12-2010

Invariant and Coinvariant Spaces for the Algebra of Symmetric Polynomials in Non-Commuting Variables

Francois Bergeron

Aaron Lauve
Loyola University Chicago, alauve@luc.edu

Invariant and coinvariant spaces for the algebra of symmetric polynomials in non-commuting variables

François Bergeron*
LaCIM
Université du Québec à Montréal
Montréal (Québec) H3C 3P8, CANADA
bergeron.francois@uqam.ca

Aaron Lauve
Department of Mathematics
Texas A&M University
College Station, TX 77843, USA
lauve@math.tamu.edu

Submitted: Oct 2, 2009; Accepted: Nov 26, 2010; Published: Dec 10, 2010
Mathematics Subject Classification: 05E05

Abstract

We analyze the structure of the algebra $\mathbb{K}\langle\mathbf{x}\rangle^{\mathfrak{S}_n}$ of symmetric polynomials in non-commuting variables in so far as it relates to $\mathbb{K}[\mathbf{x}]^{\mathfrak{S}_n}$, its commutative counterpart. Using the "place-action" of the symmetric group, we are able to realize the latter as the invariant polynomials inside the former. We discover a tensor product decomposition of $\mathbb{K}\langle\mathbf{x}\rangle^{\mathfrak{S}_n}$ analogous to the classical theorems of Chevalley, Shephard-Todd on finite reflection groups.

Résumé. Nous analysons la structure de l'algèbre $\mathbb{K}\langle\mathbf{x}\rangle^{\mathfrak{S}_n}$ des polynômes symétriques en des variables non-commutatives pour obtenir des analogues des résultats classiques concernant la structure de l'anneau $\mathbb{K}[\mathbf{x}]^{\mathfrak{S}_n}$ des polynômes symétriques en des variables commutatives. Plus précisément, au moyen de "l'action par positions", on réalise $\mathbb{K}[\mathbf{x}]^{\mathfrak{S}_n}$ comme sous-module de $\mathbb{K}\langle\mathbf{x}\rangle^{\mathfrak{S}_n}$. On découvre alors une nouvelle décomposition de $\mathbb{K}\langle\mathbf{x}\rangle^{\mathfrak{S}_n}$ comme produit tensorial, obtenant ainsi un analogues des théorèmes classiques de Chevalley et Shephard-Todd.

1 Introduction

One of the more striking results of invariant theory is certainly the following: if W is a finite group of $n\times n$ matrices (over some field $\mathbb{K}$ containing $\mathbb{Q}$), then there is a W-module decomposition of the polynomial ring $S=\mathbb{K}[\mathbf{x}]$, in variables $\mathbf{x}=\{x_1,x_2,\ldots,x_n\}$, as a tensor product

$$S \simeq S_W \otimes S^W \tag{1}$$

*F. Bergeron is supported by NSERC-Canada and FQRNT-Québec.

if and only if W is a group generated by (pseudo) reflections. As usual, S is afforded a natural W-module structure by considering it as the symmetric space on the defining vector space X^* for W, e.g., $w \cdot f(\mathbf{x}) = f(\mathbf{x} \cdot w)$. It is customary to denote by S^W the ring of W-invariant polynomials for this action. To finish parsing (1), recall that S_W stands for the **coinvariant space**, i.e., the W-module

$$S_W := S/\left\langle S_+^W \right\rangle \tag{2}$$

defined as the quotient of S by the ideal generated by constant-term free W-invariant polynomials. We give S an $\mathbb{N}$-grading by degree in the variables $\mathbf{x}$. Since the W-action on S preserves degrees, both S^W and S_W inherit a grading from the one on S, and (1) is an isomorphism of graded W-modules. One of the motivations behind the quotient in (2) is to eliminate trivially redundant copies of irreducible W-modules inside S. Indeed, if $\mathcal{V}$ is such a module and f is any W-invariant polynomial with no constant term, then $\mathcal{V}f$ is an isomorphic copy of $\mathcal{V}$ living within $\left\langle S_+^W \right\rangle$. Thus, the coinvariant space S_W is the more interesting part of the story.

The context for the present paper is the algebra $T = \mathbb{K}\langle \mathbf{x} \rangle$ of noncommutative polynomials, with W-module structure on T obtained by considering it as the tensor space on the defining space X^* for W. In the special case when W is the symmetric group $\mathfrak{S}_n$, we elucidate a relationship between the space S^W and the subalgebra T^W of W-invariants in T. The subalgebra T^W was first studied in [4, 20] with the aim of obtaining noncommutative analogs of classical results concerning symmetric function theory. Recent work in [2, 15] has extended a large part of the story surrounding (1) to this noncommutative context. In particular, there is an explicit $\mathfrak{S}_n$-module decomposition of the form $T \simeq T_{\mathfrak{S}_n} \otimes T^{\mathfrak{S}_n}$ [2, Theorem 8.7]. See [7] for a survey of other results in noncommutative invariant theory.

By contrast, our work proceeds in a somewhat complementary direction. We consider $\mathcal{N} = T^{\mathfrak{S}_n}$ as a tower of $\mathfrak{S}_d$-modules under the "place-action" and realize $S^{\mathfrak{S}_n}$ inside $\mathcal{N}$ as a subspace Λ of invariants for this action. This leads to a decomposition of $\mathcal{N}$ analogous to (1). More explicitly, our main result is as follows.

Theorem 1. *There is an explicitly constructed subspace $\mathcal{C}$ of $\mathcal{N}$ so that $\mathcal{C}$ and the place-action invariants Λ exhibit a graded vector space isomorphism*

$$\mathcal{N} \simeq \mathcal{C} \otimes \Lambda. \tag{3}$$

An analogous result holds in the case $|\mathbf{x}| = \infty$. An immediate corollary in either case is the Hilbert series formula

$$\mathrm{Hilb}_t(\mathcal{C}) = \mathrm{Hilb}_t(\mathcal{N}) \prod_{i=1}^{|\mathbf{x}|} (1 - t^i). \tag{4}$$

Here, the **Hilbert series** of a graded space $\mathcal{V} = \bigoplus_{d \geq 0} \mathcal{V}_d$ is the formal power series defined as

$$\mathrm{Hilb}_t(\mathcal{V}) = \sum_{d \geq 0} \dim \mathcal{V}_d \, t^d,$$

where $\mathcal{V}_d$ is the **homogeneous degree** d **component** of $\mathcal{V}$. The fact that (4) expands as a series in $\mathbb{N}[\![t]\!]$ is not at all obvious, as one may check that the Hilbert series of $\mathcal{N}$ is

$$\mathrm{Hilb}_t(\mathcal{N}) = 1 + \sum_{k=1}^{|\mathbf{x}|} \frac{t^k}{(1-t)(1-2\,t)\cdots(1-k\,t)}\,. \tag{5}$$

In Sections 2 and 3, we recall the relevant structural features of S and T. Section 4 describes the place-action structure of T and the original motivation for our work. Our main results are proven in Sections 5 and 6. We underline that the harder part of our work lies in working out the case $|\mathbf{x}| < \infty$. This is accomplished in Section 6. If we restrict ourselves to the case $|\mathbf{x}| = \infty$, both $\mathcal{N}$ and Λ become Hopf algebras and our results are then consequences of a general theorem of Blattner, Cohen and Montgomery. As we will see in Section 5, stronger results hold in this simpler context. For example, (4) may be refined to a statement about "shape" enumeration.

2 The algebra $S^{\mathfrak{S}}$ of symmetric functions

2.1 Vector space structure of $S^{\mathfrak{S}}$

We specialize our introductory discussion to the group $W = \mathfrak{S}_n$ of permutation matrices (writing $|\mathbf{x}| = n$). The action on $S = \mathbb{K}[\mathbf{x}]$ is simply the **permutation action** $\sigma{\cdot}x_i = x_{\sigma(i)}$ and $S^{\mathfrak{S}_n}$ comprises the familiar symmetric polynomials. We suppress n in the notation and denote the subring of symmetric polynomials by $S^{\mathfrak{S}}$. (Note that upon sending n to ∞, the elements of $S^{\mathfrak{S}}$ become formal series in $\mathbb{K}[\![\mathbf{x}]\!]$ of bounded degree; we call both finite and infinite versions "functions" in what follows to affect a uniform discussion.) A monomial in S of degree d may be written as follows: given an r-subset $\mathbf{y} = \{y_1, y_2, \ldots, y_r\}$ of $\mathbf{x}$ and a **composition** of d into r parts, $\boldsymbol{a} = (a_1, a_2, \ldots, a_r)$ $(a_i > 0)$, we write $\mathbf{y}^{\boldsymbol{a}}$ for $y_1^{a_1} y_2^{a_2} \cdots y_r^{a_r}$. We assume that the variables y_i are naturally ordered, so that whenever $y_i = x_j$ and $y_{i+1} = x_k$ we have $j < k$. Reordering the entries of a composition $\boldsymbol{a}$ in decreasing order results in a partition $\lambda(\boldsymbol{a})$ called the **shape** of $\boldsymbol{a}$. Summing over monomials $\mathbf{y}^{\boldsymbol{a}}$ with the same shape leads to the monomial symmetric function

$$m_\mu = m_\mu(\mathbf{x}) := \sum_{\lambda(\boldsymbol{a})=\mu,\ \mathbf{y}\subseteq\mathbf{x}} \mathbf{y}^{\boldsymbol{a}}.$$

Letting $\mu = (\mu_1, \mu_2, \ldots, \mu_r)$ run over all partitions of $d = |\mu| = \mu_1 + \mu_2 + \cdots + \mu_r$ gives a basis for $S_d^{\mathfrak{S}}$. As usual, we set $m_0 := 1$ and agree that $m_\mu = 0$ if μ has too many parts (i.e., $n < r$).

