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Elementary Construction of Lusztig's Canonical Basis

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Elementary construction of Lusztig's canonical basis

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ABSTRACT. In this largely expository article we present an elementary construction of Lusztig's canonical basis in type ADE. The method, which is essentially Lusztig's original approach, is to use the braid group to reduce to rank two calculations. Some of the wonderful properties of the canonical basis are already visible: that it descends to a basis for every highest weight integrable representation, and that it is a crystal basis.

1. Introduction

Fix a simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ and let $U_q^-(\mathfrak{g})$ be the lower triangular part of the corresponding quantized universal enveloping algebra. Lusztig's canonical basis B is a basis for $U_q^-(\mathfrak{g})$, unique once the Chevalley generators are fixed, which has remarkable properties. Perhaps the three most important are:

(i) For each finite dimensional irreducible representation V_λ, the non-zero elements in the image of B in $V_\lambda = U_q^-(\mathfrak{g})/I_\lambda$ form a basis; equivalently, the intersection of B with every ideal I_λ is a basis for the ideal.

(ii) B is a crystal basis in the sense of Kashiwara.

(iii) In symmetric type, the structure constants of B with respect to multiplication are Laurent polynomials in q with positive coefficients.

Much has been made of (iii), and it helped give birth to a whole new field: categorification. While this is a wonderful fact, the association of canonical bases with categorification has, I believe, obscured the fact that Lusztig's original construction is quite elementary. Using only basic properties of the braid group action on $U_q(\mathfrak{g})$ and rank 2 calculations, one can establish the existence and uniqueness of a canonical basis, and show that it satisfies both (i) and (ii). Property (iii) is mysterious with this approach, but perhaps that is to be expected, since it does not always hold is non-symmetric types (see [**Tsu10**]), and the arguments here essentially work in all finite types.

We present Lusztig's elementary construction, but with a few changes. Most notably, we have disentangled the construction from the quiver geometry Lusztig was studying at the same time. This has required modifying some arguments. In

particular, our proof of Theorem 5.1 differs from that presented by Lusztig. The results can be found in [**Lus90a, Lus90b, Lus90c, Lus90d, Lus93**].

Lusztig's canonical basis is the same as Kashiwara's global crystal basis [**Kas91**], and Kashiwara's construction is also elementary, at least in the sense that it does not use categorification. However, Kashiwara's construction is quite different from that presented here, and considerably more difficult. It is based on a complicated induction known as the "grand loop argument." Of course, Kashiwara's construction has a big advantage in that it works beyond finite type.

Leclerc [**Lec04**] has some related work, and in particular proves an analogue of our Theorem 5.1 (see [**Lec04**, Lemma 37]). Leclerc's argument is quite different from the one used here, but also avoids using quiver geometry. That work leads more naturally to the dual canonical basis.

This article is fairly self contained, the biggest exception being that we refer to Lusztig's book [**Lus93**] for one elementary but long calculation in type $\mathfrak{sl}_3$. We restrict to the ADE case for simplicity. The construction is not much harder in other finite types, but requires some more notation. The rank two calculations are also considerably more difficult in types B_2 and G_2 (see [**BFZ96**]). Those cases can also be handled by using a folding argument to understand them in terms of types A_3 and D_4 respectively (see [**BZ01, Lus11**]).

2. Notation

Let $\mathfrak{g}$ be a complex Lie algebra of type ADE, with a chosen Borel subalgebra $\mathfrak{b}$ and Cartan subalgebra $\mathfrak{h}$. Let $U_q(\mathfrak{g})$ be its quantized universal enveloping algebra defined over $\mathbb{Q}(q)$ and let $E_i, F_i, K_i^{\pm 1}$ for $i \in I$ be the standard generators. Here I indexes the nodes of the Dynkin diagram, so we can discuss elements being adjacent. Following [**Kas91, Sai94**], the defining relations are, for all $i \neq j \in I$,

$$K_i K_i^{-1} = K_i^{-1} K_i = 1, \ K_i K_j = K_j K_i, \ K_i E_i K_i^{-1} = q^2 E_i,$$

$$K_i F_i K_i^{-1} = q^{-2} F_i, \ E_i F_j - F_j E_i = 0, \ \ E_i F_i - F_i E_i = \frac{K_i - K_i^{-1}}{q - q^{-1}}.$$

(2.1)   If i is adjacent to j: $E_i^2 E_j + E_j E_i^2 = (q + q^{-1}) E_i E_j E_i,$

$$F_i^2 F_j + F_j F_i^2 = (q + q^{-1}) F_i F_j F_i,$$

$$K_i E_j K_i^{-1} = q^{-1} E_j, \ \ K_i F_j K_i^{-1} = q F_j.$$

Otherwise: $E_i E_j = E_j E_i, \ F_i F_j = F_j F_i, \ K_i E_j K_i^{-1} = E_j, \ K_i F_j K_i^{-1} = F_j.$

We use the standard triangular decomposition,

$$(2.2) \qquad\qquad U_q(\mathfrak{g}) = U_q^-(\mathfrak{g}) \otimes U_q^0(\mathfrak{g}) \otimes U_q^+(\mathfrak{g}),$$

where $U_q^-(\mathfrak{g})$ (resp. U_q^0, or U_q^+) is the subalgebra generated by the F_i (resp. $K_i^{\pm 1}$ or E_i). We also use the triangular decomposition with the order of the factors reversed. Bar involution is the $\mathbb{Q}$-algebra involution defined on generators by

$$(2.3) \qquad\qquad \bar{E}_i = E_i, \quad \bar{F}_i = F_i, \quad \bar{K}_i = K_i^{-1}, \quad \bar{q} = q^{-1}.$$

Let $\{\alpha_i\}$ be the set of simple roots for $\mathfrak{g}$. For a positive root β, define its height $\text{ht}(\beta)$ to be the sum of the coefficients when β is written as a linear combinations of simple roots. Let $(\cdot, \cdot)$ be the standard bilinear form on root space $\mathfrak{h}^*$.

