12-2013

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September 21, 2011

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An infinite dimensional algebra may depend essentially on some parameters and yet be absolutely rigid in the classical deformation theory but the variability may be captured by the cohomolgy of a naturally associated diagram of algebras as illustrated here with two examples, the function field of a sphere with four marked points and the first Weyl algebra. These show that to understand the deformation of algebras one must consider not just that of single algebras but of diagrams even if at the start one was concerned with the deformation of but a single specific algeba.

1 Introduction

An infinite dimensional algebra may be absolutely rigid in the classic deformation theory [2] while depending essentially on parameters but this dependence may be exhibited in the non-trivial deformation of some associated diagrams. This is illustrated here with two examples, the function algebra of the sphere with four marked points and the first Weyl algebra. Both are absolutely rigid but diagrams of algebras naturally built from each deform non-trivially, thereby exhibiting the dependence of these algebras on some internal parameters; we conjecture that this is the general phenomenon. The examples also demonstrate that one can not fully understand the deformation theory even of single algebras without studying the deformation of diagrams, the reason for which may be clearer whe one compares the deformation theory of algebras with that of complex manifolds.

The deformation theory of algebras identifies the space of infinitesimal deformations of an associative algebra A over a commutative unital ring k with its second Hochschild cohomology group $H^2(A, A)$ with coefficients in itself; when this vanishes A is called *absolutely rigid* and no non-trivial deformation in the classic sense of [2] is possible. However, an infinite dimensional algebra

*MG wishes to thank Prof. Fujio Kubo and Hiroshima University for their hospitality during part of the preparation of this paper

A which is absolutely rigid may depend in an essential way on one or more parameters because it is assembled from simpler ones and this assembly may vary, in analogy with how a complex manifold is assembled from coordinate patches. The 'assembly' is a diagram of algebras, i.e., a pesheaf of algebras over a small caegory.

Algebraic deformation theory arose from that of analytic manifolds. The basic discovery of Frölicher and Nijenhuis [1] was that if X is a complex analytic manifold and T its sheaf of germs of holomorphic tangent vectors then the space of infinitesimal deformations of X can be identified with $H^1(X,T)$, paving the way for the exhaustive study of analytic deformation theory by Kunihiko Kodaira and Donald C. Spencer, for references cf. [8]. Spencer gave the following exceedingly useful intuitive description of the infinitesimal deformations: Consider the manifold X as 'sewn together' from coordinate patches X_i. Deform X by unstitching them and letting them slide over each other before resewing. In every overlap $X_i \cap X_j$ the derivative of the motion of X_j relative to X_i defines a tangent vector field and these give the element of $H^1(X,T)$ which is the infinitesimal of the deformation. While $H^1(X,T)$ is actually defined by taking a direct limit over refinements of coverings one can compute it from a single covering by taking patches which are Stein manifolds or, in the purely algebraic case, by affine varieties. The latter thus play a role analogous to that of coordinate patches in the analytic case *even though these generalized coordinate patches – complex affine varieties – may themselves depend on complex parameters.* That an absolutely rigid algebra may depend on parameters is therefore not a paradox but an essential feature of deformation theory.

The algebras in our examples behave very differently under deformation but both depend on parameters – in the first case on a single one, in the second on infintely many–and are absolutely rigid in the classical theory which therefore can not detect this dependence. To each single algebra we associate a diagram of algebras constructed from the original and which therefore depends on the same parameters as the original. These diagrams, unlike the original algebras, are not absolutely rigid in the deformation theory of diagrams of algebras; infinitesimal changes in the parameters of the original algebras produce non-trivial infinitesimal deformations of the diagrams. Had changes in the parameters actually left the original algebras unchanged up to isomorphism *over the original coefficient ring* then the associated diagrams would likewise have remained invariant up to isomorphism, but that is not the case since their infinitesimal deformations are non-trivial. (The dependence on the parameters disappears, as we show explicitly when in each case one passes from the original algebra A, an algebra over k, to the algebra $A[[\hbar]]$, where $\hbar$ is a variable, this being an algebra not over the original coefficient ring but over the power series ring $k[[\hbar]]$.)

2 Diagram cohomology

A *diagram of algebras* over a small category $\mathcal{C}$ with objects $i, j, \dots$ is a contravariant functor $\mathbb{A}$ from $\mathcal{C}$ to the category of unital associative algebras, i.e., a presheaf of algebras over $\mathcal{C}$. (One can make the same definition for diagrams of other kinds of algebras; for the Lie case cf [11].) For example, the sets in an open covering $\mathcal{U}$ (closed under taking intersections) of a complex manifold $\mathcal{M}$ may be viewed as the objects of a category in which the morphisms are inclusion maps. That is, if U, V are sets in $\mathcal{U}$ then there is a unique morphism from U to V if $U \subset V$ (the inclusion itself) and no morphism otherwise. One then has a partially ordered set ('poset'); any such is viewed as a category in which there is a unique morphism $i \to j$ when $i \leq j$ and no morphism otherwise. The functor assigns to each open set the algebra of holomorphic functions on that set. In practice one takes a covering by Stein manifolds and for a smooth algebraic variety one similarly takes a covering by affine opens since the intersection of any two is again such and each such set has trivial cohomology; cf [4]. An $\mathbb{A}$-*module* $\mathbb{M}$ is a contravariant functor from $\mathcal{C}$ to the category of Abelian groups such that for each $i \in \mathrm{Ob}\,\mathcal{C}$ the group $\mathbb{M}(i)$ is an $\mathbb{A}(i)$-bimodule and for each morphism $u : i \to j$ the map $\mathbb{M}(u) : \mathbb{M}(j) \to \mathbb{M}(i)$ is a morphism of $\mathbb{A}(j)$-modules where $\mathbb{M}(i)$ is viewed as an $\mathbb{A}(j)$-module by virtue of the morphism $\mathbb{A}(u) : \mathbb{A}(j) \to \mathbb{A}(i)$.

