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An Investigation of Montmort's "Probleme de Recontres" and Generalizations

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SUMMARY

I have investigated a problem which may be phrased in many ways, such as finding the probability of answering a given number of questions correctly on a randomly-completed matching test which may have a number of extra "dud" answers. I have determined such probabilities, the average number of correct answers, and other allied results. I have also investigated a related problem involving the number of ways of choosing a different element from each of a certain collection of sets.

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INTRODUCTION

I began my project work about a year ago when I attacked a problem which I later found to be "le problème de recontres," first proposed by P. R. de Montmort.[1] I present here the results of my independent work on this problem and on some generalizations of "le problème de recontres," which I later made. Some of my later results, and especially my methods, may be original.

The basic "problème de recontres" may be phrased in many interesting ways. In its original form, the problem dealt with finding the probability that when n balls, numbered 1 to n, are drawn in a random order, no ball appears in the order indicated by its label.[2] Often considered is the number of ways to place n rooks on an n×n chessboard so that no rook is on the white diagonal or can take another.[3,4,5] I have applied the problem to the random completion of a matching test and, in one of my generalizations, to choosing systems of distinct representatives.

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[1] Warren Weaver, Lady Luck (New York: Anchor Books, Doubleday & Company, Inc., 1963), p. 135.

[2] Ibid.

[3] William Feller, An Introduction to Probability Theory and its Applications (New York: John Wiley & Sons, Inc., 1957) p. 101.

[4] John Riordan, An Introduction to Combinatorial Analysis (New York: John Wiley & Sons, Inc., 1958), pp. 164-65.

[5] Herbert John Ryser, Combinatorial Mathematics (The Mathematical Association of America, Inc., distributed by John Wiley & Sons, Inc., 1963), p. 24.

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"Le Problème de Recontres" and Closely Related Problems

Suppose a student takes a test which requires the matching of x answers with x questions. If he uses each answer once but otherwise answers randomly, what is the probability that he will answer exactly y of the questions correctly? Let us write this probability as M_x^y.

Considering first M_x^0, the true "problème de recontres,"

$$M_x^0 = \left(\frac{x-1}{x}\right)\left(\frac{1}{x-1}\cdot M_{x-2}^0 + M_{x-1}^0\right) \qquad (1)$$

for $x \geq 3$, by the following argument. M_x^0 is the product of the probabilities of (I) using a wrong answer for the first question answered and (II) answering the other questions incorrectly, given that the first answer is wrong. The probability of event (I) is $\frac{x-1}{x}$, and if this event occurs, one question (call it question two) has lost its answer. The probability of event (II) is the sum of the probabilites of (a) answering question two with the proper answer for the first question and continuing to success, and (b) answering question two with any other answer and continuing to success. The probability of answering question two with the first question's proper answer is $\frac{1}{x-1}$, and there are then x-2 answers remaining to be incorrectly placed with their x-2 questions, so that the probability of event (a) is $\frac{1}{x-1}\cdot M_{x-2}^0$. The probability of event (b) is M_{x-1}^0 since we still have x-2 answers to be incorrectly placed with their x-2 questions, while the answer to the first question plays the role of question two's answer. Thus, the probability of event (II) is $\frac{1}{x-1}\cdot M_{x-2}^0 + M_{x-1}^0$,

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and (1) is verified since the probability of event (I) is $\frac{x-1}{x}$. M_1^0 and M_2^0 are readily seen to be 1 and ½, respectively, so that successive application of (1) will yield the value of M_x^0 for any x. (See Appendix 1 for a table of values of M_x^0.)

Considering the more general case of M_x^y, there are $\binom{x}{y}$ ways to choose the y questions which will be answered correctly, where $\binom{x}{y}$ refers to the number of combinations of x objects taken y at a time. Then, since there are (x-y)! total ways to match the remaining questions and answers, there are $M_{x-y}^0 \cdot (x-y)!$ ways to do so without answering any questions correctly. Thus, out of a total of x! possible arrangements, there are $\binom{x}{y} \cdot M_{x-y}^0 \cdot (x-y)!$ ways to place the x answers so that exactly y are correct. Therefore, $M_x^y = \binom{x}{y} \cdot M_{x-y}^0 \cdot (x-y)!/x!$, and simplifying,

$$M_x^y = M_{x-y}^0/y! \qquad (2)$$

This allows us to determine M_x^y for any x and y once M_{x-y}^0 has been calculated by (1).

