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PBW Bases and Marginally Large Tableaux in Types B and C

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PBW BASES AND MARGINALLY LARGE TABLEAUX IN TYPES B AND C

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Abstract. We explicitly describe the isomorphism between two combinatorial realizations of Kashiwara's infinity crystal in types B and C. The first realization is in terms of marginally large tableaux and the other is in terms of Kostant partitions coming from PBW bases. We also discuss a stack notation for Kostant partitions which simplifies that realization.

Contents

1. Introduction

The infinity crystal $B(\infty)$ is a combinatorial object associated with a symmetrizable Kac–Moody algebra $\mathfrak{g}$. It contains information about the integrable highest weight representations of $\mathfrak{g}$ and the associate quantum group $U_q(\mathfrak{g})$. Kashiwara's original description of $B(\infty)$ used a complicated algebraic construction, but there are often simple combinatorial realizations. Here we consider two such realizations in types B_n and C_n. The first is the marginally large tableaux construction of [4, 5]. The second uses the Kostant partitions from [10], which are related to Lusztig's PBW bases [9] (see also [12]). In [3] and [11], isomorphisms between these two realizations are studied in types A_n and D_n, respectively. Our main result is a simple description of the unique isomorphism between these two realizations of $B(\infty)$ for types B_n and C_n. We also give a stack notation for Kostant partitions of these types motivated by the connection to multisegments in type A_n described in [3].

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$\beta_{i,k} = \alpha_i + \cdots + \alpha_k,$	$1 \le i \le k \le n$
$\gamma_{i,k} = \alpha_i + \cdots + \alpha_{k-1} + 2\alpha_k + 2\alpha_{k+1} + \cdots + 2\alpha_n,$	$1 \le i < k \le n$
$\beta_{i,k} = \varepsilon_i - \varepsilon_{k+1},$	$1 \le i \le k \le n-1$
$\beta_{i,n} = \varepsilon_i,$	$1 \le i \le n$
$\gamma_{i,k} = \varepsilon_i + \varepsilon_k,$	$1 \le i < k \le n$

TABLE 2.1. Positive roots of type B_n, expressed both as a linear combination of simple roots and in the canonical realization following [2].

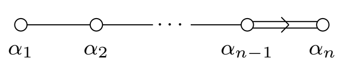
$\beta_{i,k} = \alpha_i + \cdots + \alpha_k,$	$1 \le i \le k < n$
$\gamma_{i,k} = \alpha_i + \cdots + \alpha_{n-1} + \alpha_n + \alpha_{n-1} + \cdots + \alpha_k,$	$1 \le i \le k \le n$
$\beta_{i,k} = \varepsilon_i - \varepsilon_{k+1},$	$1 \le i \le k < n$
$\gamma_{i,k} = \varepsilon_i + \varepsilon_k,$	$1 \le i \le k \le n$

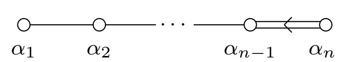
TABLE 2.2. Positive roots of type C_n, expressed both as a linear combination of simple roots and in the canonical realization following [2].

2. BACKGROUND

Let $\mathfrak{g}$ be a Lie algebra of type B_n or C_n. The Cartan matrix and Dynkin diagram are

$$B_n : (a_{ij}) = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ & & & \ddots & & & \\ 0 & 0 & 0 & \cdots & 2 & -1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -2 & 2 \end{pmatrix}, \qquad C_n : (a_{ij}) = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ & & & \ddots & & & \\ 0 & 0 & 0 & \cdots & 2 & -1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 2 & -2 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix}$$

B_n : 

C_n : 

Let $\{\alpha_1, \dots, \alpha_n\}$ be the simple roots and $\{\alpha_1^\vee, \dots, \alpha_n^\vee\}$ the simple coroots, related by the inner product $\langle \alpha_j^\vee, \alpha_i \rangle = a_{ij}$. Define the fundamental weights $\{\omega_1, \dots, \omega_n\}$ by $\langle \alpha_i^\vee, \omega_j \rangle = \delta_{ij}$. Then the weight lattice is $P = \mathbf{Z}\omega_1 \oplus \cdots \oplus \mathbf{Z}\omega_n$ and the coroot lattice is $P^\vee = \mathbf{Z}\alpha_1^\vee \oplus \cdots \oplus \mathbf{Z}\alpha_n^\vee$. Let Φ denote the roots associated to $\mathfrak{g}$, with the set of positive roots denoted Φ^+. The list of positive roots in type B_n is given in Table 2.1, and the list of positive roots in type C_n is given in Table 2.2. The Weyl group associated to $\mathfrak{g}$ is the group generated by $s_1, \dots, s_n$, where $s_i(\lambda) = \lambda - \langle \alpha_i^\vee, \lambda \rangle \alpha_i$ for all $\lambda \in P$. There exists a unique longest element of W which is denoted as w_0. For notational brevity, set $I = \{1, 2, \dots, n\}$.