To define the cohomology groups $H^n(\mathbb{A}, \mathbb{M})$ consider first the simplicial complex called the *nerve* of $\mathcal{C}$. The 0-simplices of this are just the objects i of $\mathcal{C}$. For $q > 0$ a non-degenerate q-simplex is any q-tuple of composable morphisms $\sigma = (i_0 \to i_1 \to \cdots \to i_q)$ in which no single morphism $i_r \to i_{r+1}$ is an identity morphism (although a composite of several of the morphisms is allowed to be). We will call i_0 the *domain* of σ, denoted $d\sigma$ and i_q its *codomain*, $c\sigma$. The 0-th and q-th faces of σ are given, respectively, by $\partial_0\sigma = (i_1 \to \cdots \to i_q)$, $\partial_q\sigma = (i_0 \to \cdots \to i_{q-1})$. For $0 < r < q$ define $\partial_r\sigma$ by composing the maps $i_{r-1} \to i_r$ and $i_r \to i_{r+1}$ so $\partial_r\sigma = (i_0 \to \cdots \to i_{r-1} \to i_{r+1} \to \cdots \to i_q)$. Let $C_q(\mathcal{C})$ be the set of all formal finite linear combinations of non-degenerate q-simplices and set $\partial\sigma = \sum_{r=0}^{q}(-1)^r\partial_r\sigma$ omitting any simplices which may be degenerate. Since the boundary of a degenerate simplex always vanishes, with this the C_q form a chain complex. (It is isomorphic to the quotient of the complex spanned by all simplices by the subcomplex spanned by the degenerate simplices.) Note that if $\sigma = (i_0 \to \cdots \to i_q)$ then $\mathbb{M}(d\sigma) = \mathbb{M}(i_0)$ is a module over $\mathbb{A}(i_q) = \mathbb{A}(c\sigma)$ by virtue of the composite morphism $(i_0 \to i_q)$, which we will denote by $|\sigma|$. Now let $C^{p,q}$ be the k-module of all functions on $C_q(\mathcal{C})$ which send a q-simplex σ to a cochain $\Gamma \in C^p(\mathbb{A}(c\sigma), \mathbb{M}(d\sigma))$. The image of σ under Γ will be denoted Γ^σ, but the actual cochain depends only on $|\sigma|$ and it iwll be convenient laer to label it simply by the name of this map. Setting, as before, $\sigma = (i_0 \to \cdots \to i_q)$, those faces $\partial_r\sigma$ with $0 < r < q$ all have the same domain and codomain as σ, but the first and last do not. Write briefly φ for the algebra morphism $\mathbb{A}(i_{q-1} \to i_q) : \mathbb{A}(i_q) \to \mathbb{A}(i_{q-1})$. Then $\Gamma^{\partial_q\sigma}\varphi$, defined by sending $a_1, \ldots, a_p \in \mathbb{A}(i_q)$ to $\Gamma^{\partial_q\sigma}(\varphi a_1, \ldots, \varphi a_q)$, is again in $C^p(\mathbb{A}(c\sigma), \mathbb{M}(d\sigma))$. Write T for $\mathbb{M}(i_0 \to i_1) : \mathbb{M}(i_1) \to \mathbb{M}(i_0)$. We then also have $T\Gamma^{\partial_0} \in C^p(\mathbb{A}(c\sigma), \mathbb{M}(d\sigma))$. Setting

$$\Gamma^{\partial\sigma} = T\Gamma^{\partial_0\sigma} + \sum_{r=1}^{q-1}(-1)^r\Gamma^{\partial_r\sigma} + (-1)^q\Gamma^{\partial_q\sigma}\varphi$$

we have commuting coboundaries, the algebraic (Hochschild) $\delta_{\mathrm{Hoch}} : C^{p,q} \to$

$C^{p+1,q}$ defined by sending Γ^σ to its Hochschild coboundary $\delta\Gamma^\sigma$ and the simplicial $\delta_{\text{simp}} : C^{p,q} \to C^{p,q+1}$ defined by $(\delta_{\text{simp}}\Gamma)^\sigma = \Gamma^{\partial\sigma}$. Finally, set $C^n(\mathbb{A}, \mathbb{M}) = \oplus_{p+q=n} C^{p,q}$ and define the total coboundary $\delta : C^n \to C^{n+1}$ by

$$(\delta\Gamma)^\sigma = \delta_{\text{simp}}(\Gamma)^\sigma + (-1)^{\dim\sigma}\delta_{\text{Hoch}}(\Gamma^\sigma).$$

The cohomology groups $H^*(\mathbb{A}, \mathbb{M})$ are defined to be the cohomology groups of this complex.If m is the highest dimension of any non-zero cohomology group of any algebra in the diagram, the diagram as a whole may still have non-trivial cohomology in dimensions greater than m, the limit being $m + d$ where d is the dimension of the nerve of the underlying category (the dimension of the largest simplex appearing in it). For example, denoting the diagram with $\mathbb{A}(i) = k$, the coefficient ring, and every morphism $\mathbb{A}(i \to j)$ the identity by $\mathbf{k}$, $H^*(\mathbf{k}, \mathbf{k})$ is the simplicial cohomology of the nerve of the underlying category $\mathcal{C}$ with coefficients in k. Here there is no algebra part; the cohomology is entirely simplicial.

A *deformation of a diagram* of k-algebras $\mathbb{A}$ is a diagram of $k[[\hbar]]$-algebras over the same underlying category whose reduction modulo $\hbar$ is $\mathbb{A}$, [4]. Because the cohomology of the nerve of the underlying category $\mathcal{C}$ may not be trivial, unlike the case of a single algebra one can not always identify the infinitesimal deformations of a diagram $\mathbb{A}$ with $H^2(\mathbb{A}, \mathbb{A})$; in general one must use the cohomology of the "asimplicial" subcomplex consisting of those cochains Γ where if the dimension of Γ is n and σ is an n simplex of the nerve of $\mathcal{C}$ then Γ^σ (which is just an element of $\mathbb{A}(\sigma)$ vanishes. However, if all the algebras in the diagram are commmutative (as in our first example) or if the geometric realization of the nerve of the underlinying cagegory $\mathcal{C}$ is contractible (the case in our second example, but something that could always be accomplished by adjoining a terminator to $\mathcal{C}$) then the problem does not arise, cf [4]. An infinitesimal deformation of $\mathbb{A}$ is then just the cohomology class of a 2-cocycle Γ. The latter assigns to every $i \in \text{Ob}(\mathcal{C})$ a 2-cocycle Γ^i of $\mathbb{A}(i)$ with coefficients in itself (which we may interpret as the infinitesimal of a deformation of $\mathbb{A}(i)$) and assigns to every morphism $i \to j$ a k-linear map $\phi^{ij} : \mathbb{A}(j) \to \mathbb{A}(i)$. Denoting ϕ^{ij} for the moment simply by ϕ, these are connected by the condition that