3. Braid group action and PBW bases

The following can be found in [**Lus93**]. Lusztig actually defines PBW bases for $U^+(\mathfrak{g})$, and uses a slightly different action of the braid group, but this causes no significant differences. For each $i \in I$ there is an algebra automorphism T_i of $U_q(\mathfrak{g})$ (denoted $T''_{i,1}$ in [**Lus93**]) given by

$$(3.1) \qquad T_i(F_j) := \begin{cases} F_j & i \text{ not adjacent to } j \\ F_j F_i - q F_i F_j & i \text{ adjacent to } j \\ -K_j^{-1} E_j & i = j, \end{cases}$$

$$(3.2) \qquad T_i(E_j) := \begin{cases} E_j & i \text{ not adjacent to } j \\ E_i E_j - q^{-1} E_j E_i & i \text{ adjacent to } j \\ -F_j K_j & i = j, \end{cases}$$

$$(3.3) \qquad T_i(K_j) := \begin{cases} K_j & i \text{ not adjacent to } j \\ K_i K_j & i \text{ adjacent to } j \\ K_j^{-1} & i = j. \end{cases}$$

One can directly check that these respect the defining relations of $U_q(\mathfrak{g})$, and that they satisfy the braid relations (i.e. $T_i T_j T_i = T_j T_i T_j$ for i and j adjacent, and $T_i T_j = T_j T_i$ otherwise). Each T_i performs the Weyl group reflection s_i on weights, where $U_q(\mathfrak{g})$ is graded by $\text{wt}(E_i) = -\text{wt}(F_i) = \alpha_i$, $\text{wt}(K_i) = 0$.

Fix a reduced expression $w_0 = s_{i_1} \cdots s_{i_N}$ for the longest element of the Weyl group. Let $\mathbf{i}$ denote the sequence $i_1, i_2, \ldots, i_N$. Define "root vectors"

$$\begin{aligned} F_{\mathbf{i};\beta_1} &:= F_{i_1} \\ F_{\mathbf{i};\beta_2} &:= T_{i_1} F_{i_2} \\ (3.4) \qquad F_{\mathbf{i};\beta_3} &:= T_{i_1} T_{i_2} F_{i_3} \\ &\quad\vdots \end{aligned}$$

The notation β_k in the subscripts is because, for all k,

$$(3.5) \qquad wt(F_{\mathbf{i},\beta_k}) = -s_{i_1} \cdots s_{i_k - 1} \alpha_{i_k}.$$

These are exactly the negative roots, and we index the root vectors by the corresponding positive roots β_k. When it does not cause confusion we leave off the subscript $\mathbf{i}$.

EXAMPLE 3.1. If $\mathfrak{g} = \mathfrak{sl}_3$ and $\mathbf{i}$ corresponds to the reduced expression $s_1 s_2 s_1$ then $(\beta_1, \beta_2, \beta_3) = (\alpha_1, \alpha_1 + \alpha_2, \alpha_2)$ and $(F_{\beta_1}, F_{\beta_2}, F_{\beta_3}) = (F_1, F_2 F_1 - q F_1 F_2, F_2)$.

Let

$$(3.6) \qquad B_{\mathbf{i}} := \{F_{\mathbf{i};\beta_1}^{(a_1)} F_{\mathbf{i};\beta_2}^{(a_2)} \cdots F_{\mathbf{i};\beta_N}^{(a_N)} : a_1, \ldots, a_N \in \mathbb{Z}_{\geq 0}\}.$$

Here $X^{(a)}$ is the q-divided power $X^a / ([a][a-1] \cdots [2])$, and $[n] = q^{n-1} + q^{n-3} + \cdots + q^{-n+1}$. We call the collection of exponents $\mathbf{a} = (a_1, \ldots, a_N)$ for an element of $B_{\mathbf{i}}$ its Lusztig data, and denote the element by $F_{\mathbf{i}}^{\mathbf{a}}$.

REMARK 3.2. One can define $B_{\mathbf{i}}$ for any reduced word, not just reduced expressions of w_0, and many of the results in this article still hold. In particular, this can be done outside of finite type, where there is no longest element.

LEMMA 3.3. *Fix a reduced expression* $\mathbf{i}$.
  (i) *If* i_k, i_{k+1} *are not adjacent, then reversing their order gives another reduced expression* $\mathbf{i}'$, *and the root vectors are unchanged (although they are reordered, since* $\beta'_k = \beta_{k+1}$, *and* $\beta'_{k+1} = \beta_k$*).*
  (ii) *If* $i_k = i_{k+2}$ *and is adjacent to* i_{k+1}, *then* $\beta_k + \beta_{k+2} = \beta_{k+1}$ *and*
$$F_{\beta_{k+1}} = F_{\beta_{k+2}} F_{\beta_k} - q F_{\beta_k} F_{\beta_{k+2}}.$$
  Furthermore, for the new reduced expression $\mathbf{i}'$ *where* $i_k i_{k+1} i_k$ *is replaced with* $i_{k+1} i_k i_{k+1}$, $F_{\mathbf{i}',\beta} = F_{\mathbf{i},\beta}$ *for all* $\beta \neq \beta_{k+1}$.
  (iii) *If* $\beta_k = \alpha_i$ *for some* k, i, *then* $F_{\mathbf{i};\beta_k} = F_i$. *In particular,* $F_{\beta_N} = F_{\sigma(i_N)}$, *where* σ *is the Dynkin diagram automorphism given by* $\alpha_{\sigma(i)} = -w_0 \alpha_i$.

PROOF. Part (i) and (ii) follow by applying $T_{i_{k-1}}^{-1} \cdots T_{i_1}^{-1}$ and then doing a rank two calculation. Part (iii) is an immediate consequence of (ii), since α_i is not the sum of any two positive roots, and if $i_1 = i$ then $F_{\alpha_i} = F_i$ by definition. $\square$

LEMMA 3.4. *Each root vector* $F_{\mathbf{i};\beta_k}$ *is in* $U_q^-(\mathfrak{g})$.

PROOF. Proceed by induction on the height of $\beta = \beta_k$, the case of a simple root being immediate from Lemma 3.3 (iii). So assume β is not simple. Fix i so that $(\alpha_i, \beta) > 0$. There are reduced expressions $\mathbf{i}'$ and $\mathbf{i}''$ with $i'_1 = i$ and $i''_N = \sigma(i)$, so $\beta'_1 = \beta''_N = \alpha_i$. By Matsumoto's Theorem [**Mat64**] one can move from $\mathbf{i}$ to either $\mathbf{i}'$ or $\mathbf{i}''$ by sequences of braid moves, and one of these sequences must move α_i past β. At that step $F_{\mathbf{i};\beta}$ changes. The first time $F_{\mathbf{i};\beta}$ changes Lemma 3.3 (ii) allows us to conclude by induction that $F_{\mathbf{i};\beta} \in U_q^-(\mathfrak{g})$. $\square$

LEMMA 3.5. *If* $j \geq k$, *then* $T_{i_j}^{-1} \cdots T_{i_1}^{-1} F_{\mathbf{i};\beta_k} \in U_q^{\geq 0}(\mathfrak{g})$.