Let $B(\infty)$ be the infinity crystal associated to $\mathfrak{g}$ as defined in [7]. This is a countable set along with operators e_i and f_i, which roughly correspond to the Chevalley generators of $\mathfrak{g}$. Here we use two explicit realizations of $B(\infty)$ but do not need the general definition.

2.1. **Crystal of marginally large tableaux.** Recall the fundamental crystals given below.

$$\begin{aligned} B_n : &\quad \boxed{1} \xrightarrow{1} \cdots \xrightarrow{n-1} \boxed{n} \xrightarrow{n} \boxed{0} \xrightarrow{n} \boxed{\overline{n}} \xrightarrow{n-1} \cdots \xrightarrow{1} \boxed{\overline{1}} \\ C_n : &\quad \boxed{1} \xrightarrow{1} \cdots \xrightarrow{n-1} \boxed{n} \xrightarrow{n} \boxed{\overline{n}} \xrightarrow{n-1} \cdots \xrightarrow{1} \boxed{\overline{1}} \end{aligned} \tag{2.1}$$

Define alphabets, denoted $J(B_n)$ and $J(C_n)$, to be the elements of these crystals with the natural orderings

$$\begin{aligned} J(B_n) :&\quad \left\{1 \prec \cdots \prec n-1 \prec n \prec 0 \prec \overline{n} \prec \overline{n-1} \prec \cdots \prec \overline{1}\right\}, \text{ and} \\ J(C_n) :&\quad \left\{1 \prec \cdots \prec n-1 \prec n \prec \overline{n} \prec \overline{n-1} \prec \cdots \prec \overline{1}\right\}. \end{aligned}$$

Definition 2.2. The set of marginally large tableaux, $\mathcal{T}(\infty)$, is the set of semistandard Young tableaux T with entries in $J(B_n)$ or $J(C_n)$ which satisfy the following conditions.

(1) The number of $\boxed{i}$ in the i-th row of T is exactly one more than the total number of boxes in the $(i+1)$-th row.
(2) Entries weakly increase along rows.
(3) All entries in the i-th row are $\preceq \bar{\imath}$.
(4) If T is of type B_n, then the $\boxed{0}$ does not appear more than once per row.

Definition 2.2 implies that the leftmost column of T contains $\boxed{1}$, $\boxed{2}$, …, $\boxed{n-1}$, $\boxed{n}$ in increasing order from top to bottom. We call the $\boxed{i}$ in row i *shaded boxes*. The number of shaded boxes in each row is one more than the total number of boxes in the next row.

Example 2.3. In type B_3, each $T \in \mathcal{T}(\infty)$ has the form

$$T = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1 & 1 & 1\cdots1 & 1 & 1\cdots1 & 1 & 1\cdots1 & 1\cdots1 & 1 & 2\cdots2 & 3\cdots3 & 0 & \overline{3}\cdots\overline{3} & \overline{2}\cdots\overline{2} & \overline{1}\cdots\overline{1} \\ \hline 2 & 2 & 2\cdots2 & 2 & 3\cdots3 & 0 & \overline{3}\cdots\overline{3} & \overline{2}\cdots\overline{2} \\ \cline{1-8} 3 & 0 & \overline{3}\cdots\overline{3} \\ \cline{1-3} \end{array}.$$

The notation $\boxed{i\cdots i}$ indicates any number of $\boxed{i}$ (possibly zero). Also, the $\boxed{0}$ in each row may or may not be present.

Definition 2.4. Fix $T \in \mathcal{T}(\infty)$ for $1 \le j \le n-1$ and $k \succ j \in J$. Let $\boxed{k}_j$ denote a box containing k in row j of T. Define the weight of the box by:

$$\text{Type } B_n : \quad \mathrm{wt}\left(\boxed{k}_j\right) = \begin{cases} -\beta_{j,k-1} & \text{if } k \neq 0, \\ -\beta_{j,n} & \text{if } k = 0, \end{cases} \qquad \mathrm{wt}\left(\boxed{\overline{k}}_j\right) = \begin{cases} -\gamma_{j,k} & \text{if } k \neq j, \\ -2\beta_{j,n} & \text{if } k = j. \end{cases}$$

$$\text{Type } C_n : \quad \mathrm{wt}\left(\boxed{k}_j\right) = -\beta_{j,k-1}, \qquad \mathrm{wt}\left(\boxed{\overline{k}}_j\right) = -\gamma_{j,k}.$$

Define the weight $\mathrm{wt}(T)$ of T to be the sum of the weights of all the unshaded boxes of T.