$$\phi(\Gamma^j(a, b)) - \Gamma^i(\phi a, \phi b) = (\delta\Gamma^{ij})(a, b),$$

where δ is the Hochschild coboundary. An *integral* of Γ, if one exists, is a deformation whose infinitesimal is in the class of Γ.

To illustrate the theory we give some simple examples in the following section. The underlying category of a diagram of algebras will always be obvious so we may omit mention of it.

The Cohomology Comparison Theorem cf [4] asserts that there is a functor from diagrams of algebras to single associative algebras which preserves the cohomology and the deformation theory. This allows one to transfer to the cohomology of diagrams of algebras all known properties of the cohomology of a single algebra including, e.g., its Gerstenhaber algeba structure. However, we shall not need it here. There have been important recent developments, mainly due to Lowen and Van den Berg [10] and Stancu [16, 17], who show that the appropriate setting for the theory is that of derived categories.

3 Some simple diagrams of algebras

The simplest (non-trival) diagram of algebras is just an algebra morphism

$$B \xrightarrow{\phi} A$$

; a bimodule over this diagram is a morphism of abelian groups

$$N \xrightarrow{T} M$$

where N is a B-bimodule, N an A-bimodule, and T a module morphism from N to M with the latter cosidered as a B-bimodule by virtue of the morphism ϕ. The original diagram of algebras, which by abuse of notation we may also denote simply by ϕ and likewise the module by T, is a bimodule over itself. An n-cochain Γ of ϕ with coefficients in T is a triple $\Gamma = (\Gamma^B, \Gamma^A, \Gamma^\phi)$, the first component of which is a Hochscild n-cochain of B with coefficients in N, the second an n cochain of A with coefficients in M, and the third an $n-1$ cochain of B with coefficients in M considered as a B bimodule. With coboundary $\delta\Gamma = (\delta\Gamma^B, \delta\Gamma^A, T\Gamma^B - \Gamma^A\phi - \delta\Gamma^\phi)$ this defines the complex $C^*(\phi, T)$. Note that Γ is a cocycle precisely when Γ^B and Γ^A are both cocycles and $T\Gamma^B - \Gamma^A\phi$ is a coboundary. A deformation of ϕ is a diagram $B_t \xrightarrow{\phi_t} A_t$ where B_t, A_t are deformations of B and A, respectively and $\phi_t = \phi + t\phi_1 + t^2\phi_2 + \dots$ (each ϕ_i being a k-linear map $B \to A$ extended to be $k[[t]]$ linear) is a morphism between the deformed algebras. Its infinitesimal is a two-cocycle $\Gamma \in Z^2(\phi, \phi)$, or more strictly, its cohomology class. The geometric picture is a morphism $f : X \to Y$ beween two topological spaces. If A is the ring of continuous functions on X and B that on Y, then f induces a morphism $f^* : B \to A$. The same idea applies when X and Y are analytic manifolds or varieties defined over some field k.

If one of the algebras, say B is to be held fixed then one must use the subcomplex of $C^*(\phi, T)$ consisting of those $\Gamma = (0, \Gamma^A, \Gamma^{AB})$ in which the first component is identically zero. (In the geometric situation, suppose that X and the morphism f are deformed but that we require that the space Y to which X maps is held fixed.) We are then considering only those cocycles Γ^A such that $\Gamma^A\phi$ is a coboundary. Since coboundaries are always sent to coboundaries, the cohomology is simply $\ker \phi^* : H^*(A, A) \to H^*(B, A)$. This is clearly closed under the cup product. It should, we conjecture, also be closed under the graded Lie product because of the relation of that product to obstructions. Similar consderations hold when A is held fixed. When both algebras are fixed and we are interested only in how the morphism between them can vary then the infinitesimal is just a derivation of B into A; when the morphism is an inclusion and we identify morphisms which differ only by an automorphism of B then the space of infinitesimals consists of the derivations of B into the B bimodule $A/\phi B$, cf [13, 14].

For the Weyl algebra we will use a diagram like

$$\{B \xrightarrow{\phi} A \xleftarrow{\phi'} B'\},$$

which we will denote by $\mathbb{A}$. A module over this diagram has the form

$$\mathbb{M} = \quad \{N \xrightarrow{\ T\ } M \xleftarrow{\ T'\ } N'\}$$

where N and N' are, respectively, B and B' bimodules, M is an A bimodule, T is a B bimodule morphism, M is considered a B bimodule through the morphism ϕ, and similarly for T'. An n-cochain of $\mathbb{A}$ with coefficients in $\mathbb{M}$ is a quintuple $\Gamma = (\Gamma^B, \Gamma^{B'}, \Gamma^A, \Gamma^\phi, \Gamma^{\phi'})$ with the obvious meanings. One can picture Γ in the form

$$\Gamma = \quad \{\Gamma^B \xrightarrow{\ \Gamma^\phi\ } \Gamma^A \xleftarrow{\ \Gamma^{\phi'}\ } \Gamma^{B'}\} \qquad .$$

The coboundary can then be depicted by

$$\delta\Gamma = \quad \{\delta\Gamma^B \xrightarrow{T\circ\Gamma^B - \Gamma^A\circ\phi - \delta\Gamma^\phi} \delta\Gamma^A \xleftarrow{T'\circ\Gamma^{B'} - \Gamma^A\circ\phi' - \delta\Gamma^{\phi'}} \Gamma^{B'}\} \qquad .$$

The coboundary operators inside the braces are the ordinary Hochschild coboundaries. The geometric picture is that of a space X with morphisms f and f' (which in general need not be defined on all of X) to respective spaces Y and Y'. In the very special case that we have a diagram

$$\mathbb{A} := \quad B \xrightarrow{\ f\ } A \xleftarrow{\ g\ } C$$

in which the second cohomology of every algebra with coefficients in itself vanishes and where for every derivation $\Gamma^f \in \mathrm{Der}(B, A)$ there is a derivation γ^A of A such that $\Gamma^f(b) = \gamma^A(fb)$ for all $b \in B$, it is easy to check that every 2-cocycle Γ of $\mathbb{A}$ with coefficients itself is cohomologous to one of the form $(0, 0, 0, 0, \Gamma^g)$.

