that for the special case $N = 2$, one has the standard one-dimensional situation and, taking into account the relative directions of x_1 and x_2, Eq. (3) is just the statement that the derivative $d\psi/dx$ is continuous across the vertex.

For the N-prong system, we call two functions $\psi(\vec{x})$ and $\phi(\vec{x})$ orthogonal if

$$\sum_{i=1}^{N} \int \psi_i^*(x_i)\phi_i(x_i)dx_i = 0, \tag{4}$$

where the sum indicates that all prongs are included and each integral runs over the appropriate prong length. The normalization condition, which also includes all prongs, reads

$$\sum_{i=1}^{N} \int \psi_i^*(x_i)\psi_i(x_i)dx_i = 1. \tag{5}$$

The eigenfunctions of the Schrödinger equation in a multi-prong potential not only have to satisfy the vertex conditions given in Eqs. (2) and (3), but also appropriate boundary conditions at each prong end. These conditions depend on the energy E and the specific behavior of $V_i(x_i)$ for the maximum allowed value of x_i. Eigenfunctions corresponding to bound state solutions have to vanish at the prong ends. For potentials which reach a constant value fast enough at the prong ends, one has standard plane wave solutions (e^{ikx_i}, e^{-ikx_i}) for the scattering states.

3 Bound States: Identical Prongs

We begin by discussing the interesting special case of N identical prongs, i.e. $V_i(x_i) = V(x_i)$. Note that this does not imply that the wave functions are identical on all the prongs. In fact, Eq. (3) implies the wave functions are often different, since their derivatives at the vertex have to add up to zero. This statement is the generalization of the two-prong case, where one has symmetric and antisymmetric solutions.

The general bound state solution for N identical prongs is easily obtained from the two-prong situation. For two identical prongs, one has the situation schematically shown in Fig. 2. We can define the single variable x such that $x = x_1$ for positive x, and $x = -x_2$ for negative x. Effectively, one is mapping the two prongs onto the real axis, $-\infty < x < \infty$. Clearly, this is the familiar situation of a symmetric, one-dimensional potential. Its eigenstates correspond to even and odd solutions $\psi^{(n)}(x)$ at energies E_n, labeled by a quantum number n ($n = 0, 1, 2, \cdots$).

Theorem: The eigenstates of a potential with N identical prongs can be constructed from the eigenfunctions $\psi^{(n)}(x)$ of the corresponding symmetric two-prong system, and have exactly the same eigenenergies. Explicitly,

$$\psi^{(n)}(\vec{a}, \vec{x}) \equiv \left\{ a_1 \psi^{(n)}(x_1), a_2 \psi^{(n)}(x_2), \cdots, a_N \psi^{(n)}(x_N) \right\}, \quad (x_i > 0). \tag{6}$$

where $\vec{a}$ is a compact notation for the N-tuple $(a_1, a_2, \cdots, a_N)$. For even numbered states the boundary conditions at the vertex imply $a_1 = a_2 = \cdots = a_N$; for odd numbered states one has the constraint $\sum_i a_i = 0$, and there is an $(N-1)$ fold degeneracy.

Proof: By construction, the wave function $\psi^{(n)}(\vec{a}, \vec{x})$ for an eigenenergy E_n satisfies all the boundary conditions at the prong ends. It only remains to show that the vertex conditions, Eqs. (2) and (3) are also satisfied by Eq. (6). a) For even eigenfunctions of a two-prong potential one has $\psi'(0) = 0$, and in general $\psi(0) \neq 0$. Eq. (3) is therefore trivially satisfied, and Eq. (2) implies $a_1 = a_2 = \cdots = a_N$. The eigenfunction is therefore determined upto one overall normalization constant, and the energy level is non-degenerate. b) For odd eigenfunctions of a two-prong potential, one has $\psi'(0) \neq 0$, and $\psi(0) = 0$. Now, Eq. (2) is trivially satisfied (since each wave function vanishes at the vertex) and the derivative condition Eq. (3) implies $\sum_i a_i = 0$, as claimed. This constraint is not sufficient to fully determine the constants a_i, and the levels have an $(N-1)$ fold degeneracy. The completeness of states in Eq. (6) seems intuitively reasonable, and can be established from the completeness of the eigenfunctions of a symmetric one dimensional potential.

As a first illustrative example, consider a particle of mass m moving in a harmonic

potential with three identical prongs

$$V(x_i) = \frac{1}{2}m\omega^2 x_i^2. \tag{7}$$

Each prong corresponds to the positive half of a harmonic oscillator potential. The ground state of this three-prong potential will have the same energy $E_0 = \frac{1}{2}\hbar\omega$ as the two-prong case. The un-normalized eigenfunction is given by

$$\psi^{(0)}(\vec{a}, \vec{x}) \equiv \left\{ a\psi^{(0)}(x_1), a\psi^{(0)}(x_2), a\psi^{(0)}(x_3) \right\}, \tag{8}$$

where, $\psi^{(0)} \propto e^{-\omega x^2/4}$ is the Gaussian harmonic oscillator ground state wave function. We assume $\psi^{(n)}(x)$ to be normalized to unity over one prong, so that the wave function in Eq. (6) is properly normalized. The next level at energy $E_1 = \frac{3}{2}\hbar\omega$ is doubly degenerate. Eigenfunctions at this level are given by