Note that the unique element of weight zero, denoted T_∞, is the tableau where all boxes are shaded. For example, in types B_3 and C_3,

$$T_\infty = \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 2 & 2 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array}\ .$$

Definition 2.5. Let $T \in \mathcal{T}(\infty)$.

(1) The *Far-Eastern reading* of T, denoted $\mathrm{read}_{\mathrm{FE}}(T)$, records the entries of the boxes in the columns of T from top to bottom and proceeding from right to left.

(2) The *Middle-Eastern reading* of T, denoted $\mathrm{read}_{\mathrm{ME}}(T)$, records the entries of the boxes in the rows of T from right to left and proceeding from top to bottom.

Definition 2.6. Let $T \in \mathcal{T}(\infty)$ of type B_n or C_n, and set $\mathrm{read}(T) = \mathrm{read}_{\mathrm{ME}}(T)$ or $\mathrm{read}_{\mathrm{FE}}(T)$. Consider the fundamental crystals from (2.1). For each $i \in I = \{1, 2, \dots, n\}$, the bracketing sequence $\mathrm{br}_i(T)$ is obtained by replacing each letter in $\mathrm{read}(T)$ with $)^p(^q$, where p is number of consecutive i-arrows entering and q is the number of consecutive i-arrows leaving the corresponding box in the fundamental crystal.

After determining $\mathrm{br}_i(T)$, sequentially cancel all $()$-pairs to obtain a sequence of the form $)\cdots)(\cdots($ called the *i-signature* of T. The i-signature is denoted as $\mathrm{br}_i^c(T)$.

Definition 2.7. Let $T \in \mathcal{T}(\infty)$ and $i \in I$. Define $\mathbf{0}$ as a formal object not in $\mathcal{T}(\infty)$.

(1) If there is no '$)$' in $\mathrm{br}_i^c(T)$ then set $e_iT = \mathbf{0}$. Otherwise let $\boxed{r}$ be the box in T corresponding to the rightmost '$)$' in $\mathrm{br}_i^c(T)$. Define e_iT to be the tableau obtained from T by replacing the r in $\boxed{r}$ with the predecessor in the alphabet of $\mathcal{T}(\infty)$. If this creates a column with exactly the entries $1, 2, \dots, i$, then delete that column.

(2) Let $\boxed{\ell}$ be the box in T corresponding to leftmost '$($' in $\mathrm{br}_i^c(T)$. Define f_iT to be the tableau obtained from T by replacing the ℓ in $\boxed{\ell}$ with the successor of ℓ in the alphabet of $\mathcal{T}(\infty)$. If $\boxed{\ell}$ occurs in row i and $\ell = i$, then also insert a column with the entries $1, 2, \dots, i$ directly to the left of $\boxed{\ell}$.

Example 2.8. Let $T \in \mathcal{T}(\infty)$ for $\mathfrak{g}$ of type B_3 where

$$T = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 0 & \bar{3} & \bar{2} & \bar{1} & \bar{1} \\ \hline 2 & 2 & 2 & 2 & 3 & 0 & \bar{2} & \bar{2} \\ \cline{1-8} 3 & \bar{3} & \bar{3} \\ \cline{1-3} \end{array}\ .$$

By Definition 2.6, we have

$$\begin{array}{rcccccccccccccccccccccccccc}
\mathrm{read}_{\mathrm{ME}}(T) = & \bar{1} & \bar{1} & \bar{2} & \bar{3} & 0 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \bar{2} & \bar{2} & 0 & 3 & 2 & 2 & 2 & 2 & \bar{3} & \bar{3} & 3 \\
\mathrm{br}_3(T) = & & & &)) &)(& & & & & & & & & & & & &)(& ((& & & & &)) &)) & ((\\
\mathrm{br}_3^c(T) = & & & &)) &) & & & & & & & & & & & & & & & & & & & &) & ((
\end{array}\ ,$$

so by Definition 2.7, we obtain

$$e_3T = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 0 & \bar{3} & \bar{2} & \bar{1} & \bar{1} \\ \hline 2 & 2 & 2 & 2 & 3 & 0 & \bar{2} & \bar{2} \\ \cline{1-8} 3 & 0 & \bar{3} \\ \cline{1-3} \end{array}$$

and

$$f_3T = \begin{array}{l} \boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{2}\boxed{0}\boxed{\overline{3}}\boxed{\overline{2}}\boxed{\overline{1}}\boxed{\overline{1}} \\ \boxed{2}\boxed{2}\boxed{2}\boxed{2}\boxed{2}\boxed{3}\boxed{0}\boxed{\overline{2}}\boxed{\overline{2}} \\ \boxed{3}\boxed{0}\boxed{\overline{3}}\boxed{\overline{3}} \end{array}.$$