Now notice that (2) yields $M_x^x = M_0^0/x!$ and that M_x^x should equal $\frac{1}{x!}$. Also, if we use $M_0^0 = 1$ in (1) along with $M_1^0 = 0$, we get the proper result, ½, for M_2^0. It seems quite reasonable to define $M_0^0 = 1$ and expand the domain of (1) to include x=2.

After working out the above results on my own, I found the following expression for $M_x^0 (x \geq 2)$ in print: $\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \ldots + (-1)^n/n!$.[6] I noticed this can be expanded to

$$M_x^0 = \sum_{n=0}^{x} (-1)^n/n! \text{ for all integers } x \geq 0 \qquad (3)$$

[6] Weaver, p. 136

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and I proved (3) by complete induction. The formula is clear for x=0 and x=1 so we assume (3) for all non-negative integers $x \leq k$, where k is a positive integer. Then,

$$M^0_{k+1} = \frac{k}{k+1}\left(\frac{1}{k} \cdot \sum_{n=0}^{k-1} (-1)^n/n! + \sum_{n=0}^{k} (-1)^n/n!\right) \quad \text{(by (1) and the induction hypothesis)}$$

$$= \frac{1}{k+1} \cdot \sum_{n=0}^{k-1} (-1)^n/n! + \frac{k}{k+1}\left(\sum_{n=0}^{k-1} (-1)^n/n! + (-1)^k/k!\right)$$

$$= \sum_{n=0}^{k-1} (-1)^n/n! + \frac{k}{k+1}(-1)^k/k!$$

$$= \sum_{n=0}^{k-1} (-1)^n/n! + \frac{k+1}{k+1}(-1)^k/k! - \frac{1}{k+1}(-1)^k/k!$$

$$= \sum_{n=0}^{k+1} (-1)^n/n!$$

and we are done.

(3) is usually derived using the principle of inclusion and exclusion.[7,8,9] Though my work was performed independently, the approach used up to this point is similar to one used by Euler plus additional steps suggested by Riordan and Ryser.[10,11]

Interestingly, (3) shows that M^0_x approaches $\frac{1}{e}$ as x gets large (which it actually does very quickly), since $e^t = \sum_{n=0}^{\infty} t^n/n!$. Also interesting is that regardless of the value of x, the expectation, or average number correct, in random matching of the x questions and x answers of a matching test is always 1.

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[7] Feller, pp. 90-91.

[8] Riordan, p.58.

[9] Ryser, p. 23.

[10] Riordan, p. 60.

[11] Ryser, pp. 30-31.

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I determined this as follows. The expectation will be the sum of the products formed by multiplying each possible number correct by the probability of getting that number correct. Using (2),

$$\sum_{y=0}^{x} y\cdot M_x^y = \sum_{y=1}^{x} y\cdot M_x^y = \sum_{y=1}^{x} y\cdot M_{x-y}^0/y! = \sum_{y=1}^{x} M_{x-y}^0/(y-1)! =$$

$$\sum_{y=0}^{x-1} M_{x-1-y}^0/y! = \sum_{y=0}^{x-1} M_{x-1}^y = 1 \text{ since } \sum_{y=0}^{x-1} M_{x-1}^y \text{ is just the probability}$$

of having somewhere from 0 to x-1 correct answers on a test involving x-1 questions.