PROOF.

$$(3.7) \qquad T_{i_k}^{-1} \cdots T_{i_1}^{-1} F_{\mathbf{i};\beta_k} = -K_{i_k}^{-1} E_{i_k},$$

and $(i_{k+1} \cdots, i_N, \sigma(i_1), \cdots, \sigma(i_k))$ yields another reduced expression for w_0. The claim follows from Lemma 3.4 (or more precisely an analogue with F_i and T_i replaced by E_i and T_i^{-1} respectively) since the T_i are algebra automorphisms and preserve $U_q^0(\mathfrak{g})$. $\square$

THEOREM 3.6. *For any* $\mathbf{i}$, $B_{\mathbf{i}}$ *is a* $\mathbb{Q}(q)$*-basis for* $U_q^-(\mathfrak{g})$.

PROOF. The dimension of each weight space of $U_q^-(\mathfrak{g})$ is given by Kostant's partition function, so the size of the proposed basis is correct, and it suffices to show that these elements are linearly independent. Proceed by induction on k, showing that the set of such elements where $a_j = 0$ for $j > k$ is linearly independent. The key is that

$$(3.8) \qquad T_{i_1}^{-1} F^{\mathbf{a}} = (-K_{i_1}^{-1} E_{i_1})^{(a_1)} \otimes F_{\mathbf{i}'}^{\mathbf{a}'} \in U_q^{\geq 0}(\mathfrak{g}) \otimes U_q^-(\mathfrak{g}),$$

where $\mathbf{i}' = (i_2, i_3, \ldots, i_N, \sigma(i_1))$ and $\mathbf{a}' = (a_2, a_3, \ldots, a_k, 0, \ldots, 0)$. The $F_{\mathbf{i}'}^{\mathbf{a}'}$ are linearly independent by induction, so the vectors $T_{i_1}^{-1} F_{\mathbf{i}}^{\mathbf{a}}$ are linearly independent by the triangular decomposition of $U_q(\mathfrak{g})$. The result follows since $T_{i_1}^{-1}$ is an algebra automorphism. $\square$

The following are referred to as convexity properties of PBW bases.

LEMMA 3.7. *Fix* $\mathbf{i}$ *and* $1 \leq j < k \leq N$.

    (i) *Write* $F_{\beta_k} F_{\beta_j} = \sum_{\mathbf{a}} p_{\mathbf{a}} F_{\mathbf{i}}^{\mathbf{a}}$. *If* $p_{\mathbf{a}} \neq 0$ *then the only factors that appear with non-zero exponent in* $F_{\mathbf{i}}^{\mathbf{a}}$ *are* F_{β_i} *for* $j \leq i \leq k$.

    (ii) *If* $n\beta_\ell = a_j\beta_j + \cdots + a_k\beta_k$ *for* $n, a_j, a_k > 0$ *and* $a_{j+1}, \ldots, a_{k-1} \geq 0$, *then* $j < \ell < k$.

PROOF. Since the T_i are algebra automorphisms, Lemmas 3.4 and 3.5 give

$$(3.9) \quad T_{i_{j-1}}^{-1} \cdots T_{i_2}^{-1} T_{i_1}^{-1}(F_{\beta_k} F_{\beta_j}) \in U_q^-(\mathfrak{g}) \text{ and } T_{i_k}^{-1} \cdots T_{i_2}^{-1} T_{i_1}^{-1}(F_{\beta_k} F_{\beta_j}) \in U_q^{\geq 0}(\mathfrak{g}).$$

A linear combination of PBW basis elements can only satisfy these conditions if, in all of them, the exponents of F_{β_i} are 0 unless $j \leq i \leq k$. This establishes (i).

For (ii), Notice that $s_{i_1} \cdots s_{i_{j-1}}(a_j\beta_j + \cdots + a_k\beta_k)$ is in the positive span of the simple roots, and $s_{i_1} \cdots s_{i_k}(a_j\beta_j + \cdots + a_k\beta_k)$ is in the negative span. This can only happen for $n\beta_\ell$ if $j \leq \ell \leq k$. If $\ell = j$, then for weight reasons $n > a_j$. But then $(n - a_j)\beta_\ell = 0\beta_j + a_{j+1}\beta_{j+1} + \cdots + a_k\beta_k$ leads to a contradiction as above. A similar argument rules out $\ell = k$. $\qquad\square$

LEMMA 3.8. *Assume* $\mathbf{i}, \mathbf{i}'$ *are related by a single braid move. Fix a root* β *such that* $F_{\mathbf{i},\beta} = F_{\mathbf{i}',\beta}$. *Then, for any* n,

$$\mathrm{span}\{F_{\mathbf{i}}^{\mathbf{a}} \in B_{\mathbf{i}} : F_{\mathbf{i}}^{\mathbf{a}} \neq F_{\beta}^{(n)}\} = \mathrm{span}\{F_{\mathbf{i}'}^{\mathbf{a}} \in B_{\mathbf{i}'} : F_{\mathbf{i}'}^{\mathbf{a}} \neq F_{\beta}^{(n)}\}.$$

PROOF. For two term braid moves $B_{\mathbf{i}} = B_{\mathbf{i}'}$ and the result is trivial. So assume $\mathbf{i}, \mathbf{i}'$ are related by a three term braid move affecting positions $i, i+1, i+2$. $F_{\beta_{i+1}}$ changes with such a move, so $\beta \neq \beta_{i+1}$. If $\beta \neq \beta_i, \beta_{i+2}$, then the claim is also trivial. So, it suffices to consider the cases $\beta = \beta_i, \beta_{i+2}$, and by symmetry it is enough to consider $\beta = \beta_i$. We will check that any monomial in $B_{\mathbf{i}}$ that has a non-zero exponent of F_γ for $\gamma \neq \beta$ is equal to a linear combination of monomials on $B_{\mathbf{i}'}$ that still all have a non-zero exponent for some root other than β.

If $F_{\mathbf{i}}^{\mathbf{a}} \in B_{\mathbf{i}}$ has a non-zero exponent for some $j \neq i, i+1, i+2$, every monomial that appears in its $B_{\mathbf{i}'}$ expansion will have that same exponent. If a monomial is such that the only non-zero exponents are a_i, a_{i+1}, a_{i+2}, and one of a_{i+1}, a_{i+2} is non-zero, then its weight does not equal $n\beta$, so $F_{\beta}^{(n)}$ cannot appear in its expansion in $B_{\mathbf{i}'}$. This exhausts the possibilities. $\qquad\square$

4. Equality mod q and piecewise linear bijections

Fix a reduced expression $\mathbf{i}$ for w_0, and recall from Theorem 3.6 that $B_{\mathbf{i}}$ is a basis for $U_q^-(\mathfrak{g})$. Let

$$(4.1) \qquad\qquad\qquad \mathcal{L} = \mathrm{span}_{\mathbb{Z}[q]} B_{\mathbf{i}}.$$

Part (i) of the following can be found in [**Lus93**, Proposition 41.1.4], and (ii) is part of [**Lus93**, Proposition 42.1.5]. For non-simply laced types see [**Sai94**].

THEOREM 4.1.