4 First example: The four-punctured sphere

Let k be a commutative unital ring and set $A = k[x, 1/x, 1/(x-1), 1/(x-\lambda)]$ where λ is an element of k not equal to 0 or 1; when $k = \mathcal{C}$ this is just the algebra of functions on the Riemann sphere punctured respectively at $\infty, 0, 1, \lambda$. Since the cohomology of A with coefficients in any A module vanishes in all dimensions greater than 1 it must be that when A is enlarged to $A[[\hbar]]$, where $\hbar$ is a variable, that the dependence on λ disappears. This is easy to exhibit explicitly. Denote A now more explicitly by A_λ. Then inside $A_\lambda[[\hbar]]$ we have

$$\frac{x-\lambda}{x-\lambda(1+\hbar)} = \frac{1}{1-\hbar\lambda/(x-\lambda)}, \quad \text{so} \quad \frac{1}{x-\lambda(1+\hbar)} = \frac{1}{x-\lambda}\sum_{n=0}^{\infty} \hbar^n \frac{\lambda^n}{(x-\lambda)^n}.$$

Thus $A_\lambda[[\hbar]]$ contains $A_{(1+\hbar)\lambda}[[\hbar]]$ but by the same means it is easy to see that the reverse inclusion also holds, so these rings are identical and hence isomorphic. (Notice that that the series for $1/(1-\hbar\lambda/(x-\lambda))$ is not contained in $A_\lambda \otimes_k k[[\hbar]]$, but only in the larger ring $A_\lambda[[\hbar]]$.) On the other hand, when $k = \mathbb{C}$ and x is viewed as takng on complex values, the series may be viewed as defining a mapping from the sphere punctured at $\{0, 1, \infty, \lambda\}$ to that punctured at $\{0, 1, \infty, \lambda(1+\hbar)\}$. There is no value of $\hbar$, however small, for which it can converge everywhere – these surfaces are not analytically isomoprhic – but given any neigborhood of $x = \lambda$ one can choose $\hbar$ so small that the series converges everywhere outside that neighborhood. This suggests that analytically the local moduli space is not singular, which is indeed the case.

Now set $B = k[x, 1/x, 1/(x-1)]$ and assume that λ is invertible. Then there are two monomorphisms $f, g : B \to A$ defined respectively by sending x to x and by sending x to $\lambda^{-1}x$, as pictured in the following diagram whch we will denote by $\mathbb{A}$:

$$k[x, \tfrac{1}{x}, \tfrac{1}{(x-1}] = B \quad \underset{g}{\overset{f}{\rightrightarrows}} \quad A = k[x, \tfrac{1}{x}, \tfrac{1}{(x-1}, \tfrac{1}{(x-\lambda}].$$

Geometrically, when $k = \mathcal{C}$ we have two morphisms of a four-punctured sphere into the three-punctured sphere.

Both algebras in the diagram are commutative so the infinitesimal deformations of this diagram may be indentified with its second cohomology group. The 2-cocycles may be written in the form $\Gamma = (\Gamma^B, \Gamma^A, \Gamma^f, \Gamma^g)$, where Γ^B is a 2-cocycle of B with coefficients in itself, similarly for Γ^A; Γ^f, Γ^g are 1-cochains of B with coefficients in A, and $f\Gamma^B - \Gamma^A f = \delta\Gamma^f, g\Gamma^B - \Gamma^A g = \delta\Gamma^g$. Since B and A both have vanishing cohomology in all dimensions greater than one, Γ is cohomologous to a cocycle in which $\Gamma^B = \Gamma^A = 0$, which we may now assume, and in this case the cocycle condition is just that Γ^f and Γ^g must be derivations from B into A. A derivation of any of our algebras is completely determined by its value on x which in turn can be any element of the coefficient module which here is $\mathbb{C}[x, 1/x, 1/(x-1), 1/(x-\lambda)]$. Such an element, by a classic theorem, is the sum of its principal parts, i.e, a unique linear combination of 1 and powers of $x, 1/x, 1/(x-1)$ and $1/(x-\lambda)$. The 2-cocycle Γ can be identified with the pair of elements $(\Gamma^f(x), \Gamma^g(x))$ in $A = k[x, 1/x, 1/(x-1), 1/(x-\lambda]$. We will see that the dimension of $H^2(\mathbb{A}, \mathbb{A})$ is infinite but with a natural condition on the regularity of the derivations that are allowed, drops to precisely 1, the number of moduli of the four-punctured sphere.