$$\psi^{(1)}(\vec{a}, \vec{x}) \equiv \left\{ a_1\psi^{(1)}(x_1), a_2\psi^{(1)}(x_2), a_3\psi^{(1)}(x_3) \right\}, \tag{9}$$

where, $\psi^{(1)} \propto xe^{-\omega x^2/4}$ is the wave function for the first excited state of the two-prong case. There is an infinite number of wave functions $\psi^{(1)}(\vec{a}, \vec{x})$ that correspond to the same energy $\frac{3}{2}\hbar\omega$. Each of these wave functions is characterized by a set of coefficients a_i obeying the constraint $\sum_i a_i = 0$. Due to this constraint, we see that all allowed a_i span a two-dimensional space, thus leading to a two fold degeneracy at this level. The following two functions then provide one choice for an orthonormal basis in this two-dimensional, two-fold degenerate space:

$$\psi^{(1)}(\vec{a}_1; \vec{x}), \qquad \vec{a}_1 \equiv \left(\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}} \right);$$

$$\psi^{(1)}(\vec{a}_2; \vec{x}), \qquad \vec{a}_2 \equiv \left(\frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right). \tag{10}$$

This pattern of nondegeneracy/degeneracy keeps repeating as we consider higher energy states. All even numbered states are nondegenerate and all odd numbered states have two fold degeneracy. The three lowest eigenstates are shown in Fig. 3. The normalization has been chosen such that $\int_0^\infty \psi^{(n)*}(x)\psi^{(n)}(x)dx = 1$.

Similarly, the eigenfunctions for the potential $V(x_i) = 0, \quad x_i \leq 1$ are shown by the dashed lines in Fig. 4. The wave functions are sinusoidal, and the energies are $E_n = n^2\pi^2/4 \ (n = 1, 2, \cdots)$, since they correspond to a symmetric two-prong potential which is an infinite square well of width 2.

From the above examples, it becomes clear that a potential with N identical prongs will have alternate nondegenerate and $(N-1)$ fold degenerate energy levels. Clearly, the familiar non-degeneracy property of one-dimensional potentials is maintained for $N = 2$, but not for situations with more prongs.

4 Bound States: Non-Identical Prongs

Let us now consider the general case of an N-prong system with potential $V_i(x_i)$ on the i-th prong. Let $\psi_i(x_i)$ be the solution of the Schrödinger equation with energy E along that prong:

$$-\frac{d^2\psi_i}{dx_i^2} + (V_i - E)\,\psi_i = 0, \quad (i = 1, 2, \cdots, N).$$

Of the two linearly independent solutions to this equation, let $\phi_i(x_i, E)$ be the one that vanishes at the end point of the i-th prong. This implies $\psi_i(x_i, E) = a_i\phi_i(x_i, E)$, where a_i is a constant. These $\psi_i(x_i, E)$, glued together properly so as to satisfy vertex requirements, will produce the wave function for the entire domain. From vertex conditions (2) and (3), we get

$$\psi_i(0, E) = a_i\phi_i(0, E) \equiv U, \tag{11}$$

and

$$\sum_{i=1}^{N} \psi_i'(0, E) = \sum_{i=1}^{N} a_i\phi_i'(0, E) = 0. \tag{12}$$

The energy eigenvalues E are then determined by eliminating the unknowns a_i and U from the above set of $(N + 1)$ linear equations. The eigenvalue condition is:

$$\det \begin{vmatrix} \phi_1(0, E) & 0 & \cdots & 0 & 1 \\ 0 & \phi_2(0, E) & \cdots & 0 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \phi_N(0, E) & 1 \\ \phi_1'(0, E) & \phi_2'(0, E) & \cdots & \phi_N'(0, E) & 0 \end{vmatrix} = 0. \tag{13}$$

For any given eigenvalue E_n, Eq. (11) then determines all the constants a_i in terms of one unknown constant U. The corresponding bound state wave function is:

$$\psi(\vec{x}, E_n) = U \left\{ \frac{\phi_1(x_1, E_n)}{\phi_1(0, E_n)}, \cdots, \frac{\phi_N(x_N, E_n)}{\phi_N(0, E_n)} \right\}. \tag{14}$$

If desired, the constant U can also be fixed by requiring overall normalization of the wave function.

To illustrate the procedure for determining eigenvalues, consider the example of a particle that is free to move inside a domain of three prongs of lengths l_1, l_2 and l_3. Since the potential along the prongs is zero, the wave functions along them are given by sinusoidal functions that vanish at the end points. An eigenfunction of energy $E = k^2$ has the form:

$$\psi(\vec{a}, \vec{x}) = \{a_1 \sin k(l_1 - x_1), a_2 \sin k(l_2 - x_2), a_3 \sin k(l_3 - x_3)\}. \tag{15}$$

In this case, Eq. (11) is:

$$a_1 \sin kl_1 = a_2 \sin kl_2 = a_3 \sin kl_3 = U, \tag{16}$$

where U is the common value of the wave function at the origin. The derivative condition (12) gives:

$$a_1 \cos kl_1 + a_2 \cos kl_2 + a_3 \cos kl_3 = 0. \tag{17}$$

The eigenvalue condition (13), when simplified reads

$$\cos kl_1 \, \sin kl_2 \, \sin kl_3 + \sin kl_1 \, \cos kl_2 \, \sin kl_3 + \sin kl_1 \, \sin kl_2 \cos kl_3 = 0. \tag{18}$$