The most likely number of correct answers may also be calculated. It is clear from (3) that for $x>2$, $M_{x-1}^0 \geq M_3^0 = \frac{1}{3}$; and if $y=x$, then $y>2$, and $y!M_{x-1}^0 \geq 6\cdot\frac{1}{3}>M_{x-y}^0$. If $x>2$ but $y\neq x$, then $M_{x-y}^0 \leq M_2 = \frac{1}{2}$, and as long as $y>1$, $y!M_{x-1}^0 \geq 2\cdot\frac{1}{3}> M_{x-y}^0$. Thus, for $x>2$ and $y>1$, $y!M_{x-1}^0>M_{x-y}^0$ and $M_x^1>M_x^y$. Then, the highest probability for $x>2$ is M_x^0 or M_x^1, which is M_{x-1}^0. Again from (3), it is clear that M_x^0 is greater when x is even and M_{x-1}^0 is greater when x is odd. We can check the values of M_x^y for $x\leq 2$, and we see that $y=1$ is most likely for odd x, $y=0$ is most likely for even $x\neq 2$, and $y=0$ and $y=1$ are equally likely for $x=2$.

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Recontres with "Duds"

Now consider a matching test which has not only x questions and their x "good" answers but also z extra "dud" answers to make the guessing more difficult. Let ${}^{z}M_{x}^{y}$ be the probability of getting exactly y correct when the x questions are randomly answered with x of the x+z answers. (When z=0 we are dealing with the previous problem again and may write just M_{x}^{y} .) In doing this, any number of duds, n, from 0 to min $\{x-y,z\}$ may be used. Of $\binom{x+z}{x}$ total ways of choosing x of the x+z answers for use, there are $\binom{z}{n}\binom{x}{x-n} = \binom{z}{n}\binom{x}{n}$ ways to do so with n duds and x-n good answers used, so that the probability of this occurrence is $\binom{z}{n}\binom{x}{n}/\binom{x+z}{x}$. At this point, since n duds are being used in place of n of the good answers, n questions have no chance of being answered correctly, and the probability of answering y of the remaining x-n correctly is ${}^{n}M_{x-n}^{y}$. Thus,

$$ {}^{z}M_{x}^{y} = \sum_{n=0}^{\min\{x-y,z\}} {}^{n}M_{x-n}^{y} \cdot \binom{z}{n}\binom{x}{n}/\binom{x+z}{x} \qquad (4) $$

Furthermore,

$$ {}^{z}M_{x}^{y} = \binom{x}{y} \cdot {}^{z}M_{x-y}^{0} \cdot \frac{(x-y+z)!}{(x+z)!} \qquad (5) $$

by the following argument. There are $\binom{x}{y}$ ways to choose the questions which will be answered correctly. Once this is done, there are P_{x-y}^{x-y+z} (P_{k}^{n} denotes the number of permutations of n objects taken k at a time) ways to place answers with the remaining questions and ${}^{z}M_{x-y}^{0} \cdot P_{x-y}^{x-y+z}$ ways to do so without answering any correctly. Then, the total number of successes

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is $\binom{x}{y}\cdot {}^{z}M^{0}_{x-y}\cdot P^{x-y+z}_{x-y}$, while the total number of ways to answer the x questions with the x+z answers is P^{x+z}_{x} . Thus, ${}^{z}M^{y}_{x} = \binom{x}{y}\cdot {}^{z}M^{0}_{x-y}\cdot P^{x-y+z}_{x-y}/P^{x+z}_{x}$, yielding (5).

If we use (5) with y=0, we get simply ${}^{z}M^{y}_{x} = {}^{z}M^{y}_{x}$ and we must return to (4). Nevertheless, for $y \neq 0$, (5) is very convenient if we have first calculated ${}^{z}M^{0}_{x-y}$ by (4) or some other method.