2.2 Dimension enumeration

A fundamental result in the invariant theory of $\mathfrak{S}_n$ is that $S^{\mathfrak{S}}$ is generated by a family $\{f_k\}_{1\le k\le n}$ of algebraically independent symmetric functions, having respective degrees

$\deg f_k = k$. (One may choose $\{m_k\}_{1\le k\le n}$ for such a family.) It follows that the Hilbert series of $S^{\mathfrak{S}}$ is

$$\mathrm{Hilb}_t(S^{\mathfrak{S}}) = \prod_{i=1}^{n} \frac{1}{1-t^i}. \tag{6}$$

Recalling that the Hilbert series of S is $(1-t)^{-n}$, we see from (1) and (6) that the Hilbert series for the coinvariant space $S_{\mathfrak{S}}$ is the well-known t-analog of $n!$:

$$\prod_{i=1}^{n} \frac{1-t^i}{1-t} = \prod_{i=1}^{n} (1+t+\cdots+t^{i-1}). \tag{7}$$

In particular, contrary to the situation in (4), the series $\mathrm{Hilb}_t(S)/\mathrm{Hilb}_t(S^{\mathfrak{S}})$ in $\mathbb{Q}[\![t]\!]$ *obviously* belongs to $\mathbb{N}[\![t]\!]$.

2.3 Algebra and coalgebra structures of $S^{\mathfrak{S}}$

Given partitions μ and ν, there is an explicit multiplication rule for computing the product $m_\mu \cdot m_\nu$. In lieu of giving the formula, see [2, §4.1], we simply give an example

$$m_{21} \cdot m_{11} = 3\, m_{2111} + 2\, m_{221} + 2\, m_{311} + m_{32} \tag{8}$$

and highlight two features relevant to the coming discussion.

First, we note that if $n < 4$, then the first term is equal to zero. However, if n is sufficiently large then analogs of this term always appear with positive integer coefficients. If $\mu = (\mu_1, \mu_2, \ldots, \mu_r)$ and $\nu = (\nu_1, \nu_2, \ldots, \nu_s)$ with $r \le s$, then the partition indexing the left-most term in $m_\mu m_\nu$ is denoted by $\mu \cup \nu$ and is given by sorting the list $(\mu_1, \ldots, \mu_r, \nu_1, \ldots, \nu_s)$ in increasing order; the right-most term is indexed by $\mu + \nu := (\mu_1 + \nu_1, \ldots, \mu_r + \nu_r, \nu_{r+1}, \ldots, \nu_s)$. Taking $\mu = 31$ and $\nu = 221$, we would have $\mu \cup \nu = 32211$ and $\mu + \nu = 531$.

Second, we point out that the leftmost term (indexed by $\mu \cup \nu$) is indeed a *leading term* in the following sense. An important partial order on partitions takes

$$\lambda \le \mu \qquad \text{iff} \qquad \sum_{i=1}^{k} \lambda_i \;\le\; \sum_{i=1}^{k} \mu_i \;\text{ for all } k.$$

With this ordering, $\mu \cup \nu$ is the least partition occuring with nonzero coefficient in the product of $m_\mu m_\nu$. That is, $S^{\mathfrak{S}}$ is **shape-filtered**: $(S^{\mathfrak{S}})_\lambda \cdot (S^{\mathfrak{S}})_\mu \subseteq \bigoplus_{\nu \ge \lambda \cup \mu} (S^{\mathfrak{S}})_\nu$. Here $(S^{\mathfrak{S}})_\lambda$ denotes the subspace of $S^{\mathfrak{S}}$ indexed by partitions of shape λ (the linear span of m_λ), which we point out in preparation for the noncommutative analog.

The ring $S^{\mathfrak{S}}$ is afforded a coalgebra structure with counit $\varepsilon : S^{\mathfrak{S}} \to \mathbb{K}$ and coproduct $\Delta : S^{\mathfrak{S}}_d \to \bigoplus_{k=0}^{d} S^{\mathfrak{S}}_k \otimes S^{\mathfrak{S}}_{d-k}$ given, respectively, by

$$\varepsilon(m_\mu) = \delta_{\mu,0} \quad \text{and} \quad \Delta(m_\nu) = \sum_{\lambda \cup \mu = \nu} m_\lambda \otimes m_\mu\,.$$

If $|\mathbf{x}| = \infty$, Δ and ε are algebra maps, making $S^{\mathfrak{S}}$ a graded connected Hopf algebra.

3 The algebra $\mathcal{N}$ of noncommutative symmetric functions

3.1 Vector space structure of $\mathcal{N}$

Suppose now that $\mathbf{x}$ denotes a set of non-commuting variables. The algebra $T = \mathbb{K}\langle\mathbf{x}\rangle$ of noncommutative polynomials is graded by degree. A degree d **noncommutative monomial** $\mathbf{z} \in T_d$ is simply a length d "word":

$$\mathbf{z} = z_1 z_2 \cdots z_d, \quad \text{with each} \quad z_i \in \mathbf{x}.$$

In other terms, $\mathbf{z}$ is a function $\mathbf{z} : [d] \to \mathbf{x}$, with $[d]$ denoting the set $\{1, 2, \ldots, d\}$. The permutation-action on $\mathbf{x}$ clearly extends to T, giving rise to the subspace $\mathcal{N} = T^{\mathfrak{S}}$ of noncommutative $\mathfrak{S}$-invariants. With the aim of describing a linear basis for the homogeneous component $\mathcal{N}_d$, we next introduce set partitions of $[d]$ and the type of a monomial $\mathbf{z} : [d] \to \mathbf{x}$. Let $\mathbf{A} = \{A_1, A_2, \ldots, A_r\}$ be a set of subsets of $[d]$. Say $\mathbf{A}$ is a **set partition** of $[d]$, written $\mathbf{A} \vdash [d]$, iff $A_1 \cup A_2 \cup \ldots \cup A_r = [d]$, $A_i \neq \emptyset$ $(\forall i)$, and $A_i \cap A_j = \emptyset$ $(\forall i \neq j)$. The **type** $\tau(\mathbf{z})$ of a degree d monomial $\mathbf{z} : [d] \to \mathbf{x}$ is the set partition

$$\tau(\mathbf{z}) := \{\mathbf{z}^{-1}(x) : x \in \mathbf{x}\} \setminus \{\emptyset\} \quad \text{of} \quad [d],$$

whose parts are the non-empty fibers of the function $\mathbf{z}$. For instance,

$$\tau(x_1 x_8 x_1 x_5 x_8) = \{\{1, 3\}, \{2, 5\}, \{4\}\}.$$

Note that the type of a monomial is a set partition with at most n parts. In what follows, we lighten the heavy notation for set partitions, writing, e.g., the set partition $\{\{1, 3\}, \{2, 5\}, \{4\}\}$ as 13.25.4. We also always order the parts in increasing order of their minimum elements. The **shape** $\lambda(\mathbf{A})$ of a set partition $\mathbf{A} = \{A_1, A_2, \ldots, A_r\}$ is the (integer) partition $\lambda(|A_1|, |A_2|, \ldots, |A_r|)$ obtained by sorting the part sizes of $\mathbf{A}$ in increasing order, and its **length** $\ell(\mathbf{A})$ is its number of parts (r). Observing that the permutation-action is *type preserving*, we are led to index the **monomial** linear basis for the space $\mathcal{N}_d$ by set partitions:

$$m_{\mathbf{A}} = m_{\mathbf{A}}(\mathbf{x}) := \sum_{\tau(\mathbf{z})=\mathbf{A},\ \mathbf{z}\in\mathbf{x}^{[d]}} \mathbf{z}$$

For example, with $n = 2$, we have $m_1 = x_1 + x_2$, $m_{12} = x_1^2 + x_2^2$, $m_{1.2} = x_1 x_2 + x_2 x_1$, $m_{123} = {x_1}^3 + {x_2}^3$, $m_{12.3} = {x_1}^2 x_2 + {x_2}^2 x_1$, $m_{13.2} = x_1 x_2 x_1 + x_2 x_1 x_2$, $m_{1.2.3} = 0$, and so on. (We set $m_{\boldsymbol{\emptyset}} := 1$, taking $\boldsymbol{\emptyset}$ as the unique set partition of the empty set, and we agree that $m_{\mathbf{A}} = 0$ if $\mathbf{A}$ is a set partition with more than n parts.)