As a specific case, consider the situation $l_2 = 0.8$ and $l_1 = l_3 = 1$. For this case, the solutions to Eq. (18) are found to be $k = 1.68, \pi, 3.61, 5.08, 2\pi, 7.17, 8.54, 3\pi, ...$ The solutions $k = m\pi$ $(m = 1, 2, ...)$ are a consequence of maintaining partial symmetry by taking two prongs to be identical $(l_1 = l_3 = 1)$. The corresponding wave functions are shown by the solid lines in Fig. 4. [For comparison, we have also plotted the dashed lines corresponding to the wave functions for the case of all three identical prongs of length 1]. From the figure, it is apparent that the ground state wave function at $E_0 = 1.68^2$ has no nodes. The first excited state at $E_1 = \pi^2$ has one node at the vertex point. Note that for this state, the wave function in prong 2 is zero. The second excited state is at energy $E_2 = 3.61^2$ and has two nodes, one in prong 1 and the other in prong 3. This result is very suggestive - one expects one extra node to appear for each higher eigenstate, similar to the familiar one-dimensional situation.

We have also studied the variation of the eigenvalues k systematically as the prong length l_2 is varied. The results are shown in Fig. 5. As discussed above, the solutions $k = m\pi$ are present, and when l_2 is an integral multiple of $l_1(= l_3)$, one has the interesting occurrence of degeneracy and level crossing. The curves in Fig. 5 are labeled by the number of nodes in the wave functions. At any fixed value of l_2, the number of nodes increases with energy. Note that as $l_2 \rightarrow 0$, the eigenvalues become doubly degenerate at $k = m\pi$, $(m = 1, 2, ...)$. This is intuitively clear since the limit $l_2 \rightarrow 0$ forces the wave function to vanish at the vertex, thereby effectively breaking the problem into two infinite square well potentials of widths $l_1 = 1$ and $l_3 = 1$. Finally, in Fig. 6, we take $l_1 \neq l_3$, and plot eigenvalues k. This is a completely asymmetric situation, and there is now no degeneracy of energy levels. Again note that the number of nodes in the overall wave function increases by one with increasing energy levels.

As a second example, we determine the eigenstates of a three-prong potential $V_i(x_i) = \frac{1}{4}\omega_i^2 x_i^2$, composed of three harmonic oscillators of different angular frequencies ω_i $(i = 1, 2, 3)$, and $\hbar = 2m = 1$. The wave functions which vanish at $x_i \rightarrow \infty$ are [13]

$$\psi_i(x_i) = a_i D_{\nu_i}(\sqrt{\omega_i} x_i) \ , \quad \nu_i = \frac{E}{\omega_i} - \frac{1}{2} \ , \tag{19}$$

where D_ν is a parabolic cylinder function. The eigenvalue condition [Eq. (13)] gives

$$\sqrt{\omega_1} D'_{\nu_1}(0) D_{\nu_2}(0) D_{\nu_3}(0) + \text{cyclic \ permutations} \ = \ 0 \ . \tag{20}$$

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The parabolic cylinder function and its derivative at the origin have simple expressions in terms of gamma functions [13]:

$$D_\nu(0) = 2^{\frac{\nu}{2}} \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2} - \frac{\nu}{2})} \quad , \quad D'_\nu(0) = 2^{\frac{\nu}{2} - \frac{1}{2}} \frac{\Gamma(-\frac{1}{2})}{\Gamma(-\frac{\nu}{2})} \quad . \tag{21}$$

The solutions of the eigenvalue equation are then easily found. For the choice $\omega_1 = 1.0$, $\omega_2 = 2.0$, $\omega_3 = 3.0$, one gets the four lowest eigenstates at energies $E = 0.83, 1, 94, 3.29, 3.85$. There is no degeneracy since all ω's are different.

In the above two examples, exact analytic forms for the solutions $\psi_i(x_i)$ were available. Even if this is not the case, it is easy to use numerical Runge-Kutta techniques applied to each prong.

5 Supersymmetric Quantum Mechanics on Multi-Pronged Domains

Given any one-dimensional potential $V(x)$, the powerful techniques of supersymmetric quantum mechanics can be used to generate a partner potential $\tilde{V}(x)$ with the same eigenvalues [10, 11]. This property has been extensively used to get a deeper understanding of exactly solvable potentials and for generating improved approximation methods (SWKB method [14], large N method [15], etc.) for determining eigenvalues. Supersymmetric quantum mechanics provides an elegant formalism which includes and goes substantially beyond the method of factorization previously applied to some potentials [16]. In this section, we show how the ideas of supersymmetric quantum mechanics can be applied to a given N-prong system $V(\vec{x})$ in order to generate solutions for a new N-prong potential $\tilde{V}(\vec{x})$.