    (i) $\mathcal{L}$ *is independent of* $\mathbf{i}$.

    (ii) *The basis* $B_{\mathbf{i}} + q\mathcal{L}$ *of* $\mathcal{L}/q\mathcal{L}$ *is independent of* $\mathbf{i}$.

PROOF. Any two reduced expressions are related by a sequence of braid moves, so it suffices to consider reduced expressions related by a single braid move. The case of a two-term braid move is trivial, so consider a three-term braid move involving $i_k = i, i_{k+1} = j, i_{k+2} = i$. It suffices to check that

$$(4.2) \qquad \mathrm{span}_{\mathbb{Z}[q]}\{F_{\mathbf{i};\beta_k}^{(a_k)} F_{\mathbf{i};\beta_{k+1}}^{(a_{k+1})} F_{\mathbf{i};\beta_{k+2}}^{(a_{k+2})}\} = \mathrm{span}_{\mathbb{Z}[q]}\{F_{\mathbf{i}';\beta_k'}^{(a_k)} F_{\mathbf{i}';\beta_{k+1}'}^{(a_{k+1})} F_{\mathbf{i}';\beta_{k+2}'}^{(a_{k+2})}\},$$

and that these sets coincide modulo q. Applying $T_{i_{k-1}}^{-1} \cdots T_{i_1}^{-1}$ shows that this is equivalent to the statement in the $\mathfrak{sl}_3$ case. That is an explicit (although surprisingly difficult) calculation, which can be found in [**Lus93**, Chapter 42]. □

One often wants to understand how the Lusztig data changes when one applies a braid move. That is, given $F_{\mathbf{i}}^{\mathbf{a}} \in B_{\mathbf{i}}$, one would like to know which element of $B_{\mathbf{i}'}$ is equal to it mod q. This is described by Lusztig's piecewise linear bijections from [**Lus93**, Chapter 42]. For a two term braid move involving i_k, i_{k+1}, the exponents of all F_β stay the same (although two of them change places, since the roots are reordered). For a three term braid move involving i_k, i_{k+1}, i_{k+2}, all the exponents stay the same except for a_k, a_{k+1}, a_{k+2}, and these change according to:

$$(4.3) \qquad \begin{aligned} a_k' &= \max\{a_{k+1}, a_{k+1} + a_{k+2} - a_k\}, \\ a_{k+1}' &= \min\{a_k, a_{k+2}\}, \\ a_{k+2}' &= \max\{a_{k+1}, a_{k+1} + a_k - a_{k+2}\}. \end{aligned}$$

5. Triangularity of bar involution and the canonical basis

There are two natural lexicographical orders on Lusztig data: one where $\mathbf{a} < \mathbf{b}$ if $a_1 > b_1$ or $a_1 = b_1$ and $(a_2, \ldots) < (b_2, \ldots)$, and the other where one starts by comparing a_N and b_N. Consider the partial order $\prec$ where $\mathbf{a} \preceq \mathbf{b}$ if $\mathrm{wt}(\mathbf{a}) = \mathrm{wt}(\mathbf{b})$ and $\mathbf{a}$ is less then $\mathbf{b}$ for both of these orders. It follows from Lemma 3.7(ii) that the minimal elements are those where $a_k \neq 0$ implies β_k is a simple root. Data with a unique non-zero a_k are maximal, and are in fact the unique maximal elements of weight $a_k\beta_k$.

THEOREM 5.1. *For every reduced expression $\mathbf{i}$ and every Lusztig data $\mathbf{a}$,*

$$\bar{F}_{\mathbf{i}}^{\mathbf{a}} = F_{\mathbf{i}}^{\mathbf{a}} + \sum_{\mathbf{a}' \prec \mathbf{a}} p_{\mathbf{a}'}^{\mathbf{a}}(q) F_{\mathbf{i}}^{\mathbf{a}'},$$

where the $p_{\mathbf{a}'}^{\mathbf{a}}(q)$ are Laurent polynomials in q.

PROOF. That the coefficients are Laurent polynomials follows from the form of bar and the braid group operators. The point is the unit triangularity.

If the claim is true for all $F_{\beta_j}^{(a_j)}$, then $\bar{F}_{\mathbf{i}}^{\mathbf{a}}$ would be equal to $F_{\mathbf{i}}^{\mathbf{a}}$ plus terms obtained by replacing some of the $F_{\mathbf{i};\beta}$ with lesser monomials. Lemma 3.7 implies that, once this is rearranged, all terms that appear are $\prec F_{\mathbf{i}}^{\mathbf{a}}$. Hence the minimal counter-example would have to be of the form $F_\beta^{(n)} = F_{\mathbf{i};\beta_j}^{(n)}$ for some $\mathbf{i}, j$ and n.

Proceed by induction on the height $\mathrm{ht}(\beta)$. By Lemma 3.3, $F_{\alpha_i}^{(n)} = F_i^{(n)}$ satisfies the condition (it is in fact bar-invariant), so assume $\mathrm{ht}(\beta) > 1$. Certainly

$$(5.1) \qquad \bar{F}_\beta^{(n)} = p(q) F_\beta^{(n)} + \sum_{\mathbf{a}' \prec \mathbf{a}} p_{\mathbf{a}'}^{\mathbf{a}}(q) F^{\mathbf{a}'},$$

since $F_\beta^{(n)}$ is the unique maximal element of its weight. It remains to show that $p(q) = 1$.

First consider just F_β (and please refer to Example 5.2). Do braid moves until F_β changes (this is possible as discussed in the proof of Lemma 3.4). For the braid moves where F_β does not change, by Lemma 3.8, terms $\prec F_\beta$ get sent to linear combinations of terms that are still $\prec F_\beta$, so $p(q)$ does not change. Thus we may assume that a single braid move would change F_β. Then by Lemma 3.3, $F_{\beta_j} = F_{\beta_{j+1}} F_{\beta_{j-1}} - q F_{\beta_{j-1}} F_{\beta_{j+1}}$, so

$$(5.2) \quad \begin{aligned} \bar{F}_{\beta_j} - F_{\beta_j} &= (\bar{F}_{\beta_{j+1}} - F_{\beta_{j+1}}) F_{\beta_{j-1}} + \bar{F}_{\beta_{j+1}} (\bar{F}_{\beta_{j-1}} - F_{\beta_{j-1}}) + \\ &\quad + q F_{\beta_{j-1}} F_{\beta_{j+1}} - q^{-1} \bar{F}_{\beta_{j-1}} \bar{F}_{\beta_{j+1}}. \end{aligned}$$

By induction the statement holds for $F_{\beta_{j+1}}$, so $\bar{F}_{\beta_{j+1}} - F_{\beta_{j+1}}$ is a sum of PBW monomials of weight β_{j+1}, all $\prec F_{\beta_{j+1}}$. In particular, each has a left factor F_{β_ℓ} for some $\ell < j+1$, and for weight reasons we actually must have $\ell < j$. By Lemma 3.7(i), every term in the PBW expansion of $(\bar{F}_{\beta_{j+1}} - F_{\beta_{j+1}}) F_{\beta_{j-1}}$ has a left factor F_{β_ℓ} for $\ell < j$. Similar arguments show that every term in the PBW expansion of the remaining parts has either a left factor F_{β_ℓ} for $\ell < j$ or a right factor F_{β_m} for $m > j$. Since F_{β_j} is the unique maximal PBW monomial of weight β_j the statement holds.