3.2 Dimension enumeration and shape grading

Above, we determined that $\dim \mathcal{N}_d$ is the number of set partitions of d into at most n parts. These are counted by the (length restricted) **Bell numbers** $B_d^{(n)}$. Consequently,

(5) follows from the fact that its right-hand side is the ordinary generating function for length restricted Bell numbers. See [10, §2]. We next highlight a finer enumeration, where we grade $\mathcal{N}$ by shape rather than degree.

For each partition μ, we may consider the subspace $\mathcal{N}_\mu$ spanned by those $m_{\mathbf{A}}$ for which $\lambda(\mathbf{A}) = \mu$. This results in a direct sum decomposition $\mathcal{N}_d = \bigoplus_{\mu \vdash d} \mathcal{N}_\mu$. A simple dimension description for $\mathcal{N}_d$ takes the form of a **shape Hilbert series** in the following manner. View commuting variables q_i as marking parts of size i and set $\boldsymbol{q}_\mu := q_{\mu_1} q_{\mu_2} \cdots q_{\mu_r}$. Then

$$\mathrm{Hilb}_{\boldsymbol{q}}(\mathcal{N}_d) = \sum_{\mu \vdash d} \dim \mathcal{N}_\mu \, \boldsymbol{q}_\mu, = \sum_{\mathbf{A} \vdash [d]} q_{\lambda(\mathbf{A})}. \tag{9}$$

Here, $\boldsymbol{q}_\mu$ is a marker for set partitions of shape $\lambda(\mathbf{A}) = \mu$ and the sum is over all partitions into at most n parts. Such a shape grading also makes sense for $S_d^{\mathfrak{S}}$. Summing over all $d \geq 0$ and all μ, we get

$$\mathrm{Hilb}_{\boldsymbol{q}}(S^{\mathfrak{S}}) = \sum_\mu \boldsymbol{q}_\mu = \prod_{i \geq 1}^{n} \frac{1}{1 - q_i}. \tag{10}$$

Using classical combinatorial arguments, one finds the enumerator polynomials $\mathrm{Hilb}_{\boldsymbol{q}}(\mathcal{N}_d)$ are naturally collected in the **exponential generating function**

$$\sum_{d=0}^{\infty} \mathrm{Hilb}_{\boldsymbol{q}}(\mathcal{N}_d) \, \frac{t^d}{d!} = \sum_{m=0}^{n} \frac{1}{m!} \left(\sum_{k=1}^{\infty} q_k \frac{t^k}{k!} \right)^m . \tag{11}$$

See [1, Chap. 2.3], Example 13(a). For instance, with $n = 3$, we have

$$\mathrm{Hilb}_{\boldsymbol{q}}(\mathcal{N}_6) = q_6 + 6\, q_5 q_1 + 15\, q_4 q_2 + 15\, q_4 q_1^2 + 10\, q_3^2 + 60\, q_3 q_2 q_1 + 15\, {q_2}^3,$$

thus $\dim \mathcal{N}_{222} = 15$ when $n \geq 3$. Evidently, the $\boldsymbol{q}$-polynomials $\mathrm{Hilb}_{\boldsymbol{q}}(\mathcal{N}_d)$ specialize to the length restricted Bell numbers $B_d^{(n)}$ when we set all q_k equal to 1.

In view of (10), (11), and Theorem 1, we claim the following refinement of (4).

Corollary 2. *Sending n to ∞, the shape Hilbert series of the space $\mathcal{C}$ is given by*

$$\mathrm{Hilb}_{\boldsymbol{q}}(\mathcal{C}) = \sum_{d \geq 0} d! \exp\left(\sum_{k=1}^{\infty} q_k \frac{t^k}{k!} \right) \Bigg|_{t^d} \prod_{i \geq 1} \big(1 - q_i\big), \tag{12}$$

with $(-)|_{t^d}$ standing for the operation of taking the coefficient of t^d.

This refinement of (4) will follow immediately from the isomorphism $\mathcal{C} \otimes \Lambda \to \mathcal{N}$ in Section 5, which is shape-preserving in an appropriate sense. Thus we have the expansion

$$\begin{aligned} \mathrm{Hilb}_{\boldsymbol{q}}(\mathcal{C}) \;&=\; 1 + 2\, q_2 q_1 + \big(3\, q_3 q_1 + 2\, {q_2}^2 + 3\, q_2 {q_1}^2\big) \\ &\qquad + \big(4\, q_4 q_1 + 9\, q_3 q_2 + 6\, q_3 {q_1}^2 + 10\, {q_2}^2 q_1 + 4\, q_2 {q_1}^3\big) + \cdots \end{aligned}$$

3.3 Algebra and coalgebra structures of $\mathcal{N}$

Since the action of $\mathfrak{S}$ on T is multiplicative, it is straightforward to see that $\mathcal{N}$ is a subalgebra of T. The *multiplication rule* in $\mathcal{N}$, expressing a product $m_{\mathbf{A}} \cdot m_{\mathbf{B}}$ as a sum of basis vectors $\sum_{\mathbf{C}} m_{\mathbf{C}}$, is easy to describe. Since we make heavy use of the rule later, we develop it carefully here. We begin with an example (digits corresponding to $\mathbf{B} = \mathbf{1.2}$ appear in bold):

$$\begin{aligned} m_{13.2} \cdot m_{\mathbf{1.2}} \;=\; & m_{13.2.\mathbf{4.5}} + m_{13\mathbf{4}.2.\mathbf{5}} + m_{13\mathbf{5}.2.\mathbf{4}} \\ & + m_{13.2\mathbf{4}.\mathbf{5}} + m_{13.2\mathbf{5}.\mathbf{4}} + m_{13\mathbf{5}.2\mathbf{4}} + m_{13\mathbf{4}.2\mathbf{5}} \end{aligned} \tag{13}$$

Notice that the shapes indexing the first and last terms in (13) are the partitions $\lambda(13.2) \cup \lambda(1.2)$ and $\lambda(13.2) + \lambda(1.2)$. As was the case in $S^{\mathfrak{S}}$, one of these shapes, namely $\lambda(\mathbf{A}) + \lambda(\mathbf{B})$, will always appear in the product, while appearance of the shape $\lambda(\mathbf{A}) \cup \lambda(\mathbf{B})$ depends on the cardinality of $\mathbf{x}$.

Let us now describe the multiplication rule. Given any $D \subseteq \mathbb{N}$ and $k \in \mathbb{N}$, we write D^{+k} for the set

$$D^{+k} := \{a + k : a \in D\}.$$

By extension, for any set partition $\mathbf{A} = \{A_1, A_2, \ldots, A_r\}$ we set $\mathbf{A}^{+k} := \{A_1^{+k}, A_2^{+k}, \ldots, A_r^{+k}\}$. Also, we set $\mathbf{A}_{\widehat{i}} := \mathbf{A} \setminus \{A_i\}$. Next, if $\mathcal{X}$ is a collection of set partitions of D, and A is a set disjoint from D, we extend $\mathcal{X}$ to partitions of $A \cup D$ by the rule

$$A \diamond \mathcal{X} := \bigcup_{\mathbf{B} \in \mathcal{X}} \{A\} \cup \mathbf{B}.$$

Finally, given partitions $\mathbf{A} = \{A_1, A_2, \ldots, A_r\}$ of C and $\mathbf{B} = \{B_1, B_2, \ldots, B_s\}$ of D (disjoint from C), their **quasi-shuffles** $\mathbf{A} \uplus \mathbf{B}$ are the set partitions of $C \cup D$ recursively defined by the rules:

- $\mathbf{A} \uplus \boldsymbol{\emptyset} = \boldsymbol{\emptyset} \uplus \mathbf{A} := \mathbf{A}$, where $\boldsymbol{\emptyset}$ is the unique set partition of the empty set;
- $\mathbf{A} \uplus \mathbf{B} := \bigcup_{i=0}^{s} (A_1 \cup B_i) \diamond \Big(\mathbf{A}_{\widehat{1}} \uplus (\mathbf{B}_{\widehat{i}})\Big)$, taking B_0 to be the empty set.