As in the previous section, consider a scalar N-prong potential $V(\vec{x})$ composed of potentials $V_i(x_i)$ on prong i $(i = 1, 2, ..., N)$. Its eigenvalues and eigenfunctions are then given by Eqs. (13) and (14) respectively. The un-normalized ground state wave function is

$$\psi^{(0)}(\vec{x}, E_0) \propto \left\{ \frac{\phi_1^{(0)}(x_1, E_0)}{\phi_1^{(0)}(0, E_0)}, \cdots, \frac{\phi_N^{(0)}(x_N, E_0)}{\phi_N^{(0)}(0, E_0)} \right\} . \tag{22}$$

This can be used to define the superpotential $W(\vec{x})$ whose value on prong i is

$$W_i(x_i) = -\frac{\phi_i^{(0)'}(x_i)}{\phi_i^{(0)}(x_i)} . \tag{23}$$

It is easy to check that

$$V_i(x_i) = W_i^2(x_i) - W_i'(x_i) + E_0 . \tag{24}$$

The supersymmetric partner potential is given by

$$\tilde{V}_i(x_i) = W_i^2(x_i) + W_i'(x_i) + E_0 . \tag{25}$$

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The potentials $V_i(x_i)$ taken together make up the full N-prong potential $\tilde{V}(\vec{x})$. From supersymmetric quantum mechanics, we know that the solution of the Schrödinger equation for potential $\tilde{V}_i(x_i)$ and energy E is given by [11]

$$\tilde{\phi}_i(x_i, E) = a_i \left(-\frac{d}{dx_i} + \frac{\phi_i^{(0)'}(x_i)}{\phi_i^{(0)}(x_i)} \right) \phi_i(x_i, E), \tag{26}$$

where a_i are constants. Since $\phi_i(x_i, E)$ vanishes at the prong ends, so do all the $\tilde{\phi}_i(x_i, E)$. At the vertex, one wants

$$\tilde{\phi}_i(0, E) = \tilde{\phi}_j(0, E) \quad \text{and} \quad \sum_i \tilde{\phi}'_i(0, E) = 0. \tag{27}$$

This gives an eigenvalue condition similar to Eq. (13).

$$\det \begin{vmatrix} \tilde{\phi}_1(0, E) & 0 & \cdots & 0 & 1 \\ 0 & \tilde{\phi}_2(0, E) & \cdots & 0 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \tilde{\phi}_N(0, E) & 1 \\ \tilde{\phi}'_1(0, E) & \tilde{\phi}'_2(0, E) & \cdots & \tilde{\phi}'_N(0, E) & 0 \end{vmatrix} = 0. \tag{28}$$

In general, the eigenvalues obtained from Eq. (28) will be different from those coming from Eq. (13). However, there are two important situations where the *same* eigenvalues result. This happens for the special case of two prongs ($N = 2$) and for the case of all identical prongs. These results can be understood from a physical viewpoint, since $N = 2$ is the standard one-dimensional situation treated in supersymmetric quantum mechanics, and furthermore, as we have seen in Sec. 3, these are the same eigenvalues for the identical prong case. Thus we see the machinery of supersymmetric quantum mechanics can be immediately applied to get eigenvalues and eigenfunctions of the partner potential $\tilde{V}(\vec{x})$ for the identical prong case. The equality of eigenvalues from Eqs. (13) and (28) can also be established from a mathematical viewpoint using Eqs. (26) and (27).

6   Perturbation Theory

Having solved a multi-prong problem, it is of interest to see the influence of a small perturbation of the potential in one of the prongs, say, on the energy levels. Here, for simplicity, we will only deal with an unperturbed system with three identical prongs. Consider an unperturbed system with potential $V_i(x_i) = V_0(x_i)$ on each prong. Its completely symmetric ground state

$$\psi^{(0)}(\vec{1}, \vec{x}) \equiv \frac{1}{\sqrt{3}} \left\{ \psi^{(0)}(x_1), \psi^{(0)}(x_2), \psi^{(0)}(x_3) \right\}, \tag{29}$$

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is described in terms of $\psi^{(0)}(x)$; the ground state eigenfunction of a two-prong system with the same potential $V_0(x_i)$ in each prong. We have chosen $\psi^{(0)}(x)$ to be normalized to unity over one prong, so that the wave function in Eq. (29) is properly normalized. The eigenenergy of this state is also the same as the ground state energy of the two-prong system. The first excited state for this three-prong system is doubly degenerate. We choose the following two orthonormal states as our base states:

$$\psi^{(1)}(\vec{a}_1; \vec{x}) \equiv \frac{1}{\sqrt{2}} \left\{ \psi^{(1)}(x_1), 0, -\psi^{(1)}(x_3) \right\}, \tag{30}$$

and

$$\psi^{(1)}(\vec{a}_2; \vec{x}) \equiv \frac{1}{\sqrt{6}} \left\{ \psi^{(1)}(x_1), -2\psi^{(1)}(x_2), \psi^{(1)}(x_3) \right\}. \tag{31}$$

Both of these states have exactly the same energy as the first excited state of the two-prong system.