To compute the second cohomology of the diagram $\mathbb{A}$ we must compute when a cocycle $\Gamma = (0, 0, \Gamma^f, \Gamma^g)$ is the coboundary of a 1-cochain $\gamma = (\gamma^B, \gamma^A, \gamma^f, \gamma^g)$ where (since $\Gamma^B = \Gamma^A = 0$) γ^B and γ^A must be derivations of B and of A, respectively, and γ^f, γ^g (which are just elements of$A$) are zero since we must use the 'asimplicial' subcomplex. (Alternatively, only their coboundaries enter but these vanish since the algebras are commutative so their values are irrelevant.) Thus $H^2(\mathbb{A}, \mathbb{A})$ is the k module of 2-cocycles $(0, 0, \Gamma^f, \Gamma^g)$ modulo the submodule of those for which there exist $\gamma^B \in \mathrm{Der}(B), \gamma^A \in \mathrm{Der}(A)$ such that simultaneously $\Gamma^f = f\gamma^B - \gamma^A f$, $\Gamma^g = g\gamma^B - \gamma^A g$. The coboundary γ is determined by the elements $\gamma^B(x) \in B$ and $\gamma^A(x) \in A$. Its coboundary corresponds to the pair of elements $f(\gamma^B(x)) - \gamma^A(fx), g(\gamma^B(x)) - \gamma^A(g(x))$. The first is just $\gamma^B(x) - \gamma^A(x)$, both summands considered as elements of A, while the second is $\gamma^B(\lambda^{-1}x) - \lambda^{-1}\gamma^A(x)$. So Γ is a coboundary if there exist elements $b(x) \in k[x, 1/x, 1/(x-1)], a(x) \in k[x, 1/x, 1/(x-1), 1/(x-\lambda)]$ such that (when both are considered as elements of $A = k[x, 1/x, 1/(x-1), 1/(x-\lambda)]$) $b(x) - a(x) = \Gamma^f(x), b(\lambda^{-1}x) - a(\lambda^{-1}x]) = \Gamma^g(x)$. Then $b(x) - \lambda b(\lambda^{-1}x) = \Gamma^f(x) - \lambda\Gamma^g(x)$; if we can solve this then $a(x) = b(x) - \Gamma^f(x)$ is determined. So the cocycle $\Gamma = (0, 0, \Gamma^f, \Gamma^g)$ is a coboundary precisely when we can find $b(x) \in k[x, 1/x, 1/(x-1)](= B)$ such that $b(x) - \lambda b(\lambda^{-1}x) = \Gamma^f(x) - \lambda\Gamma^g(x) \in A$. Now $\Gamma^f(x) - \lambda\Gamma^g(x)$ can be an arbitrary element of A so $H^2(\mathbb{A}, \mathbb{A})$ is isomorphic to the k-module obtained by taking the quotient of A as a k-module by the submodule consisting of those elements $c(x) \in A$ of the form $b(x) - \lambda b(\lambda^{-1}x)$ (where $\lambda \neq 1$). It is evident now that this is an infinitely generated free module. For we can solve $b((x) - \lambda b(\lambda^{-1}x) = x^N$ for all positive and negative integers N other than $N = 1$ but if $b(x) = 1/(x-1)^N$ then $b(x) - \lambda b(\lambda^{-1}x) = 1/(x-1)^N - \lambda/(\lambda^{-1}x - 1) = 1/(x-1)^N - \lambda^{N+1}/(x-\lambda)$ so we can solve $b(x) - \lambda b(\lambda^{-1}x) = c(x)$ if and only if $c(x)$ contains no term in x and the coefficients of $1/(x-1)^N$ and $1/(x-\lambda)^N$ in $c(x)$ are in the ratio $(1 : -\lambda^{N+1})$. The quotient, which is isomorphic to $H^2(\mathbb{A}, \mathbb{A})$, therefore has an infinite basis consisting of x and $1/(x-1)^N, N > 0$.

The problems is that the derivations have been allowed to be singular at $x = 1, \lambda$, while in the geometric picture the tangent vector fields are holomorphic. To follow that picture one should require regularity at $1, \lambda$; then all the $1/(x-1)^N$ are eliminated and the dimension of the space of allowable infinitesimal deformations drops to 1, as expected This is so even if wone continues to allow singularities at $0, \infty$, but to follow the geometric picture one should restrict to derivations which are regular everywhere and which vanish at $0, \infty$, i.e., which just send x to a multiple of x. The cohomology of the analogous diagram for the sphere with more than four punctures then gives the correct number of moduli. It is easy to check that $H^3(\mathbb{A}, \mathbb{A}) = 0$ so there are no obstructions to any infinitesimal deformation; the dimension of $H^2(\mathbb{A}, \mathbb{A}) = 0$ is the number of moduli.

One question arising from the foregoing is whether the Hochschild-Kosant-Rosenberg theorem [9], which asserts in particular that the cohomology of a polynomial ring $R = k[x_1, \ldots, x_n]$ with coefficients in itself is isomorphic to the exterior algebra over R on the derivations of R, can be refined to a similar assertion when certain regularity conditions are imposed on the cochains.

5 Quotients as deformations and Gröbner bases

We digress briefly to examine a common alternative approach to deformations, namely by viewing a parameterized algebra A as a quotient of a ring R by a varying ideal. One problem, if one wishes to view this as a deformation, is to display the varying quotient as a varying multiplication on a fixed underlying vector space, which may not always be possible. Another is that the procedure produces no immediate homological informaton that can be used to show that the deformed algebra (if one has a deformation) is not isomorphic to the original. Often the first problem can be overcome. Suppose, for example, that A is an affine ring, i.e., a finitely generated commutative ring over a field k. It is then a quotient $A = R/I$ with R a polynomial ring $k[x_1, \dots, x_n]$ in some finite number of variables by an ideal I which itself can be finitely generated. After choosing a term order on the monomials in R, Buchberger's algorithm produces a unique Gröbner basis. The set of (classes of) standard monomials, i.e., those not in the initial ideal of the Gröbner basis then forms a basis for the quotient algebra A. The algorithm involves only rational operations, so if we have an ideal $I_\hbar$ generated by polynomials which depend rationally on some formal parameter $\hbar$, then the polynomials in the basis will also depend rationally on $\hbar$. It follows that for all but a finite number of exceptional values of $\hbar$ in k we will have the same set of standard monomials and therefore a fixed basis for $A_\hbar = R/I_h$. If zero is not one of these exceptional values then we have a varying, necessarily associativ) multiplication on a fixed basis which for $\hbar = 0$ is the original multiplication on A, so we have a classical deformation. It would be exceedingly useful if one could extract from Buchberger's theory some cohomological information which would tell if the deformation is trivial over the original ground field.