If $\mathbf{A} \vdash [c]$ and $\mathbf{B} \vdash [d]$, we abuse notation and write $\mathbf{A} \uplus \mathbf{B}$ for $\mathbf{A} \uplus \mathbf{B}^{+c}$. As shown in [2, Prop. 3.2], the multiplication rule for $m_{\mathbf{A}}$ and $m_{\mathbf{B}}$ in $\mathcal{N}$ is

$$m_{\mathbf{A}} \cdot m_{\mathbf{B}} = \sum_{\mathbf{C} \in \mathbf{A} \uplus \mathbf{B}} m_{\mathbf{C}}\,. \tag{14}$$

The subalgebra $\mathcal{N}$, like its commutative analog, is freely generated by certain monomial symmetric functions $\{m_{\mathbf{A}}\}_{\mathbf{A} \in \mathcal{A}}$, where $\mathcal{A}$ is some carefully chosen collection of set partitions. This is the main theorem of Wolf [20]. We use two such collections later, our choice depending on whether or not $|\mathbf{x}| < \infty$.

The operation $(-)^{+k}$ has a left inverse called the **standardization** operator and denoted by "$(-)^{\downarrow}$". It maps set partitions $\mathbf{A}$ of any cardinality d subset $D \subseteq \mathbb{N}$ to set partitions of $[d]$, by defining $\mathbf{A}^{\downarrow}$ as the pullback of $\mathbf{A}$ along the unique increasing bijection from $[d]$ to D. For example, $(18.4)^{\downarrow} = 13.2$ and $(18.4.67)^{\downarrow} = 15.2.34$. The coproduct Δ and counit ε on $\mathcal{N}$ are given, respectively, by

$$\Delta(m_{\mathbf{A}}) = \sum_{\mathbf{B} \cup \mathbf{C} = \mathbf{A}} m_{\mathbf{B}^{\downarrow}} \otimes m_{\mathbf{C}^{\downarrow}} \qquad \text{and} \qquad \varepsilon(m_{\mathbf{A}}) = \delta_{\mathbf{A},\emptyset},$$

where $\mathbf{B} \cup \mathbf{C} = \mathbf{A}$ means that $\mathbf{B}$ and $\mathbf{C}$ form complementary subsets of $\mathbf{A}$. In the case $|\mathbf{x}| = \infty$, the maps Δ and ε are algebra maps, making $\mathcal{N}$ a graded connected Hopf algebra.

4 The place-action of $\mathfrak{S}$ on $\mathcal{N}$

4.1 Swapping places in T_d and $\mathcal{N}_d$

On top of the permutation-action of the symmetric group $\mathfrak{S}_{\mathbf{x}}$ on T, we also consider the "place-action" of $\mathfrak{S}_d$ on the degree d homogeneous component T_d. Observe that the permutation-action of $\sigma \in \mathfrak{S}_{\mathbf{x}}$ on a monomial $\mathbf{z}$ corresponds to the functional composition

$$\sigma \circ \mathbf{z} : [d] \xrightarrow{\mathbf{z}} \mathbf{x} \xrightarrow{\sigma} \mathbf{x}$$

(notation as in Section 3.1). By contrast, the **place-action** of $\rho \in \mathfrak{S}_d$ on $\mathbf{z}$ gives the monomial

$$\mathbf{z} \circ \rho : [d] \xrightarrow{\rho} [d] \xrightarrow{\mathbf{z}} \mathbf{x},$$

composing ρ on the right with $\mathbf{z}$. In the linear extension of this action to all of T_d, it is easily seen that $\mathcal{N}_d$ (even each $\mathcal{N}_\mu$) is an invariant subspace of T_d. Indeed, for any set partition $\mathbf{A} = \{A_1, A_2, \ldots, A_r\} \vdash [d]$ and any $\rho \in \mathfrak{S}_d$, one has

$$m_{\mathbf{A}} \cdot \rho = m_{\rho^{-1} \cdot \mathbf{A}} \tag{15}$$

(see [15, §2]), where as usual $\rho^{-1} \cdot \mathbf{A} := \{\rho^{-1}(A_1), \rho^{-1}(A_2), \ldots, \rho^{-1}(A_r)\}$.

4.2 The place-action structure of $\mathcal{N}$

Notice that the action in (15) is shape-preserving and transitive on set partitions of a given shape (i.e., $\mathcal{N}_\mu$ is an $\mathfrak{S}_d$-submodule of $\mathcal{N}_d$ for each $\mu \vdash d$). It follows that there is exactly one copy of the trivial $\mathfrak{S}_d$-module inside $\mathcal{N}_\mu$ for each $\mu \vdash d$, that is, a basis for the place-action invariants in $\mathcal{N}_d$ is indexed by partitions. We choose as basis the functions

$$\mathbf{m}_\mu := \frac{1}{(\dim \mathcal{N}_\mu)\, \mu^{!}} \sum_{\lambda(\mathbf{A}) = \mu} m_{\mathbf{A}}, \tag{16}$$

with $\mu^{!} = a_1! a_2! \cdots$ whenever $\mu = 1^{a_1} 2^{a_2} \cdots$. The rationale for choosing this normalizing coefficient will be revealed in (20).

To simplify our discussion of the structure of $\mathcal{N}$ in this context, we will say that $\mathfrak{S}$ acts on $\mathcal{N}$ rather than being fastidious about underlying in each situation that individual

$\mathcal{N}_d$'s are being acted upon on the right by the corresponding group $\mathfrak{S}_d$. We denote the set $\mathcal{N}^{\mathfrak{S}}$ of **place-invariants** by Λ in what follows. To summarize,

$$\Lambda = \operatorname{span}\{\mathbf{m}_\mu : \mu \text{ a partition of } d,\, d \in \mathbb{N}\}\,. \tag{17}$$

The pair $(\mathcal{N}, \Lambda)$ begins to look like the pair $(S, S^{\mathfrak{S}})$ from the introduction. This was the observation that originally motivated our search for Theorem 1.

We next decompose $\mathcal{N}$ into irreducible place-action representations. Although this can be worked out for any value of n, the results are more elegant when we send n to infinity. Recall that the **Frobenius characteristic** of a $\mathfrak{S}_d$-module $\mathcal{V}$ is a symmetric function

$$\mathrm{Frob}(\mathcal{V}) = \sum_{\mu \vdash d} v_\mu\, s_\mu,$$

where s_μ is a Schur function (the character of "the" irreducible $\mathfrak{S}_d$ representation $\mathcal{V}_\mu$ indexed by μ) and v_μ is the multiplicity of $\mathcal{V}_\mu$ in $\mathcal{V}$. To reveal the $\mathfrak{S}_d$-module structure of $\mathcal{N}_\mu$, we use (15) and techniques from the theory of combinatorial species.

Proposition 3. *For a partition $\mu = 1^{a_1}2^{a_2}\cdots k^{a_k}$, having a_i parts of size i, we have*

$$\mathrm{Frob}(\mathcal{N}_\mu) = h_{a_1}[h_1]\, h_{a_2}[h_2] \cdots h_{a_k}[h_k], \tag{18}$$

with $f[g]$ denoting plethysm of f and g, and h_i denoting the i^{th} homogeneous symmetric function.

Recall that the **plethysm** $f[g]$ of two symmetric functions is obtained by linear and multiplicative extension of the rule $p_k[p_\ell] := p_{k\,\ell}$, where the p_k's denote the usual power sum symmetric functions (see [12, I.8] for notation and details).

Let Par denote the combinatorial species of set partitions. So $\mathsf{Par}[n]$ denotes the set partitions of $[n]$ and permutations $\sigma\colon [n] \to [n]$ are transferred in a natural way to permutations $\mathsf{Par}[\sigma]\colon \mathsf{Par}[n] \to \mathsf{Par}[n]$. The number $\operatorname{fix}\mathsf{Par}[\sigma]$ of fixed points of this permutation is the same as the character $\chi_{\mathsf{Par}[n]}(\sigma)$ of the $\mathfrak{S}_n$-representation given by $\mathsf{Par}[n]$. Given a partition $\mu = 1^{a_1}2^{a_2}\cdots k^{a_k}$, put $z_\mu := 1^{a_1}a_1!2^{a_2}a_2!\cdots k^{a_k}a_k!$. (There are $n!/z_\mu$ permutations in $\mathfrak{S}_n$ of cycle type μ.) The **cycle index series** for Par is defined by

$$Z_{\mathsf{Par}} = \sum_{n\geq 0}\sum_{\mu\vdash n} \operatorname{fix}\mathsf{Par}[\sigma_\mu]\,\frac{p_\mu}{z_\mu}\,,$$

where σ_μ is any permutation with cycle type μ and $p_\mu := p_1^{a_1}p_2^{a_2}\cdots p_k^{a_k}$ (taking p_i as the i-th power sum symmetric function).