From (4), some interesting formulas for ${}^{z}M^{0}_{x}$ for fixed values of x can be obtained. If we use $\frac{z(z-1)(z-2)\ldots(z-n+1)}{n!}$ for $\binom{z}{n}$, $\binom{z}{n}$ will be 0 when n>z, allowing us to sum from 0 to x in (4) even if z<x. First, using (4),

$$ {}^{z}M^{0}_{0} = M^{0}_{0} = 1 \qquad (6) $$

Next, ${}^{z}M^{0}_{1} = \frac{1}{z+1}\cdot M^{0}_{1} + \frac{z}{z+1}\cdot {}^{1}M^{0}_{0}$, and since $M^{0}_{1} = 0$ and ${}^{1}M^{0}_{0} = 1$,

$$ {}^{z}M^{0}_{1} = \frac{z}{z+1} \qquad (7) $$

$$ {}^{z}M^{0}_{2} = 2\cdot M^{0}_{2}/P^{z+2}_{2} + 4z\cdot {}^{1}M^{0}_{1}/P^{z+2}_{2} + z(z-1)\cdot {}^{2}M^{0}_{0}/P^{z+2}_{2} $$

$M^{0}_{2} = \frac{1}{2}$ by (3), ${}^{1}M^{0}_{1} = \frac{1}{2}$ by (7), and ${}^{2}M^{0}_{0} = 1$ by (6), yielding

$$ {}^{z}M^{0}_{2} = (z^{2}+z+1)/P^{z+2}_{2} \qquad (8) $$

Similar equations for x=3, x=4, and x=5 are included in Appendix 2.

Another formula for ${}^{z}M^{0}_{x}$ may be developed as follows. Let us reserve a certain place for each of the z dud answers as well as for each of the x good answers. As many as z of the questions may be put in their proper places as long as we only match duds with their dummy positions. There is a total of $\binom{x+z}{n}$ ways to have n matches and $\binom{z}{n}$ ways in which the matches

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involve only the duds. Thus, the probability that n matches will involve only the duds in $\binom{z}{n}/\binom{x+z}{n}$, and

${}^{z}M_x^0 = \sum_{n=0}^{z} M_{x+z}^n \cdot \binom{z}{n}/\binom{x+z}{n}$. Since $M_{x+z}^n = M_{x+z-n}^0/n!$,

$$ {}^{z}M_x^0 = \sum_{n=0}^{z} M_{x+z-n}^0 \cdot \binom{z}{n}/P_n^{x+z} \qquad (9)$$

This is useful for small values of z. For example,

${}^{1}M_x^0 = M_{x+1}^0 + M_x^0/(x+1)$, and by (1),

$$ {}^{1}M_x^0 = \frac{x+2}{x+1} \cdot M_{x+2}^0 \qquad (10)$$

Also by (9), ${}^{2}M_x^0 = M_{x+2}^0 + \frac{2}{x+2} \cdot M_{x+1}^0 + \frac{1}{(x+2)(x+1)} \cdot M_x^0$, which can be written as

$$ {}^{2}M_x^0 = M_{x+2}^0 + \frac{1}{x+2} \cdot M_{x+1}^0 + \frac{1}{x+2} \cdot M_{x+1}^0 + \frac{1}{(x+2)(x+1)} \cdot M_x^0 \qquad (11)$$

Now, we can apply (1) to the latter two terms to obtain

${}^{2}M_x^0 = M_{x+2}^0 + \frac{1}{x+2} \cdot M_{x+1}^0 + \frac{1}{x+1} \cdot M_{x+2}^0$, and

$$ {}^{2}M_x^0 = \frac{x+2}{x+1} \cdot M_{x+2}^0 + \frac{1}{x+2} \cdot M_{x+1}^0 \qquad (12)$$

Alternatively, we may apply (1) to the first two terms of (11) as well as to the last two, and obtain

$$ {}^{2}M_x^0 = \frac{x+3}{x+2} \cdot M_{x+3}^0 + \frac{1}{x+1} \cdot M_{x+2}^0 \qquad (13)$$

Similar formulas for z=3 and z=4 are included in Appendix 2. Such formulas are convenient if one has a table of values of M_x^0 as in Appendix 1.