Now, let us include a perturbation V_I that is nonvanishing in only one prong. The energy of the system is shifted and the degeneracy of the states of Eqs. (30) and (31) is lifted. Interestingly, only one of the eigenvalues of the first excited state changes, while the other one remains the same, thus causing the split. We compute here such shifts for the ground state as well the first excited state of the system. For numerical concreteness only, in the following example, we use the harmonic oscillator potential for V_0 and αx_1^3 as the perturbation potential V_I on prong 1. Thus the unperturbed potential is

$$V_0(x_i) = \frac{1}{4}\omega^2 x_i^2 . \tag{32}$$

Following the usual route, the shift of the ground state can be computed. The first order shift, δE_0, is given by

$$\delta E_0 = \int dx_1 \psi_1^{(0)*}(x_1) V_I \psi_1^{(0)}(x_1) = \frac{4\alpha}{3\sqrt{2\pi\omega^3}} . \tag{33}$$

The next energy level has two fold degeneracy, and hence it is necessary to use the formalism of degenerate perturbation theory. We compute the matrix elements of the perturbing potential V_I using $\psi^{(1)}(\vec{a_1}; \vec{x})$ and $\psi^{(1)}(\vec{a_2}; \vec{x})$ as basis vectors. For this situation, the matrix V_I is explicitly given by

$$V_I = \begin{pmatrix} \gamma & \frac{1}{\sqrt{3}}\gamma \\ \frac{1}{\sqrt{3}}\gamma & \frac{1}{3}\gamma \end{pmatrix},$$

where, $\gamma = \frac{8\alpha}{\sqrt{2\pi\omega^3}}$. Diagonalizing the corresponding Hamiltonian, we find the eigenvalues to be 0 and $\frac{4}{3}\gamma$. Hence, at this level degeneracy is lifted and the difference in energy is $\frac{4}{3}\gamma$. The two eigenfunctions are given by

$$\frac{1}{2}\psi^{(1)}(\vec{a_1}; \vec{x}) - \frac{\sqrt{3}}{2}\psi^{(1)}(\vec{a_2}; \vec{x}) = \frac{1}{\sqrt{2}}\{0, \psi^{(1)}(x_2), -\psi^{(1)}(x_3)\} \tag{34}$$

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and

$$\frac{1}{2}\psi^{(1)}(\vec{a_1};\vec{x}) + \frac{\sqrt{3}}{2}\psi^{(1)}(\vec{a_2};\vec{x}) = \frac{1}{\sqrt{6}}\left\{2\psi^{(1)}(x_1), -\psi^{(1)}(x_2), \psi^{(1)}(x_3)\right\}. \tag{35}$$

It is interesting to note that the first state has the same energy as the unperturbed system. Thus we see that, since the perturbation was added to just one prong, it does not break the symmetry completely and one of the eigenenergies remains unchanged.

7 Scattering in a Multi-Prong System

The scattering of an incident plane wave off the vertex in a multi-prong system offers scenarios that are different from the usual scattering in one-dimensional problems. As a first example, consider a plane wave with energy E moving along Prong 1, incident upon the vertex of an N-prong system with a constant potential $V = 0$ on all prongs. This is a trivial example in the two-prong case leading to 100% transmission, and zero reflection. For more than two prongs, it is not a priori intuitively clear whether full transmission will occur or not. The wave function on prong 1 is given by

$$\psi_1(x_1) = \exp(-ikx_1) + r\exp(ikx_1), \tag{36}$$

where k is the momentum of the incident plane wave $(E = k^2)$. The wave function on all the other prongs $(i \neq 1)$ only consists of outgoing waves and is given by

$$\psi_i(x_i) = t_i \exp(ikx_i). \tag{37}$$

Imposition of the boundary conditions (2) and (3) on these wave functions at the vertex relates the reflected amplitude r to the transmission amplitudes t_i. Their explicit relationship is given by

$$(1 + r) = t_i \quad (i = 2, \cdots, N), \qquad r - 1 + \sum_{i=2}^{N} t_i = 0. \tag{38}$$

The solution is
$$r = \frac{2 - N}{N}, \quad t_i = \frac{2}{N} \quad (i = 2, \cdots, N). \tag{39}$$

The reflection and transmission coefficients are then given by

$$\mathcal{R} = \frac{(2 - N)^2}{N^2}, \qquad \mathcal{T}_i = \frac{4}{N^2} \quad (i = 2, \cdots, N).$$

Clearly, one has $\mathcal{R} + \sum_{i=2}^{N} \mathcal{T}_i = 1$, and the probability current is conserved. For $N = 2$, the reflection coefficient vanishes, as expected. However, for $N > 2$ there is always a finite amount of reflection. For example, for a three-prong potential, the reflection coefficient is $\mathcal{R} = \frac{1}{9}$. In fact $\mathcal{R}$ increases continuously with N, and approaches unity for large number of prongs, N. (This increase of $\mathcal{R}$ with N is shown in Fig. 7 by the curve labeled 1.) Thus

one has the curious result that an incident wave, when given a large choice of scattering paths, in fact prefers to be reflected back on its initial prong!

Now let us consider a more general case where the incident wave is on a prong with potential zero, whereas the remaining $(N-1)$ prongs are at constant potential V_0. Let the energy of the wave be $E > V_0$. Let us define $k^2 = E$, $k'^2 = E - V_0$ and the parameter $\zeta = \frac{k'}{k} = \sqrt{1 - \frac{V_0}{E}}$. The wave function on prong 1 is again given by $\psi_1(x_1) = \exp(-ikx_1) + r\exp(ikx_1)$, and wave functions along remaining $(N-1)$ prongs are given by $\psi_i(x_i) = t_i \exp(ik'x_i)$ $(i = 2, \cdots, N)$. Boundary conditions (2) and (3) applied at the vertex yield

$$(1 + r) = t_2 = t_3 = \cdots = t_N \equiv t, \quad -1 + r + (N-1)\zeta = 0. \tag{40}$$