Proof. Recall that the Schur and power sum symmetric functions are related by

$$s_\lambda = \sum_{\mu\vdash|\lambda|} \chi_\lambda(\sigma_\mu)\,\frac{p_\mu}{z_\mu}\,,$$

so $Z_{\mathsf{Par}} = \mathrm{Frob}(\mathsf{Par})$. Because Par is the composition $\mathsf{E} \circ \mathsf{E}_+$ of the species of sets and nonempty sets, we also know that its cycle index series is given by plethystic substitution: $Z_{\mathsf{E}\circ\mathsf{E}_+} = Z_{\mathsf{E}}[Z_{\mathsf{E}_+}]$. See Theorem 2 and (12) in [1, I.4]. Combining these two results will give the proof.

First, we are only interested in that piece of $\mathrm{Frob}(\mathsf{Par})$ coming from set partitions of shape μ. For this we need weighted combinatorial species. If a set partition has shape μ, give it the weight $q_1^{a_1} q_2^{a_2} \cdots q_k^{a_k}$ in the cycle index series enumeration. The relevant identity is

$$Z_{\mathbf{P}}(\boldsymbol{q}) = \exp \sum_{k\geq 1} \frac{1}{k}\bigg(\exp\Big(\sum_{j\geq 1} q_j^k \frac{p_{jk}}{j}\Big) - 1\bigg)$$

(cf. Example 13(c) of Chapter 2.3 in [1]). Collecting the terms of weight $\boldsymbol{q}_\mu$ gives $\mathrm{Frob}(\mathcal{N}_\mu)$. We get

$$\mathrm{coeff}_{\boldsymbol{q}_\mu}\left[Z_{\mathsf{Par}}(\boldsymbol{q})\right] = \prod_{i=1}^{k} \Big(\sum_{\lambda\vdash a_i} \frac{p_\lambda}{z_\lambda}\Big)\Big[\sum_{\nu\vdash i} \frac{p_\nu}{z_\nu}\Big].$$

Standard identities [12, (2.14') in I.2] between the h_i's and p_j's finish the proof. □

As an example, we consider $\mu = 222 = 2^3$. Since

$$h_2 = \frac{p_1^2}{2} + \frac{p_2}{2} \qquad \text{and} \qquad h_3 = \frac{p_1^3}{6} + \frac{p_1p_2}{2} + \frac{p_3}{3},$$

a plethysm computation (and a change of basis) gives

$$\begin{aligned} h_3[h_2] &= \frac{p_1^3}{6}\left[\frac{p_1^2}{2} + \frac{p_2}{2}\right] + \frac{p_1p_2}{2}\left[\frac{p_1^2}{2} + \frac{p_2}{2}\right] + \frac{p_3}{3}\left[\frac{p_1^2}{2} + \frac{p_2}{2}\right] \\ &= \frac{1}{6}\left(\frac{p_1^2}{2} + \frac{p_2}{2}\right)^3 + \frac{1}{2}\left(\frac{p_1^2}{2} + \frac{p_2}{2}\right)\left(\frac{p_2^2}{2} + \frac{p_4}{2}\right) + \frac{1}{3}\left(\frac{p_3^2}{2} + \frac{p_6}{2}\right) \\ &= s_6 + s_{42} + s_{222}. \end{aligned}$$

That is, $\mathcal{N}_{222}$ decomposes into three irreducible components, with the trivial representation s_6 being the span of $\mathbf{m}_{222}$ inside Λ.

4.3 Λ meets $S^{\mathfrak{S}}$

We begin by explaining the choice of normalizing coefficient in (16). Analyzing the **abelianization** map $\mathbf{ab} : T \to S$ (the map making the variables $\mathbf{x}$ commute), Rosas and Sagan [15, Thm. 2.1] show that $\mathbf{ab}|_{\mathcal{N}}$ satisfies:

$$\mathbf{ab}(m_{\mathbf{A}}) = \lambda(\mathbf{A})^! m_{\lambda(\mathbf{A})}\,. \tag{19}$$

In particular, $\mathbf{ab}$ maps onto $S^{\mathfrak{S}}$ and

$$\mathbf{ab}(\mathbf{m}_\mu) = m_\mu\,. \tag{20}$$

Note that **ab** is also an algebra map. The reader may wish to use (19) to compare (8) and (13). Formula (20) suggests that a natural right-inverse to $\mathbf{ab}|_{\mathcal{N}}$ is given by

$$\iota : S^{\mathfrak{S}} \hookrightarrow \mathcal{N}, \quad \text{with} \quad \iota(m_\mu) := \mathbf{m}_\mu \quad \text{and} \quad \iota(1) = 1. \tag{21}$$

This fact, combined with the observation that $\iota(S^{\mathfrak{S}}) = \Lambda$, affords a quick proof of Theorem 1 when $|\mathbf{x}| = \infty$. We explain this now.

5 The coinvariant space of $\mathcal{N}$ (Case: $|\mathbf{x}| = \infty$)

5.1 Quick proof of main result

When $|\mathbf{x}| = \infty$, the pair of maps $(\mathbf{ab}, \iota)$ have further properties: the former is a Hopf algebra map and the latter is a coalgebra map [2, Props. 4.3 & 4.5]. Together with (20) and (21), these properties make ι a **coalgebra splitting** of $\mathbf{ab} : \mathcal{N} \to S^{\mathfrak{S}} \to 0$. A theorem of Blattner, Cohen, and Montgomery immediately gives our main result in this case.

Theorem 4 ([5], Thm. 4.14)**.** *If $H \xrightarrow{\pi} \overline{H} \to 0$ is an exact sequence of Hopf algebras that is split as a coalgebra sequence, and the splitting map ι satisfies $\iota(\bar{1}) = 1$, then H is isomorphic to a* crossed product $A \# \overline{H}$*, where A is the* left Hopf kernel *of π. In particular, $H \simeq A \otimes \overline{H}$ as vector spaces.*

For the technical definition of crossed products, we refer the reader to [5, §4]. We mention only that: (i) the crossed product $A \# \overline{H}$ is a certain algebra structure placed on the tensor product $A \otimes \overline{H}$; and (ii) the **left Hopf kernel** is the subalgebra

$$A := \{h \in H : (\mathrm{id} \otimes \pi) \circ \Delta(h) = h \otimes \bar{1}\}.$$

We take $H = \mathcal{N}$, $\overline{H} = S^{\mathfrak{S}}$, and $\pi = \mathbf{ab}$. Since our ι is a coalgebra splitting, the coinvariant space $\mathcal{C}$ we seek seems to be the left Hopf kernel of **ab**. Before setting off to describe $\mathcal{C}$ more explicitly, we point out that the left Hopf kernel is graded: the maps Δ, id, and **ab** are graded, as is the map $\mathcal{C} \# \Lambda \xrightarrow{\simeq} \mathcal{N}$ used in the proof of Theorem 4 (which is simply $a \otimes \overline{h} \mapsto a \cdot \iota(\overline{h})$). Theorem 1 follows immediately from this result.

5.2 Atomic set partitions.

Recall the main result of Wolf [20] that $\mathcal{N}$ is freely generated by some collection of functions. We announce our first choice for this collection now, following the terminology of [3]. Let Π denote the set of all set partitions (of $[d]$, $\forall\, d \geq 0$). The **atomic set partitions** $\dot{\Pi}$ are defined as follows. A set partition $\mathbf{A} = \{A_1, A_2, \ldots, A_r\}$ of $[d]$ is *atomic* if there does not exist a pair (s, c) $(1 \leq s < r, 1 \leq c < d)$ such that $\{A_1, A_2, \ldots, A_s\}$ is a set partition of $[c]$. Conversely, $\mathbf{A}$ is not atomic if there are set partitions $\mathbf{B}$ of $[d']$ and $\mathbf{C}$ of $[d'']$ splitting $\mathbf{A}$ in two: $\mathbf{A} = \mathbf{B} \cup \mathbf{C}^{+d'}$. We write $\mathbf{A} = \mathbf{B}|\mathbf{C}$ in this situation. A **maximal splitting** $\mathbf{A} = \mathbf{A}'|\mathbf{A}''|\cdots|\mathbf{A}^{(t)}$ of $\mathbf{A}$ is one where each $\mathbf{A}^{(i)}$ is atomic. For example, the partition 17.235.4.68 is atomic, while 12.346.57.8 is not. The maximal splitting of the latter would be $12|124.35|1$, but we abuse notation and write $12|346.57|8$ to improve legibility.