In addition to the convenient formulas for fixed values of z or x, a nice general formula for ${}^{z}M_x^0$ similar to (3) can be obtained using the principle of inclusion and exclusion. Let

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p_n be the probability that a given n questions will be correctly answered. S_n will be p_n multiplied by the number of ways to choose n questions. Whereas there are $(x+z)!$ total ways to match the questions and answers, there are $(x+z-n)!$ ways to match the remaining questions and answers when n questions have been correctly answered. Thus $p_n = \frac{(x+z-n)!}{(x+z)!}$, and $S_n = \binom{x}{n}\frac{(x+z-n)!}{(x+z)!}$. By the principle of inclusion and exclusion ${}^{z}M_x^0 = \sum_{n=0}^{x} (-1)^n S_n$, so

$$ {}^{z}M_x^0 = \frac{x!}{(x+z)!} \sum_{n=0}^{x} (-1)^n \cdot \frac{(x+z-n)!}{n!(x-n)!} \qquad (14)$$

Thus, (3) is a special case of (14). (See Appendix 3 for a table of values of ${}^{z}M_x^0$.)

Notice that $\frac{x!}{(x+z)!}\cdot\frac{(x+z-n)!}{(x-n)!} = \frac{(x-n+z)(x-n+z-1)\ldots(x-n+1)}{(x+z)(x+z-1)\ldots(x+1)}$ is the quotient of two monic polynomials in x of degree z and has a limit of 1 as x goes to infinity. Thus, we can see from (14) that $\lim_{x\to\infty} {}^{z}M_x^0 = \lim_{x\to\infty} \sum_{n=0}^{x} (-1)^n/n! = \frac{1}{e}$ for any z. Also, $\lim_{z\to\infty} \frac{(x+z-n)!}{(x+z)!}$ is 0 whenever $n>0$, so $\lim_{z\to\infty} {}^{z}M_x^0 = \frac{x!}{(x+z)!}(-1)^0 \cdot \frac{(x+z-0)!}{0!(x-0)!} = 1$ as might be expected.

The expectation for a matching test with duds can be found just as it was without duds by using (5) instead of (2). This yields $\frac{x}{x+z}$ as does another simple argument. The probability of a given question being answered correctly is $\frac{1}{x+z}$, and since there are x questions, we expect an average of $\frac{x}{x+z}$ correctly answered questions.

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For ${}^{1}M_{x}^{y}$, the most likely event is $y=0$ when $x \neq 1$. This is seen as follows. By (5) and (10), ${}^{1}M_{x}^{y} = \frac{x-y+2}{x+1} \cdot M_{x-y+2}^{0}/y!$. $M_{x-y+2}^{0}/y!$ is at least as great for $y=0$ or $y=1$ as for any other value of y, so ${}^{1}M_{x}^{y}$ clearly is greatest for $y=0$ or $y=1$. It is greatest for $y=0$ when $\frac{x+2}{x+1} \cdot M_{x+2}^{0} > M_{x+1}^{0}$, which is true when $(x+2)(M_{x+1}^{0} + (-1)^{x+2}/(x+2)!) > (x+1)M_{x+1}^{0}$. This inequality can be written as $(-1)^{x+2}/(x+1)! > -M_{x+1}^{0}$ and is clearly true for even values of x. For odd $x>1$, we have $(-1)^{x+2}/(x+1)! \geq -\frac{1}{24} \geq \frac{1}{e} > -M_{x+1}$. It is easily checked that $y=0$ and $y=1$ are equally likely when $x=1$. For $z>1$, there is an even greater tendendy to have fewer correct answers. Thus $y=0$ is always most likely for $z>1$.

Systems of Distinct Representatives

"Le problème de recontres" may also be generalized by considering the number of ways to choose a system of distinct representatives (SDR's) from k sets of n elements ($n,k \geq 1$) determined as follows. The numbers from 1 to k are arranged in order in a circle, with the i'th set consisting of i and the next $n-1$ consecutive numbers in the circle. This number of SDR's will be denoted by A_k^n . For A_4^3, for instance, we are considering sets {1,2,3}, {2,3,4}, {3,4,1}, {4,1,2}. If $k<n$, numbers will occur more than once in a single set as in {1,2,1}, {2,1,2} ($n=3$, $k=2$). Nevertheless, we will consider that there are four different SDR's composed of a 1 from the first set and a 2 from the second set.