Now consider $F_{\beta_j}^{(n)}$. We know $\bar{F}_{\beta_j} - F_{\beta_j}$ is a sum of terms $\prec F_{\beta_j}$, so

$$(5.3) \quad \bar{F}_{\beta_j}^{(n)} - F_{\beta_j}^{(n)} = (F_{\beta_j} + (\bar{F}_{\beta_j} - F_{\beta_j}))^{(n)} - F_{\beta_j}^{(n)}$$

is a linear combination of terms of the form

$$(5.4) \quad F_{\beta_j}^k (\text{ a PBW monomial } M \prec F_{\beta_j}) R,$$

where the precise form of R is irrelevant. Each M has a left factor F_{β_ℓ} for $\ell < j$. Applying Lemma 3.7(i) repeatedly, every term in the PBW expansion of $F_{\beta_j}^k M R$ also has a left factor $F_{\beta_{\ell'}}$ for some $\ell' < j$, so is $\prec F_{\beta_j}^{(n)}$. $\qquad\square$

EXAMPLE 5.2. Consider $\mathfrak{sl}_4$ and the reduced expression $w_0 = s_3 s_1 s_2 s_1 s_3 s_2$. The corresponding order on positive roots is

$$(5.5) \quad \beta_1 = \alpha_3, \ \beta_2 = \alpha_1, \ \beta_3 = \alpha_1 + \alpha_2 + \alpha_3, \ \beta_4 = \alpha_2 + \alpha_3, \ \beta_5 = \alpha_1 + \alpha_2, \ \beta_6 = \alpha_2.$$

Applying braid moves until the relevant F_{β_k} changes, and using Lemma 3.3, gives $F_{\beta_4} = F_2 F_3 - q F_3 F_2$, and $F_{\beta_3} = F_{\beta_4} F_1 - q F_1 F_{\beta_4}$. Then

$$(5.6) \quad \bar{F}_{\beta_4} - F_{\beta_4} = (q - q^{-1}) F_3 F_2,$$

which is certainly $\prec F_{\beta_4}$. We also have

$$(5.7) \quad \bar{F}_{\beta_3} - F_{\beta_3} = \bar{F}_{\beta_4} F_1 - F_{\beta_4} F_1 + q F_1 F_{\beta_4} - q^{-1} F_1 \bar{F}_{\beta_4}.$$

This is simpler than (5.2) because F_1 is bar invariant. Inductively, the right side is

$$(5.8) \quad (\text{terms} \prec F_{\beta_4}) F_1 + F_1 (\text{ something }).$$

The terms $\prec F_{\beta_4}$ all have factors F_{β_k} for $k < 3$, a property which is preserved under right multiplication by Lemma 3.7, so all terms that appear when one rearranges are $\prec F_{\beta_3}$. Here the only term $\prec F_{\beta_4}$ is $F_3 F_2$ so this can also be verified directly.

THEOREM 5.3. *There is a unique basis B of $U_q^-(\mathfrak{g})$ such that*

(i) *B is contained in $\mathcal{L}$, $B + q\mathcal{L}$ is a basis for $\mathcal{L}/q\mathcal{L}$, and this agrees with $B_{\mathbf{i}} + q\mathcal{L}$ for some (equivalently any by Theorem 4.1) $\mathbf{i}$.*

(ii) *B is bar invariant.*

Furthermore, the change of basis from any $B_{\mathbf{i}}$ to B is unit-triangular.

PROOF. This proof can be found in [**Lec04**, §5.1] and [**DDPW08**, Lemma 0.27] in slightly different settings. Fix $\mathbf{i}$ and proceed by induction on the partial order $\prec$, proving that there is such a basis for $V_{\mathbf{a}} = \text{span}\{F^{\mathbf{a}'}\}_{\mathbf{a}' \preceq \mathbf{a}}$. The case when $\mathbf{a}$ is minimal holds since Theorem 5.1 shows that $F^{\mathbf{a}}$ itself is bar-invariant.

So, fix a non-minimal $\mathbf{a}$. By Theorem 5.1,

$$(5.9) \qquad \bar{F}^{\mathbf{a}} = F^{\mathbf{a}} + \sum_{\mathbf{a}' \prec \mathbf{a}} p_{\mathbf{a}'}^{\mathbf{a}}(q) b^{\mathbf{a}'}$$

for various Laurent polynomials $p_{\mathbf{a}'}^{\mathbf{a}}(q)$, where the $b^{\mathbf{a}'}$ are the inductively found elements of B. But $\bar{\bar{F}}^{\mathbf{a}} = F^{\mathbf{a}}$, which implies that each $p_{\mathbf{a}'}^{\mathbf{a}}(q)$ is of the form

$$(5.10) \qquad p_{\mathbf{a}'}^{\mathbf{a}}(q) = q f_{\mathbf{a}'}^{\mathbf{a}}(q) - q^{-1} f_{\mathbf{a}'}^{\mathbf{a}}(q^{-1}),$$

where each $f_{\mathbf{a}'}^{\mathbf{a}}(q)$ is a polynomial. Set

$$(5.11) \qquad b^{\mathbf{a}} = F^{\mathbf{a}} + \sum_{\mathbf{a}' \prec \mathbf{a}} q f_{\mathbf{a}'}^{\mathbf{a}}(q) b^{\mathbf{a}'}.$$

Replacing $F^{\mathbf{a}}$ with $b^{\mathbf{a}}$ does not change $\mathcal{L}$ and $b^{\mathbf{a}} = F^{\mathbf{a}} \mod q\mathcal{L}$. Then

$$(5.12) \qquad \begin{aligned} \bar{b}^{\mathbf{a}} &= F^{\mathbf{a}} + \sum_{\mathbf{a}' \prec \mathbf{a}} (q f_{\mathbf{a}'}^{\mathbf{a}}(q) - q^{-1} f_{\mathbf{a}'}^{\mathbf{a}}(q^{-1})) b^{\mathbf{a}'} + \sum_{\mathbf{a}' \prec \mathbf{a}} q^{-1} f_{\mathbf{a}'}^{\mathbf{a}}(q^{-1}) b^{\mathbf{a}'} \\ &= F^{\mathbf{a}} + \sum_{\mathbf{a}' \prec \mathbf{a}} q f_{\mathbf{a}'}^{\mathbf{a}}(q) b^{\mathbf{a}'} = b^{\mathbf{a}}, \end{aligned}$$

so we have found the desired element.