It follows from [3, Corollary 9] that $\mathcal{N}$ is freely generated by the atomic monomial functions $\{m_{\mathbf{A}} : \mathbf{A} \in \dot{\Pi}\}$. We now introduce an order on Π that will make this explicit. First we introduce the *restricted growth function* associated to a set partition (see Section 6.1): if $\mathbf{A} = \{A_1, A_2, \ldots, A_r\} \vdash [d]$, define $\mathsf{w}(\mathbf{A}) \in \mathbb{N}^d$ by

$$\mathsf{w}(\mathbf{A}) = w_1 w_2 \cdots w_d, \qquad \text{with} \qquad w_i := k \iff i \in A_k. \tag{22}$$

For example, $\mathsf{w}(13.24) = 1212$ and $\mathsf{w}(17.235.4.68) = 12232414$. Now, given two atomic set partitions $\mathbf{A} \vdash [c]$ and $\mathbf{B} \vdash [d]$, we put:

- $\mathbf{A} \succ \mathbf{B}$ when $c > d$; or
- $\mathbf{A} \succ \mathbf{B}$ when $c = d$ and $\mathsf{w}(\mathbf{A}) >_{\mathsf{lex}} \mathsf{w}(\mathbf{B})$.

Finally, given two set partitions $\mathbf{A}$ and $\mathbf{B}$, put $\mathbf{A} > \mathbf{B}$ if $\lambda(\mathbf{A}) <_{\mathsf{lex}} \lambda(\mathbf{B})$ in the usual lexicographic order on integer partitions. If $\lambda(\mathbf{A}) = \lambda(\mathbf{B})$, then determine maximal splittings of $\mathbf{A}$ and $\mathbf{B}$, view them as words in the atomic set partitions and use the lexicographic order induced by $\succ$. The following chain of set partitions of shape 3221 illustrates our total ordering on Π:

$$1|23|45|678 < 13.2|456|78 < 13.24|568.7 < 13.24|578.6 < 17.235.4.68 < 17.236.4.58\,.$$

In fact, $1|23|45|678$ is the unique minimal element of Π of shape 3221.

Define the **leading term** of a sum $\sum_{\mathbf{C}} \alpha_{\mathbf{C}}\, m_{\mathbf{C}}$ to be the monomial $m_{\mathbf{C}_0}$ such that $\mathbf{C}_0$ is greatest (according to $>$ above) among all $\mathbf{C}$ with $\alpha_{\mathbf{C}} \neq 0$. Combined with (14), our definition of $>$ makes it clear that the leading term of $m_{\mathbf{A}} \cdot m_{\mathbf{B}}$ is $m_{\mathbf{A}|\mathbf{B}}$ and that $\mathcal{N}$ is freely generated by the atomic monomial functions. Moreover, it is clear that multiplication in $\mathcal{N}$ is shape-filtered. Since the left Hopf kernel $\mathcal{C}$ is a subalgebra, $\mathcal{C}$ is shape-filtered as well. Finally, the isomorphism $\mathcal{C} \,\#\, \Lambda \xrightarrow{\simeq} \mathcal{N}$ constructed in the proof of Theorem 4 is also shape-filtered. These facts give Corollary 2 immediately.

5.3 Explicit description of the Hopf algebra structure of $\mathcal{C}$

We begin by partitioning $\dot{\Pi}$ into two sets according to length,

$$\dot{\Pi}_{(1)} := \big\{\, \mathbf{A} \in \dot{\Pi} : \ell(\mathbf{A}) = 1 \big\} \qquad \text{and} \qquad \dot{\Pi}_{(>1)} := \big\{\, \mathbf{A} \in \dot{\Pi} : \ell(\mathbf{A}) > 1 \big\}.$$

It is easy to find elements of the left Hopf kernel $\mathcal{C}$. For instance, if $\mathbf{A}$ and $\mathbf{B}$ belong to $\dot{\Pi}_{(1)}$, then the Lie bracket $[m_{\mathbf{A}}, m_{\mathbf{B}}]$ belongs to $\mathcal{C}$. Indeed,

$$\begin{aligned}
\Delta\left([m_{\mathbf{A}}, m_{\mathbf{B}}]\right) &= \Delta\left(m_{\mathbf{A}|\mathbf{B}} - m_{\mathbf{B}|\mathbf{A}}\right) \\
&= m_{\mathbf{A}|\mathbf{B}} \otimes 1 \;+\; m_{\mathbf{A}} \otimes m_{\mathbf{B}} \;+\; m_{\mathbf{B}} \otimes m_{\mathbf{A}} \;+\; 1 \otimes m_{\mathbf{A}|\mathbf{B}} \\
&\quad - m_{\mathbf{B}|\mathbf{A}} \otimes 1 \;-\; m_{\mathbf{B}} \otimes m_{\mathbf{A}} \;-\; m_{\mathbf{A}} \otimes m_{\mathbf{B}} \;-\; 1 \otimes m_{\mathbf{B}|\mathbf{A}} \\
&= \left(m_{\mathbf{A}|\mathbf{B}} - m_{\mathbf{B}|\mathbf{A}}\right) \otimes 1 + 1 \otimes \left(m_{\mathbf{A}|\mathbf{B}} - m_{\mathbf{B}|\mathbf{A}}\right).
\end{aligned}$$

Since $\mathbf{ab}(m_{\mathbf{A}|\mathbf{B}}) = \mathbf{ab}(m_{\mathbf{B}|\mathbf{A}})$, we have

$$(\mathrm{id} \otimes \mathbf{ab}) \circ \Delta \left([m_{\mathbf{A}}, m_{\mathbf{B}}] \right) \;=\; [m_{\mathbf{A}}, m_{\mathbf{B}}] \otimes 1$$

as desired. Similarly, the difference of monomial functions $m_{13.2} - m_{12.3}$ belongs to $\mathcal{C}$. The leading term here is indexed by $13.2 \in \dot{\Pi}_{(>1)}$. These two simple examples essentially exhaust the different ways in which an element can belong to $\mathcal{C}$. The following discussion makes this precise.

From [3, Theorem 15], we learn that $\mathcal{N}$ is cofree cocommutative with minimal cogenerating set indexed by the Lyndon words in $\dot{\Pi}$. (This result and the previously mentioned freeness result may also be deduced from the techniques developed in [9].) Since single letters are Lyndon words, we know there are primitive elements associated to each atomic set partition. Recall that an element h in a Hopf algebra is **primitive** if $\Delta(h) = h \otimes 1 + 1 \otimes h$. Let $\mathrm{Prim}(\mathcal{N})$ denote the set of primitive elements in $\mathcal{N}$—a Lie algebra under the commutator bracket.

Bearing the free and cofree cocommutative results in mind, a classical theorem of Milnor and Moore [13] guarantees that $\mathcal{N}$ is isomorphic to the universal enveloping algebra $\mathfrak{U}(\mathfrak{L}(\dot{\Pi}))$ of the free Lie algebra $\mathfrak{L}(\dot{\Pi})$ on the set $\dot{\Pi}$. In the isomorphism $\mathfrak{L}(\dot{\Pi}) \xrightarrow{\simeq} \mathrm{Prim}(\mathcal{N})$, one may map $\mathbf{A} \in \dot{\Pi}_{(1)}$ to $m_{\mathbf{A}}$ since these monomial functions are already primitive. The choice of where to send $\mathbf{A} \in \dot{\Pi}_{(>1)}$ is the subject of the next proposition.

Proposition 5. *For each $\mathbf{A} \in \dot{\Pi}_{(>1)}$, there is a primitive element $\tilde{m}_{\mathbf{A}}$ of $\mathcal{N}$,*

$$\tilde{m}_{\mathbf{A}} = m_{\mathbf{A}} - \sum_{\mathbf{B} \in \Pi} \alpha_{\mathbf{B}} \, m_{\mathbf{B}},$$

satisfying: (i) if $\mathbf{B} \in \dot{\Pi}$ or $\lambda(\mathbf{B}) \neq \lambda(\mathbf{A})$, then $\alpha_{\mathbf{B}} = 0$; and (ii) $\sum_{\mathbf{B}} \alpha_{\mathbf{B}} = 1$.

Proof. Suppose $\mathbf{A} \in \dot{\Pi}_{(>1)}$. A primitive $\tilde{m}_{\mathbf{A}}$ exists by the Milnor–Moore theorem, as explained above.