"Le problème de recontres" is involved whenever $n=k-1$. We

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are matching the numbers from 1 to k with the k sets, but each number is prohibited from a different one of the sets, so $M_k^0 \cdot k! = A_k^{k-1}$ for $k \geq 2$. A_k^{k-2} with $k \geq 3$ is a case of the "problème de ménages" for which the solution may be established by a tidy recurrence argument developed by I. Kaplansky.[12]

Let us now try fixing n. A_k^1 and A_k^2 are trivial; the former is always 1, and the latter is always 2. For A_k^3 and A_k^4, the following recursions hold:

$$A_k^3 = 2A_{k-1}^3 - A_{k-3}^3 \qquad (15)$$

$$A_k^4 = 2A_{k-1}^4 - A_{k-4}^4 \qquad (16)$$

My proof for (15) is as follows. A_k^3 can be broken down into two smaller problems where we have k-1 sets remaining to choose from after choosing a number from the first set and removing this number from the possible choices for the other sets. The first problem is B_{k-1}, where we have chosen one of the end numbers from the first set. We have the same problem regardless of which end number is chosen. The second problem is C_{k-1}, where we have chosen the middle number of the first set. We have $A_k^3 = 2B_{k-1} + C_{k-1}$ with $B_0=C_0=1$. In the B_k problem, there are just two possible choices remaining in the last set. Depending upon which we choose, there are C_{k-1} ways, or just one way to continue. By defining $C_k=0$ for $k<0$, we can write $B_k=C_{k-1}+1$ for $k \geq 0$. Similarly, $C_k=C_{k-1}+C_{k-2}$ for $k \geq 1$. (The C_{k-2} arises because one of the choices forces another.)

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[12] Ryser, pp. 32-35.

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Thus, for $k \geq 1$, $A_k^3 = 2(C_{k-1}+1) + C_{k-1} = C_{k-2}+2+C_k$.

Then, for $k \geq 4$, $2A_{k-1}^3 - A_{k-3}^3 = 2(C_{k-1}+C_{k-3}+2) - (C_{k-3}+C_{k-5}+2) = 2C_{k-1}+C_{k-3}-C_{k-5}+2 = 2C_{k-1}+C_{k-4}+2 = C_{k-1}+C_{k-2}+C_{k-3}+C_{k-4}+2 = C_k+C_{k-2}+2 = A_k^3$. Now, we must simply notice that $A_1^3 = 3$, $A_2^3 = 5$, and $A_3^3 = 6$ in order to make use of (15). A similar but more difficult process yields (16).

It is now true that for $1 \leq n \leq 4$, $A_k^n = 2A_{k-1}^n - A_{k-n}^n$, but this pattern does not continue for $n>4$. The recurrences get considerably more complicated. After finding my recursions, I discovered that researchers at the University of California, Los Alamos Scientific Laboratory, had considered an equivalent problem and devised a general method to find a recursion formula for A_k^n with any fixed value of n (although they do not define A_k^n for $k<n$).[13] Their recurrences for $n=3$ and $n=4$ are given in the forms $A_{k+2}^3 = A_{k+1}^3 + A_k^3 - 2$ and $A_{k+3}^4 = A_{k+2}^4 + A_{k+1}^4 + A_k^4 - 4$, which may be verified by my method. Their recurrences for $n=5$ and $n=6$ are included in Appendix 4, and all the recurrences actually work for $k<n$ according to my definition of A_k^n. Also included in Appendix 4 are tables of some of the values of A_k^n which I computed using my own recursions and/or the University of Chicago's DEC-20 computer. All my calculations agree with the more extensive tables of Metropolis, Stein, and Stein.[14]

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[13] N. Metropolis, M.L. Stein, and P.R. Stein, "Permanents of Cyclic (0,1) Matrices," Journal of Combinatorial Theory, 7 (December 1969), pp. 291-306.

[14] Ibid., pp. 315-317.