Solving these equations, we find $r = \frac{1 - \zeta(N-1)}{1 + \zeta(N-1)}$ and $t = \frac{2}{1 + \zeta(N-1)}$. One can show that r and t obey $1 = r^2 + \zeta(N-1)t^2$, as required by conservation of probability. The reflection coefficient $\mathcal{R}$, which is now given by

$$\mathcal{R} = \left[\frac{1 - \zeta(N-1)}{1 + \zeta(N-1)}\right]^2,$$

has a rather interesting behavior as a function of ζ and N. In Fig. 7, we have plotted $\mathcal{R}$ as a function of N for several values of ζ. $\mathcal{R}$ vanishes for $\zeta = \frac{1}{(N-1)}$. Thus for some special values of $\frac{V_0}{E}$, we have complete transmission through the vertex. As N increases, $\mathcal{R}$ slowly approaches unity for all ζ.

As yet another interesting example we consider scattering from a T-stub with N open ended prongs and one prong of finite length l (the stub). We consider a case where potentials along all the prongs are zero. A plane wave of momentum k (energy $E = k^2$) incident upon the vertex from prong 1, sets up stationary waves in the stub, and outgoing plane waves in the remaining $N-1$ prongs. The wave function on prong 1 is given by $\exp(-ikx_1) + r\exp(ikx_1)$. Along the stub, the wave function is given by $A\sin k(l - x_s)$, where x_s is the coordinate along the stub and A is a constant. Wave functions along all other prongs are given by $t\exp(ikx_j)$, $(j = 2, \cdots, N)$. From boundary conditions (2) and (3), we get:

$$1 + r = A\sin kl = t; \quad ik(r - 1) + ik(N-1)t - Ak\cos kl = 0.$$

Solving them, we get

$$r = \frac{-(N-2) - \cot kl}{N + i\cot kl}, \quad t = \frac{2}{N + i\cot kl}. \tag{41}$$

The reflection coefficient is given by $\mathcal{R} \equiv |r|^2 = \frac{\cot^2 kl + (N-2)^2}{\cot^2 kl + N^2}$. Clearly, from the above expression, one gets $|r|^2 + (N-1)|t|^2 = 1$. Interestingly, we find that for $kl = n\pi$, i.e. energy $E = \frac{n^2\pi^2}{l^2}$, the reflection coefficient $\mathcal{R} \to 1$. At these energies, a standing wave is set up jointly in the stub and in prong 1, and the particle never enters the other prongs.

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Let us generalize the discussion of scattering to an N-prong vertex with identical potentials $V(x_i)$ on all prongs. Define two linearly independent solutions $f(x_i)$ and $g(x_i)$ of the Schrödinger equation using asymptotic boundary conditions as $x_i \to \infty$:

$$f(x_i) \to \exp(-ikx_i), \quad g(x_i) \to \exp(ikx_i). \tag{42}$$

The vertex conditions are $f(0) + r\, g(0) = t\, g(0)$, and $f'(0) + r\, g'(0) + (N-1)\, t\, g'(0) = 0$. The solution is

$$r = -\frac{1}{N}\frac{f'(0)}{g'(0)} - \frac{(N-1)}{N}\frac{f(0)}{g(0)}, \quad t = -\frac{1}{N}\frac{f'(0)}{g'(0)} + \frac{1}{N}\frac{f(0)}{g(0)}. \tag{43}$$

Conservation of probability requires $|r|^2 + (N-1)|t|^2 = 1$ for all values of N. This is satisfied provided $\left|\frac{f(0)}{g(0)}\right| = \left|\frac{f'(0)}{g'(0)}\right| = 1$. Here again, we see that as the number of prongs $N \to \infty$, the reflection coefficient $\mathcal{R} = |r|^2$ approaches unity.

Having analyzed scattering on a general identical N-prong domain, we now investigate whether we can extract any further information using supersymmetry. In one dimensional quantum mechanics, supersymmetry relates reflection and transmission coefficients of one potential with those of its supersymmetric partner potential[12]. We find that this relationship holds also for the case of N identical prongs.

We have just seen that for the potential $V(x_i)$ on all prongs, the scattering is described by Eq. (43). The partner potential is given by Eq. (25). Let $\tilde{f}$ and $\tilde{g}$ be the solutions of the partner potential which behave like $\tilde{f}(x_i) \to \exp(-ikx_i)$, $\tilde{g}(x_i) \to \exp(ikx_i)$, as $x_i \to \infty$. Functions $\tilde{f}$ and $\tilde{g}$ are given by

$$\tilde{f}(x_i) = \frac{1}{W_\infty + ik}\left\{\frac{-d}{dx_i} + W(x_i)\right\}f(x_i), \quad \tilde{g}(x_i) = \frac{1}{W_\infty - ik}\left\{\frac{-d}{dx_i} + W(x_i)\right\}g(x_i), \tag{44}$$

where W_∞ is the value of the superpotential at infinity. From these expressions, we can show that $\frac{\tilde{f}(0)}{\tilde{g}(0)} = \left(\frac{W_\infty - ik}{W_\infty + ik}\right)\frac{f'(0)}{g'(0)}$, and $\frac{\tilde{f}'(0)}{\tilde{g}'(0)} = \left(\frac{W_\infty - ik}{W_\infty + ik}\right)\frac{f(0)}{g(0)}$. The reflection and transmission amplitudes for the potential $\tilde{V}(x_i)$ are given by

$$\tilde{r} = \left(\frac{W_\infty - ik}{W_\infty + ik}\right)\left(-\frac{1}{N}\frac{f(0)}{g(0)} - \frac{(N-1)}{N}\frac{f'(0)}{g'(0)}\right), \quad \tilde{t} = \left(\frac{W_\infty - ik}{W_\infty + ik}\right)\left(-\frac{1}{N}\frac{f(0)}{g(0)} + \frac{1}{N}\frac{f'(0)}{g'(0)}\right).$$