(i). Since $\mathcal{N} = \bigoplus_{\mu} \mathcal{N}_{\mu}$ is a coalgebra grading by shape, we may assume $\lambda(\mathbf{B}) = \lambda(\mathbf{A})$ for any nonzero coefficients $\alpha_{\mathbf{B}}$. Now, since there are linearly independent primitive elements in $\mathcal{N}$ associated to every atomic set partition, we may use Gaussian elimination and our ordering on Π to ensure that $\alpha_{\mathbf{B}} = 0$ for any $\mathbf{B} \in \dot{\Pi}$.

(ii). Define linear maps $\Delta_+^j : \mathcal{N}_+ \to \mathcal{N} \otimes \mathcal{N}$ recursively by

$$\Delta_+(h)^1 := \Delta(h) - h \otimes 1 - 1 \otimes h,$$
$$\Delta_+^{j+1}(h) := (\Delta_+ \otimes \mathrm{id}^{\otimes j}) \circ \Delta_+^j(h) \quad \text{for} \;\; j > 0.$$

Assume that (i) is satisfied for $\tilde{m}_{\mathbf{A}}$ and that $\mathbf{A} = \{A_1, A_2, \ldots, A_r\}$. Since $\Delta_+(\tilde{m}_{\mathbf{A}}) = 0$, we have $\Delta_+^j(m_{\mathbf{A}}) = \Delta_+^j(\sum_{\mathbf{B}} \alpha_{\mathbf{B}} m_{\mathbf{B}})$ for all $j > 1$. Now,

$$\Delta_+^r(m_{\mathbf{A}}) = \sum_{\sigma \in \mathfrak{S}_r} m_{A_{\sigma 1}\downarrow} \otimes m_{A_{\sigma 2}\downarrow} \otimes \cdots \otimes m_{A_{\sigma r}\downarrow}.$$

Indeed, the same holds for any $\mathbf{B}$ with $\lambda(\mathbf{B}) = \lambda(\mathbf{A})$:

$$\Delta_+^r\Big(\sum_{\mathbf{B}} \alpha_{\mathbf{B}} m_{\mathbf{B}}\Big) = \Big(\sum_{\mathbf{B}} \alpha_{\mathbf{B}}\Big) \sum_{\sigma\in\mathfrak{S}_r} m_{A_{\sigma 1}\downarrow} \otimes m_{A_{\sigma 2}\downarrow} \otimes \cdots \otimes m_{A_{\sigma r}\downarrow}.$$

Conclude that $\sum_{\mathbf{B}} \alpha_B = 1$. ☐

We say an element $h \in \mathcal{N}_\mu$ has the *"zero-sum"* property if it satisfies (ii) from the proposition. Put $\tilde{m}_{\mathbf{A}} := m_{\mathbf{A}}$ for $\mathbf{A} \in \dot{\Pi}_{(1)}$. We next describe the coinvariant space $\mathcal{C}$.

Corollary 6. *Let $\mathfrak{C}$ be the Lie ideal in $\mathfrak{L}(\dot{\Pi})$ given by $\mathfrak{C} = \big[\mathfrak{L}(\dot{\Pi}), \mathfrak{L}(\dot{\Pi})\big] \oplus \dot{\Pi}_{(>1)}$. If $\varphi : \mathfrak{U}(\mathfrak{L}(\dot{\Pi})) \to \mathcal{N}$ is the Milnor–Moore isomorphism given by putting $\varphi(\mathbf{A}) := \tilde{m}_{\mathbf{A}}$ for all $\mathbf{A} \in \dot{\Pi}$ and extending multiplicatively, then the left Hopf kernel $\mathcal{C}$ is the Hopf subalgebra $\varphi(\mathfrak{U}(\mathfrak{C}))$.*

Proof. We first show that $\varphi(\mathfrak{U}(\mathfrak{C})) \subseteq \mathcal{C}$. We certainly have $\tilde{m}_{\mathbf{A}} \in \mathcal{C}$ for all $\mathbf{A} \in \dot{\Pi}_{(>1)}$, since the zero-sum property means $\mathbf{ab}(\tilde{m}_{\mathbf{A}}) = 0$. Next suppose $f \in \big[\mathfrak{L}(\dot{\Pi}), \mathfrak{L}(\dot{\Pi})\big]$ is a sum of Lie brackets $[\mathbf{A}] = [[\ldots[\mathbf{A}', \mathbf{A}''], \ldots], \mathbf{A}^{(t)}]$. In this case, $\varphi(f) \in \mathcal{C}$ because each $\varphi([\mathbf{A}])$ is primitive and $\mathbf{ab}$ is an algebra map. Indeed, $\mathbf{ab}([\tilde{m}_{\mathbf{A}'}, \tilde{m}_{\mathbf{A}''}]) = 0$. The inclusion follows, since $\mathfrak{U}(\mathfrak{C})$ is generated by elements of these two types.

It remains to show that $\mathcal{C} \subseteq \varphi(\mathfrak{U}(\mathfrak{C}))$. To begin, note that $\mathfrak{L}(\dot{\Pi})/\mathfrak{C}$ is isomorphic to the abelian Lie algebra generated by $\dot{\Pi}_{(1)}$. The universal enveloping algebra of this latter object is evidently isomorphic to $S^{\mathfrak{S}}$. (Send $\mathbf{A} = \{[d]\}$ to m_d.) The Poincaré–Birkhoff–Witt theorem guarantees that the map $\varphi(\mathfrak{U}(\mathfrak{C})) \otimes S^{\mathfrak{S}} \to \mathcal{N}$ given by $a \otimes b \mapsto a \cdot \iota(b)$ is onto $\mathcal{N}$. Conclude that $\mathcal{C} \subseteq \varphi(\mathfrak{U}(\mathfrak{C}))$, as needed. ☐

Before turning to the case $|\mathbf{x}| < \infty$, we remark that we have left unanswered the question of finding a systematic procedure (e.g., a closed formula in the spirit of Möbius inversion) that constructs a primitive element $\tilde{m}_{\mathbf{A}}$ for each $\mathbf{A} \in \dot{\Pi}_{(>1)}$. This is accomplished in [11].

6 The coinvariant space of $\mathcal{N}$ (Case: $|\mathbf{x}| \leq \infty$)

6.1 Restricted growth functions

We repeat our example of Section 3.3 in the case $n = 3$. The leading term with respect to our previous order would be $m_{13.2.4.5}$, except that this term does not appear because 13.2.4.5 has more than $n = 3$ parts:

$$m_{13.2} \cdot m_{\mathbf{1.2}} = 0 + m_{13\mathbf{4}.2.\mathbf{5}} + m_{13\mathbf{5}.2.\mathbf{4}} + m_{13.2\mathbf{4}.\mathbf{5}} + m_{13.2\mathbf{5}.\mathbf{4}} + m_{13\mathbf{5}.2\mathbf{4}} + m_{13\mathbf{4}.2\mathbf{5}}\,.$$

Fortunately, the map w from set partitions to words on the alphabet $\mathbb{N}_{>0}$ reveals a more useful leading term, underlined below:

$$m_{\mathbf{121}} \cdot m_{12} \;=\; 0 + m_{\mathbf{121}13} + m_{\mathbf{121}31} + m_{\mathbf{121}23} + m_{\mathbf{121}32} + m_{\mathbf{121}21} + \underline{m_{\mathbf{121}12}}\,. \qquad (23)$$

Notice that the words appearing on the right in (23) all begin by 121 and that the concatenation $\underline{121}\,\underline{12}$ is the lexicographically smallest word appearing there. This is generally true and easy to see: if $\mathsf{w}(\mathbf{A}) = u$ and $\mathsf{w}(\mathbf{B}) = v$, then uv is the lexicographically smallest element of $\mathsf{w}(\mathbf{A} \sqcup\!\sqcup \mathbf{B})$.

The map w maps set partitions to **restricted growth functions**, i.e., the words $w = w_1 w_2 \cdots w_d$ satisfying $w_1 = 1$ and $w_i \le 1 + \max\{w_1, w_2, \ldots, w_{i-1}\}$ for all $2 \le i \le d$. We call them restricted growth words here. See [16, 17, 19] and [6, 8] for some of their combinatorial properties and applications. These words are also known as *"rhyme scheme words"* in the literature; see [14] and [18, A000110]. Before looking for a coinvariant space $\mathcal{C}$ within $\mathcal{N}$, we first fix the representatives of Λ. Consider the partition $\mu = 3221$. Of course, $\mathbf{m}_\mu$ is the sum of all set partitions of shape μ, but it will be nice to have a single one in mind when we speak of $\mathbf{m}_\mu$. A convenient choice turns out to be 123.45.67.8: if we use the length plus lexicographic order on $\mathsf{w}(\Pi)$, then it is easy to see that $\mathsf{w}(123.45.67.8) = 11122334$ is the minimal element of Π of shape 3221. We are led to introduce the words

$$\mathsf{w}(\mu) := 1^{\mu_1} 2^{\mu_2} \cdots k^{\mu_k}$$

associated to partitions $\mu = (\mu_1, \mu_2, \cdots, \mu_k)$; we call such restricted growth words **convex words** since $\mu_1 \ge \mu_2 \ge \cdots \ge \mu_k$.

