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Some Applications of Sophisticated Mathematics to Randomized Computing

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Mathematics and Randomized Computing

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Randomized Algorithms

Use random bits (typically pseudorandom no. generator in practice) to make decisions about what to do.

Two types of randomized alg.:

- Las Vegas
- Monte Carlo

Randomized Alg. Example

Primality Testing

Input: p

Output: "prime" or "composite"

Idea: Repeatedly pick random a such that $0<a<p$, and compute $a^{p-1} \bmod p$. If result is ever $\neq 1$, then p is composite. If always 1, then p is probably prime.

Three Mathematical Results

- Chernoff bound on sums of random variables
- Lovász Local Lemma
- Weil's Theorem

Chernoff Bound

Very general.
Let S be the sum of k independent observations of a random variable X.

Let $m(a) = \inf_{t \in \mathbb{R}} E(e^{t(X-a)})$.

Then, for $a \geq E(X)$,

$P(S_k \geq ka) \leq [m(a)]^k$.

Special Case of Chernoff Bound

S_k is a binomial distribution
= sum of Bernoulli trials.

$P(X=1)=q$

$P(X=0)=1-q$

$E(e^{t(X-a)})=(1-q)e^{-at}+qe^{t(1-a)}$

Find inf by computing $\frac{d}{dt}$, etc.

After more work,

$$P(S_k \geq ka) \leq e^{-\left(\frac{a}{q}-1\right)^2 kq/2}$$

for $a \geq q$.

An Application

A variation of Sample Sort parallel sorting alg. Begins w. each of p processors holding $\frac{n}{p}$ elements. Each proc. sends each elt. to a random one of the p procs. Expect ith proc. P_i to send $\approx n/p^2$ elts. to P_j $\forall i,j$.

What prob. of imbalance (degrading running time)? Prob. P_i sends cn/p^2 elts. to P_j is

$$\leq e^{-(c-1)^2 n/(2p^2)}.$$

$(k = n/p,\ q = 1/p,\ a = c/p)$

Very small for e.g. $c = 2$ & n large.

Lovász Local Lemma

Let $A_1, \ldots, A_m$ be events, each of which depends on $\leq b$ others.

If $P(A_i) \leq p \ \forall i$ and $4pb < 1$, then $P(A_1 \cup A_2 \cup \cdots \cup A_m) < 1$.

