



10-1982

Ideals and Centralizing Mappings in Prime Rings

Joseph H. Mayne

Loyola University Chicago, jmayne@luc.edu

Recommended Citation

Mayne, JH. "Ideals and Centralizing Mappings in Prime Rings." *Proceedings of the American Mathematical Society* 86(2), 1982.

This Article is brought to you for free and open access by the Faculty Publications at Loyola eCommons. It has been accepted for inclusion in Mathematics and Statistics: Faculty Publications and Other Works by an authorized administrator of Loyola eCommons. For more information, please contact ecommons@luc.edu.



This work is licensed under a [Creative Commons Attribution-Noncommercial-No Derivative Works 3.0 License](https://creativecommons.org/licenses/by-nc-nd/3.0/).
© American Mathematical Society, 1982.

IDEALS AND CENTRALIZING MAPPINGS IN PRIME RINGS

JOSEPH H. MAYNE

ABSTRACT. Let R be a prime ring and U be a nonzero ideal of R . If T is a nontrivial automorphism or derivation of R such that $u^T - u^T u$ is in the center of R and u^T is in U for every u in U , then R is commutative. If R does not have characteristic equal to two, then U need only be a nonzero Jordan ideal.

If R is a ring, a mapping T of R to itself is called *centralizing* on a subset S of R if $ss^T - s^T s$ is in the center of R for every s in S . There has been considerable interest in centralizing automorphisms and derivations defined on rings. Miers [4] has studied these mappings defined on C^* -algebras. In [5] Posner proved that if a prime ring has a nontrivial centralizing derivation, then the ring must be commutative. The same result was obtained for centralizing automorphisms in [3]. In this paper it is shown that the automorphism or derivation need only be centralizing and invariant on a nonzero ideal in the prime ring in order to ensure that the ring is commutative. Also, if R is of characteristic not two, then the mapping need only be centralizing and invariant on a nonzero Jordan ideal. For derivations this gives a short proof of a result related to that of Awtar [1, Theorem 3]. Awtar proved that if R is a prime ring of characteristic not two with a nontrivial derivation and a nonzero Jordan ideal U such that the derivation is centralizing on U , then U is contained in the center of R .

Jeffrey Bergen deserves many thanks for his suggestions concerning the results and proofs in this paper.

From now on assume that R is a prime ring and let Z be the center of R . Let $[x, y] = xy - yx$ and note the important identity $[x, yz] = y[x, z] + [x, y]z$. The following lemmas will be used in the proofs of the main results.

LEMMA 1. *If $b[a, r] = 0$ for all r in R , then $b = 0$ or a is in Z .*

PROOF. Assume that $b[a, r] = 0$ for all r in R . Replace r by xy to obtain $b[a, xy] = bx[a, y] + b[a, x]y = bx[a, y] = 0$ for all x and y in R . Since R is prime, $b = 0$ or $[a, y] = 0$ for all x in R .

LEMMA 2. *If D is a derivation of R such that $u^D = 0$ for all u in a nonzero right ideal U of R , then $r^D = 0$ for all r in R .*

PROOF. Let u be a nonzero element in U and x be an element in R . Then ux is in U and $0 = (ux)^D = u^D x + u(x^D) = u(x^D)$. Now replace x by sr to obtain $0 = u(sr)^D = [u(s^D)]r + us(r^D) = us(r^D)$ for all r and s in R . Since R is prime and u is nonzero, $r^D = 0$ for all r in R .

Received by the editors October 21, 1981 and, in revised form, January 12, 1982.

1980 *Mathematics Subject Classification.* Primary 16A70; Secondary 16A12, 16A68, 16A72.

Key words and phrases. Ideal, centralizing automorphism, centralizing derivation, prime ring, commutative.

© 1982 American Mathematical Society
0002-9939/82/0000-0069/\$01.75

LEMMA 3. *If T is a homomorphism of R such that $u^T = u$ for all u in a nonzero right ideal U of R , then $r^T = r$ for every r in R .*

PROOF. Let u be a nonzero element in U and r, s be in R . Since U is a right ideal, us and usr are in U . Then $(usr)^T = usr = (us)^T r^T = usr^T$. Hence $us(r - r^T) = 0$ for all s and r in R . Thus $r = r^T$ for all r in R .

LEMMA 4. *If R contains a nonzero commutative right ideal U , then R must be commutative.*

PROOF. Let u be in U and assume that u^2 is not zero. Such an element exists for if not, then by a variation of Levitzki's theorem [2, Lemma 1.1], R has a nonzero nilpotent ideal and this is impossible in a prime ring. U is a right ideal and so ur and us are in U for every r and s in R . Since U is commutative, $u^2 sr = u(us)r = us(ur) = ur(us) = u(ur)s = u^2 rs$. Hence $u^2[r, s] = 0$ for all r and s in R . By Lemma 1, every r in R is in Z . Therefore R is commutative.

THEOREM. *Let R be a prime ring and U be a nonzero ideal of R . If R has a nontrivial automorphism or derivation T such that $uu^T - u^T u$ is in the center of R and u^T is in U for every u in U , then R is commutative.*

PROOF. By Lemma 2 or Lemma 3, T is nontrivial on U . Since U is a nonzero ideal in a prime ring, U is itself a prime ring. U is then commutative by the author's result in [3] for automorphisms or by Posner's result [5] for derivations. By Lemma 4, R is commutative.

COROLLARY. *If U is a nonzero Jordan ideal in a prime ring R of characteristic not two and T is a nontrivial automorphism or derivation of R which is centralizing and invariant on U , then R is commutative.*

PROOF. Every nonzero Jordan ideal in a prime ring of characteristic not two contains a nonzero ideal [2, Theorem 1.1]. Apply the theorem to this ideal.

REFERENCES

1. R. Awtar, *Lie and Jordan structures in prime rings with derivations*, Proc. Amer. Math. Soc. **41** (1973), 67-74.
2. I. Herstein, *Topics in ring theory*, Univ. of Chicago Press, Chicago, Illinois, 1969.
3. J. Mayne, *Centralizing automorphisms of prime rings*, Canad. Math. Bull. **19** (1976), 113-115.
4. C. R. Miers, *Centralizing mappings of operator algebras*, J. Algebra **59** (1979), 56-64.
5. E. Posner, *Derivations in prime rings*, Proc. Amer. Math. Soc. **8** (1957), 1093-1100.

DEPARTMENT OF MATHEMATICAL SCIENCES, LOYOLA UNIVERSITY, CHICAGO, ILLINOIS 60626