Note, for the special case of two-prongs, these equations agree with reference [12]. We find the following relationship among r, t and their partners $\tilde{r}$ and $\tilde{t}$:

$$\tilde{r} = \left(\frac{W_\infty - ik}{W_\infty + ik}\right)r, \quad \tilde{t} = -\left(\frac{W_\infty - ik}{W_\infty + ik}\right)t.$$

Note that $|\tilde{r}|^2 = |r|^2$, which implies that for the identical prong case, the reflection coefficients are the same for any potential and its supersymmetric partner.

Now, using an explicit example, we provide a concrete demonstration of the usefulness of the machinery that we just developed. For the sake of simplicity, we shall work with

the example of a free particle in an N-prong system for which reflection and transmission coefficients are given by Eq. (39). The superpotential for this problem is given by $W(x) = \tanh x$. This superpotential generates two distinct potentials related by supersymmetry. They are $V(x) \equiv W^2(x) - W'(x) = 1 - 2\,\text{sech}^2 x$ and $\tilde{V}(x) \equiv W^2(x) + W'(x) = 1$. The first potential holds one bound state at energy $E = 0$.

Amplitudes $\tilde{r}$ and $\tilde{t}$ for the free particle system are $\frac{2-N}{N}$ and $\frac{2}{N}$, respectively. Amplitudes r and t for the potential $V(x) = 1 - 2\,\text{sech}^2 x$, are then given by

$$r = \left(\frac{1+ik}{1-ik}\right)\left[-\frac{1}{N}\frac{\tilde{f}(0)}{\tilde{g}(0)} - \frac{N-1}{N}\frac{\tilde{f}'(0)}{\tilde{g}'(0)}\right] = \left(\frac{1+ik}{1-ik}\right)\left(\frac{N-2}{N}\right) \ , \quad t = -\left(\frac{1+ik}{1-ik}\right)\left(\frac{2}{N}\right) \ .$$

Thus, knowing the reflection and transmission amplitudes $\tilde{r}$ and $\tilde{t}$ for the free particle system, supersymmetry allows us to determine amplitudes r and t for the rather non-trivial potential $V(x) = 1 - 2\,\text{sech}^2 x$. Note that coefficients $\tilde{\mathcal{R}} = |\tilde{r}|^2$ and $\tilde{\mathcal{T}} = |\tilde{t}|^2$ are equal to coefficients for the partner potential, $\mathcal{R} = |r|^2$ and $\mathcal{T} = |t|^2$ respectively.

8 Tunneling in a Multi-Prong System

Tunneling is another sector of great interest that shows a marked difference from the two-prong case. Consider a system with N identical prongs, each one having one minimum (well-like structure, similar to the two-prong case shown in Fig. 2). In this section, we look at the tunneling of a localized wave packet from one prong to the others. For simplicity, we start with a three-prong system. The generalization to higher number of prongs is then straightforward. Let $\Phi_i(\vec{x})$ denote a wave packet localized in the i-th prong. Such a packet can be approximated by the following linear combination of eigenstates of the system:

$$\Phi_i(\vec{x}) = \frac{1}{3}\left(\psi^{(0)}(\vec{1}, \vec{x}) + \psi^{(1)}(\vec{a_i}, \vec{x})\right), \tag{45}$$

where $\vec{1} \equiv (1,1,1)$ and $\vec{a_i}$ denotes a three-tuple that has a 2 in the i-th position and -1 in the rest. For example, $\vec{a_1}$ is given by the three-tuple $(2, -1, -1)$.

The amplitude for the localized state $\Phi_i(\vec{x})$ to tunnel out of the prong 1 is given by:

$$\mathcal{A}(t) = \langle\psi_f|\exp(-iHt)|\psi_i\rangle \tag{46}$$

where

$$\psi_i = \Phi_1(\vec{x}) \quad \text{and} \quad \psi_f = \frac{1}{\sqrt{\alpha_2^2 + \alpha_3^2}}\left(\alpha_2\Phi_2(\vec{x}) + \alpha_3\Phi_3(\vec{x})\right);$$