Uniqueness is clear, since as the induction proceeds there is no choice. $\qquad \square$

REMARK 5.4. The basis B from Theorem 5.3 is Lusztig's canonical basis (see [**Lus90b**, Theorem 3.2]). As in the above proof, it can be indexed as $B = \{b^{\mathbf{a}}\}$ where the $\mathbf{a}$ are Lusztig data with respect to a fixed reduced expression of w_0. However, as in §4, the indexing changes depending on the reduced expression.

6. Properties of the canonical basis

6.1. Descent to modules.

THEOREM 6.1. *Fix a dominant integral weight λ and write $V_\lambda = U_q^-(\mathfrak{g})/I_\lambda$. Then $B \cap I_\lambda$ spans I_λ. Equivalently, $\{b + I_\lambda : b \in B, b \notin I_\lambda\}$ is a basis for V_λ.*

PROOF. Write λ using fundamental weights, $\lambda = \sum c_i \omega_i$. It is well known that

$$(6.1) \qquad I_\lambda = \sum_{i \in I} U_q^-(\mathfrak{g}) F_i^{c_i+1}.$$

Thus it suffices to show that $B \cap U_q^-(\mathfrak{g})F_i^n$ spans $U_q^-(\mathfrak{g})F_i^n$ for all n.

Fix a reduced expression $\mathbf{i}$ with $i_N = \sigma(i)$, so that $F_{\beta_N} = F_i$. Then it is clear that $B_{\mathbf{i}} \cap U_q^-(\mathfrak{g})F_i^n$ spans $U_q^-(\mathfrak{g})F_i^n$. The change of basis from $B_{\mathbf{i}}$ to B is upper triangular, so the canonical basis elements corresponding to elements in $B_{\mathbf{i}} \cap U_q^-(\mathfrak{g})F_i^n$ are all still in $U_q^-(\mathfrak{g})F_i^n$, giving a spanning set. $\qquad \square$

6.2. Crystal combinatorics. In a sense we already have a combinatorial object that could be called a crystal. With that point of view the underlying set is the basis $B + q\mathcal{L}$ of $\mathcal{L}/q\mathcal{L}$. To perform a crystal operator f_i, choose a reduced expression $\mathbf{i}$ where $i_1 = i$. On $B_\mathbf{i}$, define

$$(6.2) \qquad f_i(F_i^{(a_1)} F_{\beta_2}^{(a_2)} \cdots F_{\beta_N}^{(a_N)}) = F_i^{(a_1+1)} F_{\beta_2}^{(a_2)} \cdots F_{\beta_N}^{(a_N)}.$$

This descends to an operation on $B_\mathbf{i} + q\mathcal{L} = B + q\mathcal{L}$. One must use different reduced expressions to define each f_i, and the full structure is somewhat complex.

Since B itself can be hard to work with, we often choose a reduced expression $\mathbf{i}$, and think of the crystal operators as acting on $B_\mathbf{i} + q\mathcal{L}$ (which is of course equivalent). With this point of view, the crystal operator f_i acts as follows (see §7 for an example).

- Perform a series of braid moves to get a new reduced expression $\mathbf{i}'$ with $i_i' = i$, and use the piecewise linear functions to find the $F_{\mathbf{i}'}^{\mathbf{a}'} \in B_{\mathbf{i}'}$ which is equal to $F_\mathbf{i}^\mathbf{a}$ mod q.
- Add 1 to a_1'.
- Perform a series of braid moves to get $\mathbf{i}'$ back to $\mathbf{i}$ and use the piecewise linear bijections to find the corresponding $F_\mathbf{i}^{\bar{\mathbf{a}}} \in B_\mathbf{i}$. Then $f_i(F_\mathbf{i}^\mathbf{a}) = F_\mathbf{i}^{\bar{\mathbf{a}}}$.

We now show that the structure defined above matches Kashiwara's crystal $B(\infty)$ from [**Kas91**]. This has previously been observed by Lusztig [**Lus90c**] (see also [**GL93, Lus11**]) and by Saito [**Sai94**]. We give a somewhat different proof.

We first review Kashiwara's construction of $B(\infty)$, roughly following [**Kas91**, §3]. For each $i \in I$, elementary calculations show that, for any $X \in U_q^-(\mathfrak{g})$,

$$(6.3) \qquad E_i X = P K_i^{-1} + Q K_i + X E_i$$

for some $P, Q \in U_q^-(\mathfrak{g})$. Define $e_i' : U_q^-(\mathfrak{g}) \to U_q^-(\mathfrak{g})$ by $e_i'(X) = P$. As a vector space,

$$(6.4) \qquad U_q^-(\mathfrak{g}) \cong \mathbb{Q}(q)[F_i] \otimes \ker(e_i'),$$

where the isomorphism is multiplication. Define operators $\widetilde{F}_i$ (the Kashiwara operators) by, for all $Y \in \ker(e_i')$ and $n \geq 0$,

$$(6.5) \qquad \widetilde{F}_i(F_i^{(n)} Y) = F_i^{(n+1)} Y.$$

Let $\mathbb{Q}[q]_0$ be the ring of rational functions which are regular at $q = 0$, and let $\mathcal{L}(\infty)$ to be the $\mathbb{Q}[q]_0$ lattice generated by all sequences of $\widetilde{F}_i$ acting on $1 \in U_q^-(\mathfrak{g})$. There is a unique basis $B(\infty)$ for $\mathcal{L}(\infty)/q\mathcal{L}(\infty)$ such that the residues of all the $\widetilde{F}_i$ act by partial permutations. This basis, along with the residues of the $\widetilde{F}_i$, is $B(\infty)$.

THEOREM 6.2. *Let B be the canonical basis from Theorem 5.3. Then $\mathcal{L}(\infty) = \operatorname{span}_{\mathbb{Q}[q]_0} B$, and $B(\infty) = B + q\mathcal{L}(\infty)$. Furthermore, the crystal operators $\widetilde{F}_i$ mod q coincide with the operators described at the beginning of §6.2.*

Before proving Theorem 6.2 we need some preliminary Lemmas.