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I noticed early that A_k^n is even whenever n is even since there are equal numbers of SDR's starting with symmetrically located numbers in the first set. I conjecture that A_k^n is divisible by four when n is. I also suspect that for odd values of n, A_k^n is even if and only if the greatest common divisor of n and k is not 1. Though I have not yet devised proofs, these conjectures are supported by my numerical calculations and those of Metropolis, Stein, and Stein.

By the method I used for A_k^3, I also found simple recurrences for the number of SDR's of a collection of sets of 3 elements, where the elements do not runn all the way around in a circle. If we have k sets in which the i'th set just contains, i, i+1, and i+2, then the recursion is of the form $A_k = A_{k-1} + A_{k-2} + k+1$ or $A_k = 2A_{k-1} - A_{k-3} +1$. The first three numbers in the sequence are 3, 7 and 14. We may also consider the problem just mentioned with one change: the last set consists of k, k+1, and 1. Then the recursion is of the form $A_k = A_{k-1} + A_{k-2} + 2$ or $A_k = 2A_{k-1} - A_{k-3}$. The latter is the same recurrence as for A_k^3, but here the first three numbers of the sequence are 3, 6, and 11. By considering situations such as these for all n, we may develop a problem even broader than A_k^n.

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CONCLUSION

I have rediscovered some classic arguments relating to "le problème de recontres" and found multiple expressions for the more general probability of ${}^{z}M_{x}^{y}$, with some possibly original results. My approach to the problems dealing with systems of distinct representatives is different from that of Metropolis, Stein, and Stein, but my recurrences and numerical calculations involving A_k^n are corroborated by their work. I also have made other discoveries involving SDR's, as well as expectation, limits, and most likely event for a matching test, some of which might be new or at least be derived by novel methods. The results I have found may be used not only with matching tests and systems of distinct representatives, but also for many other problems of a similar nature.

As a result of my investigations, additional questions come to mind. I plan to attempt to determine the number of ways to choose an SDR from the sets described for A_k^n so that there are agreements in y places with the SDR 1,2,3,...,k. I also plan to investigate in more detail divisibility patterns for A_k^n and seek proofs or counterexamples for my conjectures. In addition, I would like to explore the problem I suggested involving a generalization of A_k^n.

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APPENDIX 1

Values of M_x^0

x	M_x^0	
0		1
1		0
2	$\frac{1}{2}$	.5
3	$\frac{1}{3}$	.3333333333
4	$\frac{3}{8}$	.375
5	$\frac{11}{30}$	.3666666667
6	$\frac{53}{144}$	.3680555556
7	$\frac{103}{280}$	.3678571429
8	$\frac{2119}{5760}$	.3678819444
9	$\frac{16687}{45360}$	.3678791887
10	$\frac{16481}{44800}$	.3678794643
11	$\frac{1468457}{3991680}$	.3678794392
12	$\frac{16019531}{43545600}$	.3678794413
13		.3678794412
14		.3678794412

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APPENDIX 2

Formulas for ${}^{z}M_{x}^{0}$ with fixed values of x or z

$${}^{z}M_{0}^{0} = 1$$

$${}^{z}M_{1}^{0} = \frac{z}{z+1}$$

$${}^{z}M_{2}^{0} = (z^{2}+z+1)/P_{2}^{z+2}$$

$${}^{z}M_{3}^{0} = (z^{3}+3z^{2}+5z+2)/P_{3}^{z+2}$$

$${}^{z}M_{4}^{0} = (z^{4}+6z^{3}+17z^{2}+20z+9)/P_{4}^{z+4}$$

$${}^{z}M_{5}^{0} = (z^{5}+10^{4}+45z^{3}+100z^{2}+109z+44)/P_{5}^{z+5}$$

$${}^{1}M_{x}^{0} = \frac{x+2}{x+1}\cdot M_{x+2}^{0}$$

$${}^{2}M_{x}^{0} = \frac{x+2}{x+1}\cdot M_{x+2}^{0} + \frac{1}{x+2}\cdot M_{x+1}^{0}$$

$${}^{3}M_{x}^{0} = \frac{x+4}{x+2}\cdot M_{x+3}^{0} + \frac{x+2}{(x+1)(x+3)}\cdot M_{x+2}^{0}$$

$${}^{4}M_{x}^{0} = \frac{x^{2}+7x+7}{(x+1)(x+3)}\cdot M_{x+4}^{0} + \frac{x^{2}+5x+3}{(x+1)(x+2)(x+4)}\cdot M_{x+3}^{0}$$