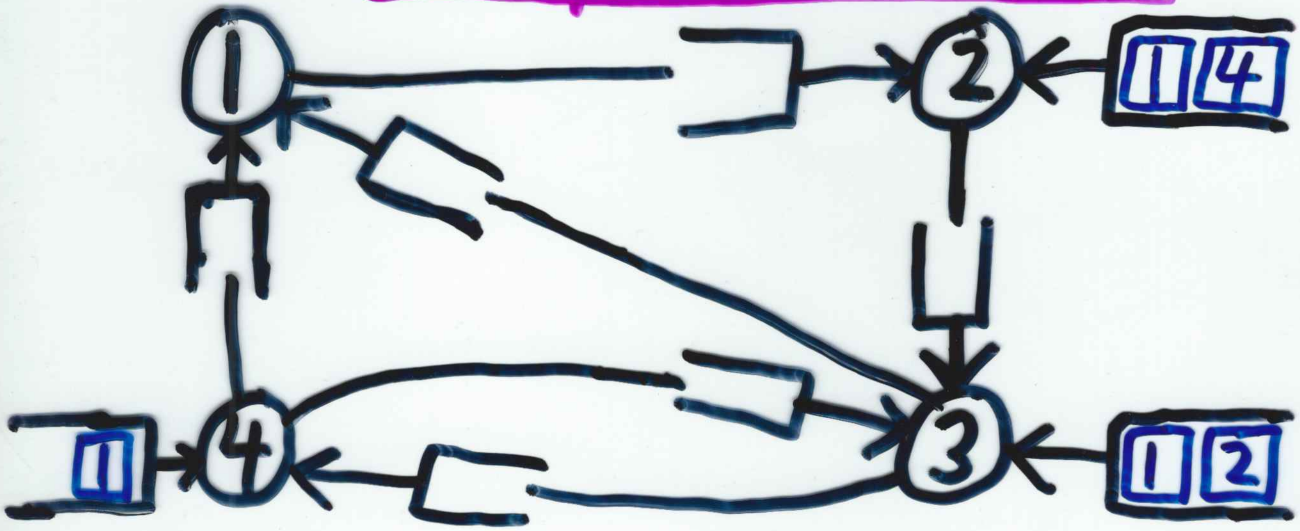
An Application

Used repeatedly to show that any set of packet paths in a network w. congestion c and dilation d can be realized in $O(c+d)$ time steps.

c = max. over edges of no. of packets that cross.

d = max. no. of edges traversed by a packet.

Graph Model



- Before routing, packets in initial queues at nodes (processors) where generated.
- Packet can traverse edge and enter queue at end when edge queue not full
- d_e time to traverse edge e

One step in the $O(c+d)$ Proof

WLOG, assume $c=d$.

Give each packet an initial delay from $[1, \alpha c]$, and let packets go w/o further delay.

<u>Claim</u>: $\exists$ a choice of delay such that $\leq \lg c$ packets traverse any given edge during any given interval of $\lg c$ time steps.

<u>Pf.</u>: Consider random delays.

Bad event for each edge: $> \lg c$ packets during some interval of $\lg c$ time steps.

Pf. Cont.'d

$$p \le (1+\alpha)c\binom{c}{\lg c}\left(\frac{\lg c}{\alpha c}\right)^{\lg c}$$

$$b \le cd = c^2$$

$4pb < 1$ for large enough const. α.

(Note: $\binom{n}{k} \le \left(\frac{en}{k}\right)^k$.)

Weil Application

Variation of Lehmer's alg. for $\sqrt{\ }$ mod p

Idea: Find x s.t. $\Delta = x^2 - a$ is a non-square mod p. Let $u = x + \sqrt{\Delta}$. In $\mathbb{Z}_p[\sqrt{\Delta}] = \mathbb{Z}_{p^2}$, compute and return $v = u^{(p+1)/2}$.

($\mathbb{Z}_{p^2}$ is the field of elts. of the form $x + y\sqrt{\Delta}$ w. $x, y \in \mathbb{Z}_p$ and Δ a nonsquare in $\mathbb{Z}_p$.)

Works because $v^2 = (u^{(p+1)/2})^2 = u^{p+1}$ $\equiv u\bar{u} = x^2 - \Delta = a$.

Exer.

(Note: $u^{(p+1)/2} \in \mathbb{Z}_p$. Exer.)

Choosing x

Make random picks. Prob. $= \frac{1}{2}$ that $x^2 - a$ is square (failure). With k indep. random picks, failure prob. $= 1/2^k$.

But typical computation better modeled as random x_1, then $x_2 = ax_1 + b \bmod c$, $x_3 = ax_2 + b \bmod c$, etc.

Actually can get good results w. $x, x+1, x+2, \ldots, x+k-1$. Fail only if $\exists\ y_1, \ldots, y_k$ such that

$$x^2 - a = y_1^2;\ (x+1)^2 - a = y_2^2;\ \ldots;\ (x+k-1)^2 - a = y_k^2.$$

Bounding Failure Prob.

Can show failure unlikely by bounding no. of solutions on curve.

Weil's Theorem: Let $\overline{F}_p$ denote the algebraic closure of $\mathbb{F}_p$. Let C be a curve, def. by eqns. whose coefficients lie in $\mathbb{F}_p$, whose projective closure $\overline{C}$ (over $\overline{F}_p$) is nonsingular. Then N, the no. of pts. in $\overline{C}$ w. coordinates in $\mathbb{F}_p$ satisfies $|p+1-N| \leq 2g\sqrt{p}$, where g is the genus of C.