LEMMA 6.3. *Fix $i \in I$, a reduced expression $\mathbf{i}$, and a positive root β with $(\beta, \alpha_i) \leq 0$. Then there is a sequence of braid moves, none of which affect the relative positions of α_i and β in the corresponding order on roots, with the last move being a three term braid move with β the middle root (so that F_β changes).*

PROOF. Fix j, k so that $\beta_j = \alpha_i$ and $\beta_k = \beta$. Without loss of generality $j < k$. The prefix $w = s_{i_1} \cdots s_{i_j}$ satisfies $w^{-1}\alpha_i = -\alpha_j$, which is a negative root, so w has a reduced expression of the form $s_i \cdots$. One can perform a sequence of braid moves relating these two reduced expressions which do not change the position of β. Thus we may assume $i_1 = i$. Since $(\beta, \alpha_i) \leq 0$ and $(\beta, \rho) > 0$, we must have $(\beta, \alpha_\ell) > 0$ for some other ℓ.

If $(\alpha_i, \alpha_\ell) = 0$, then there are reduced expressions for w_0 of the form

$$(6.6) \qquad\qquad s_i s_\ell \cdots \qquad \text{and} \qquad s_i \cdots s_{\sigma(\ell)},$$

and both can be reached from $\mathbf{i}$ by performing braid moves that do not change the position of α_i. Certainly the relative positions of β and α_ℓ are different in these two expressions, so one of these sequences moves β past α_ℓ. Since $(\beta, \alpha_\ell) > 0$, at that step β is the middle root for a 3 term braid move.

If $(\alpha_i, \alpha_\ell) = -1$, then there are reduced expressions for w_0 of the form

$$(6.7) \qquad\qquad s_i s_\ell s_i \cdots \qquad \text{and} \qquad s_i \cdots s_{\sigma(\ell)},$$

and the same argument works. $\square$

LEMMA 6.4. *Fix a reduced expression* $\mathbf{i}$, *and let* j *be such that* $\beta_j = \alpha_i$ *is a simple root. For all* $k > j$,

$$E_i F_{\beta_k} - F_{\beta_k} E_i \in U_q^-(\mathfrak{g}) K_i.$$

PROOF. Proceed by induction on the height $\mathrm{ht}(\beta_k)$, the case where β_k is a simple root $\alpha_\ell \neq \alpha_i$ being trivial since $E_i F_{\beta_k} - F_{\beta_k} E_i = 0$ by Serre's relations.

So, assume $\mathrm{ht}(\beta_k) \geq 2$. If $(\beta_k, \alpha_i) \leq 0$, then by Lemma 6.3 we can do a sequence of braid moves that don't change the relative positions of α_i and β_k and so that the last is a three term move with β in the middle. At that step, by Lemma 3.3,

$$(6.8) \qquad\qquad F_{\beta_k} = F_{\beta_{k+1}} F_{\beta_{k-1}} - q F_{\beta_{k-1}} F_{\beta_{k+1}},$$

where $\mathrm{ht}(\beta_{k-1}), \mathrm{ht}(\beta_{k+1}) < \mathrm{ht}(\beta_k)$. The claim holds for $F_{\beta_{k-1}}$ and $F_{\beta_{k+1}}$ by induction, and so it easily follows for F_{β_k}.

If $(\beta_k, \alpha_i) > 0$, perform any sequence of braid moves until β_k is the middle term of a three term move. If α_i has not moved past β_k the result follows as in the previous paragraph. Otherwise at the step where α_i moves past β_k, we see that β_k is the middle term of a three term move affecting the roots $\alpha_i, \beta_k, \beta_k - \alpha_i$, so, again using Lemma 3.3,

$$E_i F_{\beta_k} - F_{\beta_k} E_i = E_i \left(F_{\beta_k - \alpha_i} F_i - q F_i F_{\beta_k - \alpha_i} \right) - \left(F_{\beta_k - \alpha_i} F_i - q F_i F_{\beta_k - \alpha_i} \right) E_i$$

$$(6.9) \qquad\qquad = F_{\beta_k - \alpha_i} \frac{K_i - K_i^{-1}}{q - q^{-1}} - q \frac{K_i - K_i^{-1}}{q - q^{-1}} F_{\beta_k - \alpha_i} + \text{terms in } U_q^- K_i.$$

The fact that the other terms are in $U_q^- K_i$ uses induction. The claim follows since $K_i^{-1} F_{\beta_k - \alpha_i} K_i = q^{-1} F_{\beta_k - \alpha_i}$. $\square$

LEMMA 6.5. *Fix* i *and* $\mathbf{i}$ *such that* $i_1 = i$. *Then*

$$\ker e_i' = \mathrm{span}\{F_{\beta_2}^{(a_2)} \cdots F_{\beta_N}^{(a_N)}\};$$

that is, the span of PBW basis elements where the exponent of F_i *is 0. In particular,* $\widetilde{F}_i$ *acts on* $B_{\mathbf{i}}$ *by simply increasing the exponent of* F_i *by 1.*

$$
\begin{array}{ccccccc}
x = & F_1^{(2)} & F_{12}^{(3)} & F_{123}^{(1)} & F_2^{(3)} & F_{23}^{(3)} & F_3^{(2)} \\[2mm]
& F_1^{(2)} & F_{12}{}^{(3)} & F_{123}^{(1)} & \color{red}{F_3^{(3)}} & \color{red}{F_{32}^{(2)}} & \color{red}{F_2^{(4)}} \\[2mm]
& F_1^{(2)} & \color{red}{F_3{}^{(1)}} & \color{red}{F_{312}^{(3)}} & \color{red}{F_{12}^{(1)}} & F_{32}^{(2)} & F_2^{(4)} \\[2mm]
& \color{red}{F_3^{(1)}} & \color{red}{F_1{}^{(2)}} & F_{312}^{(3)} & F_{12}^{(1)} & F_{32}^{(2)} & F_2^{(4)} \\[2mm]
& \color{red}{F_3^{(2)}} & F_1{}^{(2)} & F_{312}^{(3)} & F_{12}^{(1)} & F_{32}^{(2)} & F_2^{(4)} \\[2mm]
& \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\[2mm]
f_3(x) = & F_1^{(2)} & F_{12}{}^{(3)} & F_{123}^{(1)} & \color{blue}{F_2^{(2)}} & \color{blue}{F_{23}^{(4)}} & F_3^{(2)}
\end{array}
$$

FIGURE 1. Calculation of $f_3(x)$. Lines 2-4 each show the PBW monomial obtained after applying a braid move and the corresponding piecewise linear bijection. The factors colored red have changed. The notation e.g. F_{312} just means $F_{\alpha_1+\alpha_2+\alpha_3}$, but we distinguish between it and F_{123} since root vectors depend on the reduced expression, and they are in fact different. We skip the steps of applying braid moves and piecewise linear bijections to get back to the original reduced expression.