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APPENDIX 3

Values of ${}^{z}M_x^0$

x \ z	0	1	2	3	4	5
0	1	1	1	1	1	1
1	0	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{4}{5}$	$\frac{5}{6}$
2	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{7}{12}$	$\frac{13}{20}$	$\frac{7}{10}$	$\frac{31}{42}$
3	$\frac{1}{3}$	$\frac{11}{24}$	$\frac{8}{15}$	$\frac{71}{120}$	$\frac{67}{105}$	$\frac{227}{336}$
4	$\frac{3}{8}$	$\frac{53}{120}$	$\frac{181}{360}$	$\frac{31}{56}$	$\frac{143}{240}$	$\frac{1909}{3024}$
5	$\frac{11}{30}$	$\frac{103}{240}$	$\frac{607}{1260}$	$\frac{3539}{6720}$	$\frac{178}{315}$	$\frac{18089}{30240}$

x \ z	0	1	2	3	4	5
0	1	1	1	1	1	1
1	0	.5	.66667	.75	.8	.83333
2	.5	.5	.58333	.65	.7	.73810
3	.33333	.45833	.53333	.59167	.63810	.67560
4	.375	.44167	.50278	.55358	.59583	.63128
5	.36667	.42917	.48175	.52664	.56508	.59818

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APPENDIX 4

Recursions and Numerical Values for A_k^n

n=3

$$A_{k+2}^3 = A_{k+1}^3 + A_k^3 - 2^* \quad \text{or} \quad A_k^3 = 2A_{k-1}^3 - A_{k-3}^3$$

k	A_k^3
1	3
2	5
3	6
4	9
5	13
6	20
7	31
8	49
9	78
10	125
11	201
12	324
13	523
14	845
15	1366

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APPENDIX 4 (Continued)

n=4

$$A^4_{k+3} = A^4_{k+2} + A^4_{k+1} + A^4_k - 4^* \quad \text{or} \quad A^4_k = 2A^4_{k-1} - A^4_{k-4}$$

k	A^4_k
1	4
2	8
3	16
4	24
5	44
6	80
7	144
8	264
9	484
10	888
11	1632
12	3000
13	5516
14	10144
15	18656

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APPENDIX 4 (Continued)

n=5

$$A^5_{k+10} = 2A^5_{k+9} + 2A^5_{k+8} - 2A^5_{k+6} - 8A^5_{k+5} - 6A^5_{k+4} - 2A^5_{k+3} + 2A^5_{k+1} + A^5_k + 24^*$$

k	A^5_k
1	5
2	13
3	29
4	65
5	120
6	265
7	579
8	1265
9	2783
10	6208
11	13909
12	31337
13	70985
14	161545
15	369024

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APPENDIX 4 (Continued)

n=6

$$A^6_{k+15} = 2A^6_{k+14} + 2A^6_{k+13} + A^6_{k+12} - 4A^6_{k+10} - 18A^6_{k+9} - 16A^6_{k+8} - 12A^6_{k+7} - 10A^6_{k+6} - 4A^6_{k+5} + 4A^6_{k+4} + 3A^6_{k+3} + 2A^6_{k+2} + 2A^6_{k+1} + A^6_k + 96^*$$

k	A^6_k
1	6
2	18
3	48
4	130
5	326
6	720
7	1854
8	4738
9	12072
10	30818
11	79118
12	204448
13	528950
14	1370674
15	3557408

* Taken from Metropolis, Stein, and Stein.

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