Buchberger's theory is applicable also to certain classes of non-commutative algebras but where the reduced Gröbner basis may be infinite. In this case, even though we may have a fixed complement to the ideal $I_\hbar$ there can be infinitely many exceptional values in any neighborhood of $\hbar = 0$. A representation of $A_\hbar$ as a classical deformation may then be meaningless even though it can actually be evaluated at all but the exceptional values. We will see this with the Weyl algebra which is quasicommutative (commutative with respect to a Yang-Baxter operator) and where left ideals have finite Gröbner bases, but where the free non-commutative polynomial ring $\mathbb{C}\{x, y\}$, of which the Weyl algebra is a quotient, is not quasicommutative.

The algebra $\mathbb{C}[x, 1/x, 1/(x-1), 1/(x-\lambda)]$ is a quotient of the polynomial ring $R = \mathbb{C}[x, y, z, w]$ by the ideal $I(\lambda)$ generated by $xy-1, (x-1)z-1, (x-\lambda)w-1$ and varying the ideal by varying λ can in this case indeed be viewed as a deformation. Taking as term order total degree with reverse lexicographic order to break ties (frequently the fastest, computationally), the resulting Gröbner basis (using Maple) is $\lambda zw + z - wz - w, \lambda yw + y - w, wx - \lambda w - 1, zy + y - z, zx - z - 1, xy - 1$. For simplicity we have not divided by λ when that is the leading coefficient. Here λ has been treated as a variable but it is clear from the form of the result that the only exceptional values of λ are 0 and 1 (and technically, ∞). For every other value of λ there is a neighborhood in which $R/I(\lambda)$ is indeed a deformation in our sense of the algebra of the four-punctured sphere. The relations $xy - 1$ and $(x-1)z - 1$ have not changed; the subalgebra generated by x, y and z is that of the three-punctured sphere, independent of λ. However, the Gröbner basis contains a new relation amongst x, y and z, namely $zy + y - z$. This is an immediate consequence of the original defining relations but is necessary: the Gröbner basis shows that the initial ideal is generated by zw, yw, wx, zy, zx, xy so the standard monomials S, which span a complementary vector space $\mathbb{C}S$ to $I(\lambda)$ for all but the exceptional values of λ, are those not divisible by any of these. But this simply says that S consists of all pure powers of either x, y, z or w and 1. We have simply recaptured the decomposition of an element of $k[x, 1/x1/(x-1), 1/(x-\lambda)]$ into its principal parts, but so far there is no indication that the quotient algebra depends essentially on the parameter λ.

6 Second example: The Weyl algebra

Throughout this section the coefficient ring k is assumed to be a field of characteristic zero (but in some obvious places could be more general). The (first) Weyl algebra $W_1 = k\{x,y\}/(xy - yx - 1)$ can be exhibited in various ways as a deformation of the polynomial ring $k[x,y]$, of which the simplest is the 'normal' form: Letting $*$ denote the deformed multiplication

$$a * b = ab + \hbar\partial_x a \partial_y b + \frac{1}{2!}\hbar^2\partial_x^2 a\partial_y^2 b + \cdots = \mu[\exp(\hbar\partial_x \otimes \partial_y)(a \otimes b)],$$

where on the right μ denotes the original multiplication.If ϕ and ψ are commuting derivations of an algebra A of characteristic zero then setting $a * b = \mu\exp(\hbar\phi \otimes \psi)(a \otimes b)$ defines a deformation of A; more generally, given commuting derivations $\phi_1, \ldots, \phi_r, \psi_1, \ldots, \psi_r$ replace $\phi \otimes \psi$ by $\sum \phi_i \otimes \psi_i$, [2, 3]. In particular, $\phi\otimes\psi$ can be replaced by $\phi\wedge\psi = (\phi\otimes\psi - \psi\otimes\phi)/2$ to give a "Moyal deformation" [12] but the idea is already be implicit in Groenewold [6]. This gives $[x,y]_* = \hbar$; taking $\hbar = 1$ gives the Weyl algebra. In most physical applications the ground ring is $\mathbb{R}$ or $\mathbb{C}$, the algebra A being deformed is one of functions, and the deformed product is of the form $a*b = ab + \hbar F_1(a,b) + \hbar^2 F_2(a,b) + \ldots$, and the F_i are (bi)differential operators of increasing orders, and one hopes for convergence at least for sufficiently small real or complex values of $\hbar$. Here we are dealing with polynomials so there is no problem of convergence since the series for any particular product will terminate as a polynomial will ultimately be annihilated by a sufficiently high-order differential operator. Note that if both $k[x,y]$ and W_1 are viewed as singly graded with $\deg x = 1$ and $\deg y = -1$ then the Moyal deformation has preserved the grading.

In contrast to the case of the four-punctured sphere, we will show that the second cohomology of a diagram naturally associated to W_1 is infinite dimensional. However, we show first that viewing q as a function of $\hbar$ with $q(0) = 1$, $W_1[[\hbar]]$ and $W_q[[\hbar]]$ are formally isomorphic when we $q = 1 + \hbar$. It will follow that the cohomology of W_q, like that of W_1 vanishes in all positive dimensions. One approach is to vary the inclusion morphism of $k[y]$ into W_1, i.e., to find an element y' of the form $y + \hbar\mu_1 + \hbar^2\eta_2 + \ldots, \eta_i \in W_1[[\hbar]]$ such that the relation $[x,y'] = xy' - y'x = 1$ is equivalent to having $[x,y] = 1 - \hbar xy$; this would give $(1+\hbar)xy - yx = 1$ or $W_q[[\hbar]]$ with $q = 1 + \hbar$. As in the example of the four-punctured sphere, if we take k to be $\mathbb{R}$ or $\mathbb{C}$ then the power series will have zero radius of convergence but remarkably can be evaluated for $\hbar \in \mathbb{C}$ as long as $1 + \hbar$ is not a root of unity.