PROOF. Certainly $E_i F_{\beta_2}^{(a_2)} \cdots F_{\beta_N}^{(a_N)}$ is equal to $F_{\beta_2}^{(a_2)} \cdots F_{\beta_N}^{(a_N)} E_i$ plus a sum of terms each of which is a PBW monomial but with one root vector F_β replaced by $E_i F_\beta - F_\beta E_i$. By Lemma 6.4 each of these is in $U_q^-(\mathfrak{g}) K_i$. Therefore, by definition, each $F_{\beta_2}^{(a_2)} \cdots F_{\beta_N}^{(a_N)}$ is in $\ker e_i'$. It follows from (6.4) that the span of these vectors has the correct graded dimension, so is the whole kernel. $\square$

PROOF OF THEOREM 6.2. Fix i, and choose $\mathbf{i}$ such that $i_1 = i$. By Lemma 6.5, $\widetilde{F}_i$ acts by partial permutations on the basis $B_{\mathbf{i}}$. By a simple inductive argument, this implies that $\mathrm{span}_{\mathbb{Q}[q]_0} B_{\mathbf{i}} = \mathrm{span}_{\mathbb{Q}[q]_0} B$ is the lattice generated by all sequences of $\widetilde{F}_i$ acting on $1 \in U_q^-(\mathfrak{g})$. That is, it is $\mathcal{L}(\infty)$. It also shows that $\widetilde{F}_i$ acts on $B_{\mathbf{i}}$ as in (6.2), and hence agrees with the crystal operators described at the beginning of this section. $\square$

7. Example: Crystal operators from piecewise linear bijections

As in §6.2, one can develop crystal theory entirely within Lusztig's setup, where the underlying set is $B_{\mathbf{i}} + q\mathcal{L}$ for a fixed $\mathbf{i}$. To illustrate, take $\mathfrak{g} = \mathfrak{sl}_4$ and the reduced expression $w_0 = s_1 s_2 s_3 s_1 s_2 s_1$. The corresponding order on positive roots is

(7.1) $\qquad \alpha_1, \quad \alpha_1 + \alpha_2, \quad \alpha_1 + \alpha_2 + \alpha_3, \quad \alpha_2, \quad \alpha_2 + \alpha_3, \quad \alpha_3.$

Consider

(7.2) $\qquad\qquad x = F_1^{(2)} F_{12}^{(3)} F_{123}^{(1)} F_2^{(2)} F_{23}^{(4)} F_3^{(2)} \in B_{\mathbf{i}}.$

Here we use e.g. F_{23} to mean $F_{\alpha_2+\alpha_3}$. Applying f_1 is easy: just increase the exponent of F_1 to (3). Figure 1 shows the calculation of $f_3(x)$.

For this reduced expression things work out nicely: at most two exponents change when one applies an f_i, and, as discussed in [**CT15**], there is a straightforward relationship with the well known crystal structure on semi-standard Young

tableaux. There are reduced expressions with similar behavior in types D_n, E_6, and E_7 (see [**SST**]).

In general the relationship with standard combinatorial models is more complicated. For instance, for the reduced expression $w_0 = s_1 s_3 s_2 s_1 s_3 s_2$,

$$(7.3) \qquad f_2(F_1^{(2)} F_3^{(3)} F_{123}^{(3)} F_{23}^{(2)} F_{12}^{(3)} F_2^{(2)}) = F_1^{(2)} F_3^{(3)} F_{123}^{(2)} F_{23}^{(3)} F_{12}^{(4)} F_2^{(2)}.$$

Notice that 3 exponents have changed.

References

[BFZ96] Berenstein, Arkady; Fomin, Sergey; Zelevinsky, Andrei. Parametrizations of canonical bases and totally positive matrices. *Adv. Math.* **122** (1996), no. 1, 49–149.

[BZ01] Berenstein, Arkady; Zelevinsky, Andrei. Tensor product multiplicities, canonical bases and totally positive varieties. *Invent. Math.* **143** (2001), no. 1, 77–128. arXiv:math/9912012

[CT15] Claxton, John; Tingley, Peter. Young tableaux, multisegments, and PBW bases. *Sém. Lothar. Combin.* **73** (2014/15), Art. B73c, 21 pp. arXiv:1503.08194

[DDPW08] Deng, Bangming; Du, Jie; Parshall, Brian; Wang, Jianpan. *Finite dimensional algebras and quantum groups.* Mathematical Surveys and Monographs, 150. American Mathematical Society, Providence, RI, 2008.

[GL93] Grojnowski, Ian; Lusztig, George. A comparison of bases of quantized enveloping algebras. *Linear algebraic groups and their representations (Los Angeles, CA, 1992).* Contemp. Math., **153**, 11–19. Amer. Math. Soc., Providence, RI, 1993.

[Kas91] Kashiwara, Masaki. On crystal bases of the q-analogue of universal enveloping algebras. *Duke Math. J.* **63** (1991), no. 2, 465–516.

[Lec04] Leclerc, Bernard. Dual canonical bases, quantum shuffles and q-characters. *Math. Z.* **246** (2004), Issue 4, pp 691–732. arXiv:math/0209133v3

[Lus90a] Lusztig, George. Quantum groups at roots of 1. *Geom. Dedicata* **35** (1990), no. 1-3, 89–113.

[Lus90b] Lusztig, George. Canonical bases arising from quantized enveloping algebras. *J. Amer. Math. Soc.* **3** (1990), no. 2, 447–498.

[Lus90c] Lusztig, George. *Canonical bases arising from quantized enveloping algebras. II.* Progr. Theoret. Phys. Suppl. (1990), no. 102, 175–201 (1991), Common trends in mathematics and quantum field theories (Kyoto, 1990).

[Lus90d] Lusztig, George. Finite-dimensional Hopf algebras arising from quantized universal enveloping algebra. *J. Amer. Math. Soc.* **3** (1990), no. 1, 257–296.

[Lus93] Lusztig, George. *Introduction to quantum groups.* Volume 110 of Progress in Mathematics. Birkhaüser Boston Inc., Boston, MA, 1993.

[Lus11] Lusztig, George. Piecewise linear parametrization of canonical bases. (English summary) *Pure Appl. Math.* Q. 7 (2011), no. 3, Special Issue: In honor of Jacques Tits, 783–796. arXiv:0807.2824

[Mat64] Matsumoto, Hideya. Générateurs et relations des groupes de Weyl généralisés. *C. R. Acad. Sci.* Paris **258** (1964) 3419–3422.

[Sai94] Saito, Yoshihisa. PBW basis of quantized universal enveloping algebras. *Publ. Res. Inst. Math. Sci.* **30** (1994), no. 2, 209–232.

[SST] Salisbury, Ben; Schultze, Adam; Tingley, Peter. Combinatorial descriptions of the crystal structure on certain PBW bases. To appear in the conference proceedings of FPSAC 2016. arXiv:1603.09013

[Tsu10] Tsuchioka, Shunsuke. Answer to *When does Lusztig's canonical basis have non-positive structure coefficients?* mathoverflow.net/questions/39934/

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