$\alpha_2 + \alpha_3 = 1$. Substituting explicit expressions for ψ_f and ψ_i in Eq. (46), we get

$$\mathcal{A}_{\alpha_2\alpha_3} = \frac{1}{3\sqrt{\alpha_2^2 + \alpha_3^2}}\int d\vec{x}\,[\alpha_2\Phi_2(\vec{x}) + \alpha_3\Phi_3(\vec{x})]\left[e^{-iE_0 t}\psi^{(0)}(\vec{1}, \vec{x}) + e^{-iE_1 t}\psi^{(1)}(\vec{a_1}, \vec{x})\right]$$

$$
= \frac{1}{9\sqrt{\alpha_2^2 + \alpha_3^2}} \int d\vec{x} \left[\alpha_2 \psi^{(1)}(\vec{a_2}, \vec{x}) + \alpha_3 \psi^{(1)}(\vec{a_3}, \vec{x}) + \psi^{(0)}(\vec{1}, \vec{x}) \right]
$$

$$
\times \left[e^{-iE_0 t} \psi^{(0)}(\vec{1}, \vec{x}) + e^{-iE_1 t} \psi^{(1)}(\vec{a_1}, \vec{x}) \right]
$$

$$
= \frac{e^{-iE_1 t}}{9\sqrt{\alpha_2^2 + \alpha_3^2}} \left[\alpha_2 \vec{a_1} \cdot \vec{a_2} + \alpha_3 \vec{a_1} \cdot \vec{a_3} + 3 e^{i\delta E t} \right], \quad \delta E \equiv E_1 - E_0 ,
$$

where we have used $\int d\vec{x}\, \psi^{(1)}(\vec{a_1}, \vec{x}) \psi^{(1)}(\vec{a_2}, \vec{x}) = \vec{a_1} \cdot \vec{a_2} = \sum_i (\vec{a_1})_i (\vec{a_2})_i$. Substituting $\vec{a_1} \cdot \vec{a_3} = \vec{a_1} \cdot \vec{a_2} = -3$, we get

$$
\mathcal{A}_{\alpha_2 \alpha_3} = \frac{e^{-iE_1 t}}{3\sqrt{\alpha_2^2 + \alpha_3^2}} \left[-1 + e^{i\delta E t} \right].
$$

The probability of tunneling is then given by

$$
\mathcal{P}_{\alpha_2 \alpha_3} = |\mathcal{A}_{\alpha_2 \alpha_3}|^2 = \frac{4}{9(\alpha_2^2 + \alpha_3^2)} \sin^2 \left(\frac{\delta E\, t}{2} \right);
$$

The maximum value for this tunneling probability is $\frac{8}{9}$, and that occurs when $\alpha_2 = \alpha_3 = \frac{1}{2}$. It is worth noting here that unlike the two-prong case, the wave packet never completely goes out of the prong 1.

When we generalize this situation to N-prong case, we find that the probability amplitude for a wave packet $\Phi_1(\vec{x})$ of tunneling out of prong 1 is given by

$$
\mathcal{A}_{\vec{\alpha}} = \frac{2ie^{-i(E_0 + \delta E/2)} \sin(\delta E t/2)}{N \left(\sum_{j=2}^{N} \alpha_j^2 \right)^{1/2}} ,
$$

where $\vec{\alpha} \equiv \{\alpha_2, \alpha_3, \cdots, \alpha_N\}$ with $\frac{\alpha_i}{\left(\sum_{j=2}^{N} \alpha_j^2 \right)}$ giving the amplitude for finding the packet in the i-th prong $\left(\frac{\sum_i \alpha_i \Phi_i(\vec{x})}{\left(\sum_{j=2}^{N} \alpha_j^2 \right)} \right)$. This amplitude is maximum for the symmetric case for which all α_i are equal. In this case the probability of tunneling is given by

$$
\mathcal{P} = \frac{4(N-1)}{N^2} \sin^2 \left(\frac{\delta E\, t}{2} \right).
$$

Thus the probability of tunneling decreases when more alternatives are available, and goes to zero as the number of prongs becomes very large. This is similar to the scattering situation discussed in Sec. 7.

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Figure Captions

Figure 1: Schematic diagram of an N-prong potential. The position on any prong i is specified by the positive coordinate x_i, with $x_i = 0$ being the vertex. The angles between prongs play no role in any computations.

Figure 2: Example of a potential with two identical prongs. The positive variables x_1, x_2 together span the real x-axis $(-\infty < x < \infty)$.

Figure 3: The three lowest eigenfunctions [Eqs. (8), (9) and (10)] for a three-prong harmonic oscillator potential $V(x_i) = \frac{1}{2}m\omega^2 x_i^2$. The ground state $\psi^{(0)}$ is non-degenerate, whereas the first excited state $\psi^{(1)}$ is doubly degenerate.

Figure 4: The three lowest energy eigenstates (un-normalized) for a three-prong free particle potential. The solid lines correspond to $l_1 = l_3 = 1$, $l_2 = 0.8$. The dashed lines correspond to the identical prong case $l_1 = l_2 = l_3 = 1$.

Figure 5: The eigenvalues k (energy $E = k^2$) for a three-prong free particle potential as a function of l_2, the length of prong 2. The lengths of the other two prongs are kept fixed and *equal* $l_1 = l_3 = 1$. Note the pattern of degeneracy and level crossings. The curves are labeled by the number of nodes in the wave function (in all prongs).

Figure 6: The eigenvalues k (energy $E = k^2$) for a three-prong free particle potential as a function of l_2, the length of prong 2. The lengths of the other two prongs are kept fixed and *unequal* $l_1 = 1$, $l_3 = \sqrt{2}$. There is now no degeneracy. The curves are labeled by the number of nodes in the wave function (in all prongs).

Figure 7: A plot of the reflection coefficient $\mathcal{R}$ versus the number of prongs N. The incident wave with energy E is on prong 1 with zero potential. All other prongs are taken to be at a constant potential V_0. The curves are labeled by the parameter $\zeta \equiv \sqrt{1 - \frac{V_0}{E}}$.

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