One can solve for the η_i recursively. There there is obviously some choice but taking the simplest one yields a power series in $\hbar$ beginning

$$y' = y + \hbar\eta_1 + \hbar^2\eta_2 + \cdots = y + \hbar(\frac{1}{2}xy^2) + \hbar^2(\frac{1}{3}x^2y^3) + \hbar^3(\frac{1}{4}x^3y^4 - \frac{1}{2}x^2y^3 + \frac{1}{4}xy^2) + \ldots.$$

Here η_{r+1} is obtained from η_r as follows: Writing η_r as a pseudopolynomial linear combination of monomials of the form $x^i y^j$ replace y^n everywhere by $\frac{n}{n+1}xy^{n+1} - \frac{n-1}{2}y^n$; then $\eta_{r+1} = x\hat{\eta}_r$. The coefficients, while small at the start, will ultimately grow rapidly. However, only monomials of the form $x^m y^{m+1}, m = 0, 1, 2\dots$ appear in y' so we can reorder the series into one of the form $y + a_1 xy^2 + a_2 x^2 y^3 + \dots$, the coefficients a_i being power series in $\hbar$. A simple recursion then gives

$$y' = y + a_1(\hbar)xy^2 + a_2(\hbar)x^2y^3 + \dots, \qquad \text{where } \ a_r(\hbar) = \frac{\hbar^{r+1}}{(1+\hbar)^{r+1} - 1}.$$

This series is singular whenever $1+\hbar$ is a root of unity, showing that $W_1[[\hbar]]$ is not isomorphic $W_q[[\hbar]]$ when q is a root of unity, a stronger assertion than that W_1 is not isomorphic to W_q. The latter was evident from the start: The equation $qxy - yx = 1$ yields inductively that $q^n x^n y - yx^n = (1+q+q^2+\dots+q^{n-1})x^{n-1}$ for these imply that $q^{n+1}x^{n+1}y - yx^{+1} = (q^n x^n y - yx^n)x + q^n x^n(qxy - yx) = ((1+q+q^2+\dots+q^{n-1})x^{n-1})x + q^n x^n$. Therefore, if q is a primitive nth root of unity in k then x^n is central in W_q, and similarly so is y^n, so the algebra may be viewed as a module over the polynomia ring $k[x^n, y^n]$. It is close in some respects to being a matrix ring. If q is a primitive nth root of unity and ξ, η are variables, then the matrices

$$\begin{aligned} X &= \xi \operatorname{diag}(1, q, q^2, \dots, q^{N-1}) \\ Y &= \xi^{-1}((q-1)^{-1} \operatorname{diag}(1, q^{n-1}, q^{n-2}, \dots, q) + \eta(e_{12} + e_{23} + \dots e_{n-1,n} + e_{n1})) \end{aligned}$$

satisfy

$$qXY - YX = \mathbf{1}_n, \quad X^n = \xi^n \mathbf{1}_n, \quad Y^n = \xi^{-n}((q-1)^n + \eta^n)\mathbf{1}_n$$

so there is a monomorphism of W_q into the ring of $n \times n$ matrices over $k[\xi, \xi^{-1}, \eta]$.

There is another isomorphism $W_1[[\hbar]] \cong W_{1+\hbar}[[\hbar]]$ given by the second author and Zhang, [5]. Writing D for d/dx one has (as operators) $xD - Dx = -1$, so x and $-D$ satisfy the same relation as do x and y in W_1. The element

$$\zeta = \frac{(1/x)(e^{\hbar xD} - 1)}{e^{\hbar} - 1}$$

then has the property that $\zeta x = e^{\hbar} x \zeta + 1$, for note that

$$\zeta x = x^{-1}\frac{e^{\hbar xD} - 1}{e^{\hbar} - 1}x = (e^{\hbar x(x^{-1}Dx)} - 1)/(e^{\hbar} - 1) \quad \text{and} \quad x^{-1}Dx = D + x^{-1},$$

from which the equation follows. Therefore, setting $y_\hbar = x^{-1}(e^{-\hbar xy} - 1)/(e^{\hbar} - 1)$ one has $e^{\hbar} xy_\hbar - y_\hbar x = 1$. Here $e^{\hbar}$ takes the place of q and the expression explodes, as before, when q is a root of unity.

Digressing briefly, one can express $(xD)^n$ as a linear combination of the $x^k D^k, k = 1, \dots, n$ using the Stirling numbers of the second kind (the number of partitions of a set of n elements into k non-empty subsets),

$$S(n,k) = \frac{1}{k!}\sum_{i=0}^{k-1}(-1)^i\binom{k}{i}(k-i)^n.$$

With this one has (cf [15, p.144], [18])

$$(xD)^n = \sum_{k=1}^{n} S(n,k)x^kD^k.$$

(One can show by induction that

$$D^qx^r = x^rD^q + qrx^{r-1}D^{q-1} + \binom{q}{2}r(r-1)x^{r-2}D^{q-2} + \dots$$

which together with the recursion $S(n+1,k+1) = S(n,k) + (k+1)S(n,k+1)$ yields the formula for the $S(n,k)$.) With this, $e^{-\hbar xy}$ can be expressed as a series in the monomials x^ny^n with coefficients which are polynomials in $\hbar$. The Stirling numbers with alternating signs (as required in the formula for $y_\hbar$) also appear as entries in certain matrices. Let Λ be the infinite diagonal matrix with diagonal entries -1, -2, -3,...., let N be the infinite lower triangular matrix with 1 everywhere on the first diagonal below the main diagonal and 0 elsewhere, and set $S = \Lambda + N$. The recursion formula shows that $(-1)^{n+r}S(n,r)$ is the rth entry in the first column of S^{n-1}. Therefore, letting e_1 be the infinite column vector with 1 in the first entry and 0 elsewhere, we have

$$e^{-\hbar xy} = (xy, x^2y^2, x^3y^3, \dots)\cdot S^{-1}(e^{\hbar S}-1)\cdot e_1.$$

The eigenvectors of S are the columns of e^N (the ith column corresponding to the eigenvalue $-i$) so $S = e^N\Lambda e^{-N}$. Substituting this into the foregoing shows that when $e^{-\hbar xy}$ is written in terms of $xy, x^2y^2, x^3y^3\dots$ the expression actually converges very rapidly. Writing $S_n(r)$ for $(-1)^{n+r}S(n,r)$, another way to calculate these numbers is to observe that $(-1)^{n-1}e^x(Dx)^{n-1}e^{-x} = [(1-D)x]^{n-1}\cdot 1 = S_n(n)x^n + S_n(n-1)x^{n-1} + S_n(n-2)x^{n-2} + \dots + S_n(1)$, where $S_n(0) = 0$. As with the formula for y' above, difficulty arises only when $q = e^\hbar$ is a root of unity. Note that by Rodrigues' formula the Laguerre polynomials, which appear in the quantization of the hydrogen atom, are given by $L_n(x) = (1/n!)e^xD^nx^ne^{-x}$. With the foregoing one can also use the Stirling numbers to rewrite the polynomials $p_n(x) = e^x(Dx)^ne^{-x}$ in terms of Laguerre polynomials.