6.2 Proof of main theorem

We say that a restricted growth word is **non-splittable** if $w_i \cdots w_{n-1} w_n$ is not a restricted growth word for any $i > 1$. The **maximal splitting** of a restricted growth word w is the maximal deconcatenation $w = w' | w'' | \cdots | w^{(r)}$ of w into non-splittable words $w^{(i)}$. For example, 12314 is non-splittable while 11232411 is a string of four non-splittable words $1|12324|1|1$.

It is easy to see that if a, b, c, and d are non-splittable, then $ac = bd$ if and only if $a = b$ and $c = d$. Together with the remarks on $\mathbf{A} \sqcup\!\sqcup \mathbf{B}$ following (23), this implies that if $\{u_1, u_2, \ldots, u_r\}$ and $\{v_1, v_2, \ldots, v_s\}$ are two sets of non-splittable words, then

$$m_{u_1} m_{u_2} \cdots m_{u_r} \qquad \text{and} \qquad m_{v_1} m_{v_2} \cdots m_{v_s}$$

share the same leading term (namely, $m_{u_1|u_2|\cdots|u_r}$) if and only if $r = s$ and $u_i = v_i$ for all i. In other words, our algebra $\mathcal{N}$ is *non-splittable word–filtered* and freely generated by the monomial functions $\{m_{\mathsf{w}(\mathbf{A})} : \mathsf{w}(\mathbf{A}) \text{ is non-splittable}\}$. This is one of the collections of monomial functions originally chosen by Wolf [20].

We aim to index $\mathcal{C}$ by the restricted growth words that don't end in a convex word. Toward that end, we introduce the notion of **bimodal words.** These are words with a maximal (but possibly empty) convex prefix, followed by one non-splittable word. The **bimodal decomposition** of a restricted growth word w is the expression of w as a product $w = w' | w'' | \cdots | w^{(r)} | w^{(r+1)}$, where $w', w'', \ldots, w^{(r)}$ are bimodal and $w^{(r+1)}$ is a possibly empty convex word (which we call a **tail**). For a given word w, this decomposition is accomplished by first splitting w into non-splittable words, then recombining, from left to right, consecutive non-splittable words to form bimodal words. For instance, the maximal splitting of 1122212 into non-splittable words is $1|1222|12$. The first two factors combine to make one bimodal word; the last factor is a convex tail: $1122212 \mapsto \widehat{1\,1222}\,\widehat{12}$. Similarly,

$$1231231411122311 \mapsto 123|12314|1|1|1223|1|1 \mapsto \widehat{123\,12314}\;\widehat{1\,1\,1223\,1}\;\acute{1}.$$

Suppose now that u and v are restricted growth words and that the bimodal decomposition of u is tail-free. Then by construction, the bimodal decomposition of uv is the concatenation of the respective bimodal decompositions of u and v. We are ready to identify $\mathcal{C}$ as a subalgebra of $\mathcal{N}$.

Theorem 7. *Let $\mathcal{C}$ be the subalgebra of $\mathcal{N}$ generated by $\{m_v : v \text{ is bimodal}\}$. Then $\mathcal{C}$ has a basis indexed by restricted growth words w whose bimodal decompositions are tail-free. Moreover, the map $\varphi : \mathcal{C} \otimes \Lambda \to \mathcal{N}$ given by $m_{w'} m_{w''} \cdots m_{w^{(r)}} \otimes \mathbf{m}_\mu \mapsto m_{w'|w''|\cdots|w^{(r)}|\mathsf{w}(\mu)}$ is a vector space isomorphism.*

Proof. The advertised map is certainly onto, since $\{m_w : w \in \mathsf{w}(\Pi)\}$ is a basis for $\mathcal{N}$ and every restricted growth word has a bimodal decomposition $w'|w''|\cdots|w^{(r)}|\mathsf{w}(\mu)$. It remains to show that the map is one-to-one.

Note that the monomial functions $\{m_v : v \text{ is bimodal}\}$ are algebraically independent: certainly, the leading term in a product $m_{v_1} m_{v_2} \cdots m_{v_s}$ (with v_i bimodal) is $m_{v_1|v_2|\cdots|v_s}$; now, since every word has a unique bimodal decomposition, no (nontrivial) linear combination of products of this form can be zero. Finally, the leading term in the simple tensor $m_{w'} m_{w''} \cdots m_{w^{(r)}} \otimes \mathbf{m}_\mu$ is the basis vector $m_{w'|w''|\cdots|w^{(r)}} \otimes m_{\mathsf{w}(\mu)}$, so no (nontrivial) linear combination of these will vanish under the map φ. □

References

[1] François Bergeron, Gilbert Labelle, and Pierre Leroux. *Combinatorial species and tree-like structures*, volume 67 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1998. Translated from the 1994 French original by Margaret Readdy, With a foreword by Gian-Carlo Rota.

[2] Nantel Bergeron, Christophe Reutenauer, Mercedes Rosas, and Mike Zabrocki. Invariants and coinvariants of the symmetric groups in noncommuting variables. *Canad. J. Math.*, 60(2):266–296, 2008.

[3] Nantel Bergeron and Mike Zabrocki. The Hopf algebras of symmetric functions and quasi-symmetric functions in non-commutative variables are free and co-free. *J. Algebra Appl.*, 8(4):581–600, 2009.

[4] George M. Bergman and Paul M. Cohn. Symmetric elements in free powers of rings. *J. London Math. Soc. (2)*, 1:525–534, 1969.

[5] Robert J. Blattner, Miriam Cohen, and Susan Montgomery. Crossed products and inner actions of Hopf algebras. *Trans. Amer. Math. Soc.*, 298(2):671–711, 1986.

[6] David Bremner and Lars Schewe. Edge-graph diameter bounds for convex polytopes with few facets. preprint, arXiv:0809.0915v2.

[7] Edward Formanek. Noncommutative invariant theory. In *Group actions on rings (Brunswick, Maine, 1984)*, volume 43 of *Contemp. Math.*, pages 87–119. Amer. Math. Soc., Providence, RI, 1985.

[8] Ira Gessel, Jonathan Weinstein, and Herbert S. Wilf. Lattice walks in $\mathbf{Z}^d$ and permutations with no long ascending subsequences. *Electron. J. Combin.*, 5:Research Paper 2, 11 pp. (electronic), 1998.

[9] Florent Hivert, Jean-Christophe Novelli, and Jean-Yves Thibon. Commutative combinatorial Hopf algebras. *J. Algebraic Combin.*, 28(1):65–95, 2008.

[10] Martin Klazar. Bell numbers, their relatives, and algebraic differential equations. *J. Combin. Theory Ser. A*, 102(1):63–87, 2003.

[11] Aaron Lauve and Mitja Mastnak. The primitives and antipode in the Hopf algebra of symmetric functions in noncommuting variables. preprint, arXiv:1006.0367v3.

[12] Ian G. Macdonald. *Symmetric functions and Hall polynomials*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, second edition, 1995. With contributions by A. Zelevinsky, Oxford Science Publications.

[13] John W. Milnor and John C. Moore. On the structure of Hopf algebras. *Ann. of Math. (2)*, 81:211–264, 1965.

[14] John Riordan. A budget of rhyme scheme counts. In *Second International Conference on Combinatorial Mathematics (New York, 1978)*, volume 319 of *Ann. New York Acad. Sci.*, pages 455–465. New York Acad. Sci., New York, 1979.

[15] Mercedes H. Rosas and Bruce E. Sagan. Symmetric functions in noncommuting variables. *Trans. Amer. Math. Soc.*, 358(1):215–232 (electronic), 2006.

[16] Bruce E. Sagan. Pattern avoidance in set partitions. *Ars Combin.*, 94:79–96, 2010.

[17] Rodica Simion. Combinatorial statistics on noncrossing partitions. *J. Combin. Theory Ser. A*, 66(2):270–301, 1994.

[18] Niel J. A. Sloane. The on-line encyclopedia of integer sequences. published electronically at http://www.research.att.com/~njas/sequences/.

[19] Michelle Wachs and Dennis White. p, q-Stirling numbers and set partition statistics. *J. Combin. Theory Ser. A*, 56(1):27–46, 1991.

[20] Margarete C. Wolf. Symmetric functions of non-commutative elements. *Duke Math. J.*, 2(4):626–637, 1936.