Before computing the infinitesimal deformations of the diagram

$$\mathbb{W}_1 := \quad B = k[x] \xrightarrow{f} W_1(=A) \xleftarrow{g} k[y] = C$$

naturally associated to the Weyl algebra we compute for comparison those of

$$\mathbb{W}_0 := \quad B = k[x] \xrightarrow{f} k[x,y] \xleftarrow{g} k[y] = C$$

since $\mathbb{W}_1$ is a deformation of $\mathbb{W}_0$. (Note that in both diagrams the rings $B = k[x]$ and $C = k[y]$, although isomorphic, are treated distinctly since their images under the inclusion morphisms are algebraically independent, unlike the case of the four-punctured sphere. There the analogous diagram would have been rigid.) Every 2-cocycle of $k[x,y]$ is cohomologous to on of the form $a\partial_x \smile \partial_y$, $a \in k[x,y]$. Now $(a\partial_x \smile \partial_y)f = a\partial_x f \smile \partial_y f = 0$ since $\partial_y f = 0$ and similarly $(a\partial_x \smile \partial_y)g = 0$. From this it is easy to check that every 2-cocycle $\Gamma = (\Gamma^B, \Gamma^C, \Gamma^{k[x,y}, \Gamma^f, \Gamma^g)$ of $\mathbb{W}_0$) with coefficients in itself is cohomologous to one of the form $(0, 0, a\partial_x \smile \partial_y, \Gamma^f, G^g)$ where now we must have $\Gamma^f \in \mathrm{Der}(B, k[x,y]), \Gamma^g \in \mathrm{Der}(B, k[x,y])$ and the latter cocycle is in fact a coboundary whenever $a = 0$. Therefore, the infinitesimal deformations of $\mathbb{W}_0$ can be identified with the elements a of $k[x,y]$, the same as the infinitesimal deformations of $k[x,y]$, a free module of rank one over $k[x,y]$.

However, unlike $k[x,y]$ all of whose cohomology is killed by deformation to W_1, not all the cohomology of the diagram $\mathbb{W}_{\not{\mathbb{k}}}$ disappears when it is deformed to the diagram $\mathbb{W}_1$. Since all the algebras in this diagram are now absolutely rigid, every two cocycle is cohomologous to one of the form $\Gamma = (0, 0, 0, \Gamma^f, G^g)$ (a special case of the comment at the end of Secton 3), the only restriction on which is that Γ^f, Γ^g be derivations of $k[x]$ into W and of $k[y]$ into W, respectively. Such a cocycle is a coboundary precisely when there are derivations $\gamma^B, \gamma^C, \gamma^{W_1}$ of B, C, W_1, respectively, such that $\Gamma^f = f\gamma^B - \gamma^{W_1}f$, $\Gamma^g = g\gamma^C - \gamma^{W_1}g$. Every derivation of W_1 is inner, so there is $\chi in W_1$ such that $\Gamma^{W_1} = \mathrm{ad}\,\chi$. Then $\Gamma^f(x) = \alpha(x) - [\chi, x]$ for some polynomial α and similarly $\Gamma^g(y) = \beta(y) - [\chi, y]$ for some polynomial β. The first equality can always be satisfied by choosing ξ suitably (as could the second), so every two-cocycle Γ is cohomologous to one of the form $(0, 0, 0, 0, \Gamma^g)$ where now we can only replace Γ^g by $\Gamma^g - (\gamma^C g - \mathrm{ad}\,\chi)$ with ξ having the property that $[\xi, x]$ is a polynomial in x; in effect, ξ must be a linear combination of monomials of the form $x^r y$ and therefore $[\xi, x]$ is an arbitrary linear combination of the same monomials. Finally, $\Gamma^g(y)$ is an arbitrary polynomial in y, so the infinitesimal deformations of $\mathbb{W}_1$ can be identified with the linear combinations of those monomials of the form $x^i y^j$ omitting those which are just powers of y (including 1) or of the form $x^r y$ for some r. The interpretation of these pseudopolynomials as infinitesimal deformations is clear: if μ is one of them then we should like to replace y by some y' which is a power series in $\hbar$ beginning $y' = y + \hbar\mu + \hbar^2\mu_2 + \hbar^3\mu_3 + \dots$. This is exactly what we did at the beginning of this example where we had

$$\mu = \frac{1}{2}xy^2, \quad \mu_2 = \frac{1}{3}x^2y^3, \quad \mu_3 = \frac{1}{4}x^3y^4 - \frac{1}{2}x^2y^3 + \frac{1}{4}xy^2, \dots.$$

As with the four-punctured sphere it is easy to check that $H^3(\mathbb{W}_{\not{\mathbb{k}}}, \mathbb{W}_{\not{\mathbb{k}}}) = 0$ so there are no obstructions to any infinitesimal. This is also evident from the fact that we could choose any power series in $\hbar$ for y' and obtain a deformation and as long as the leading term did not represent a oboundary it would be non-trivial. The series exhibited were simply the ones that produced W_q where q was $1 + \hbar$ or $e^\hbar$. We began the example by showing that the deformation was trivial as a deformation of $W_1\hbar$, as the classical deformation of a single algebra asserts it must be, but the diagram cohomology shows that it is not trivial viewed over the original ground field k. Both examples demonstrate that to understand the deformation of algebras one must consider not just that of single algebras but of diagrams even if at the start one was concrned with the deformation of but a single specific algeba.

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