Example 2.9. Let $T \in \mathcal{T}(\infty)$ for $\mathfrak{g}$ of type C_3 where

$$T = \begin{array}{l} \boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{2}\boxed{3}\boxed{3}\boxed{\overline{3}}\boxed{\overline{2}}\boxed{\overline{1}} \\ \boxed{2}\boxed{2}\boxed{2}\boxed{2}\boxed{3}\boxed{\overline{3}}\boxed{\overline{3}}\boxed{\overline{1}} \\ \boxed{3}\boxed{\overline{3}}\boxed{\overline{3}} \end{array}.$$

By Definition 2.6, we have

$$\begin{array}{rccccccccccccccccccccccccc} \mathrm{read}_{\mathrm{ME}}(T) = & \overline{1} & \overline{2} & \overline{3} & 3 & 3 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \overline{1} & \overline{3} & \overline{3} & 3 & 2 & 2 & 2 & 2 & \overline{3} & \overline{3} & 3 \\ \mathrm{br}_3(T) = & & &) & \color{red}{(} & \color{red}{(} & & & & & & & & & & & \color{red}{)} & \color{red}{)} & \color{red}{(} & & & & & \color{red}{)} & \color{blue}{)} & \color{blue}{(} \\ \mathrm{br}_3^c(T) = & & &) & & & & & & & & & & & & & & & & & & & & & \color{blue}{)} & \color{blue}{(}\ , \end{array}$$

so by Definition 2.7, we obtain

$$e_3T = \begin{array}{l} \boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{2}\boxed{3}\boxed{3}\boxed{\overline{3}}\boxed{\overline{2}}\boxed{\overline{1}} \\ \boxed{2}\boxed{2}\boxed{2}\boxed{3}\boxed{\overline{3}}\boxed{\overline{3}}\boxed{\overline{1}} \\ \boxed{3}\boxed{\overline{3}} \end{array}$$

and

$$f_3T = \begin{array}{l} \boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{2}\boxed{3}\boxed{3}\boxed{\overline{3}}\boxed{\overline{2}}\boxed{\overline{1}} \\ \boxed{2}\boxed{2}\boxed{2}\boxed{2}\boxed{2}\boxed{3}\boxed{\overline{3}}\boxed{\overline{3}}\boxed{\overline{1}} \\ \boxed{3}\boxed{\overline{3}}\boxed{\overline{3}}\boxed{\overline{3}} \end{array}.$$

Theorem 2.10 ([5]). *Using* $\mathrm{read}_{\mathrm{FE}}(T)$ *and the operations defined in Definition 2.7,* $\mathcal{T}(\infty)$ *is a crystal isomorphic to* $B(\infty)$*.* ∎

It turns out that using $\mathrm{read}_{\mathrm{ME}}$ in place of $\mathrm{read}_{\mathrm{FE}}$ is more convenient for us, and we can do this because of the following:

Proposition 2.11. *Let* $\mathcal{T}(\infty)$ *be the set of marginally large tableaux of type* B_n *or* C_n*. Then the crystal structures on* $\mathcal{T}(\infty)$ *using either* $\mathrm{read}_{\mathrm{FE}}$ *or* $\mathrm{read}_{\mathrm{ME}}$ *are identical.*

Proof. Fix $T \in \mathcal{T}(\infty)$ and $i \in I$. By the definition of e_i and f_i, we must show that the leftmost '(' and the rightmost ')' in $\mathrm{br}_i^c(T)$ correspond to the same box for the two different readings. We need only consider the positions of the $\boxed{i}$, $\boxed{i+1}$, $\boxed{\overline{i}}$, $\boxed{\overline{i+1}}$, and $\boxed{0}$ (if $i = n$ and T is of type B_n). By Definition 2.2, these all occur in the first $i+1$ rows.

The unshaded boxes are read in the same order under the two readings (since there cannot be two in the same column, and if one box is to the left of another it is also weakly below it). Thus the two bracketing sequences are identical until the first shaded $\boxed{i}$ is read. We will call that part of the sequences the prefix. After that, the sequences are as follows, where we use $\ell_{i,j}$ to denote the number of $\boxed{j}$ in row i (and if $i = n$, $\ell_{i+1,?}$ is taken to be 0):

$$\text{Middle-Eastern: } \cdots (^{\ell_{i,i}}\, (^{\ell_{i+1,\overline{i+1}}}\,)^{\ell_{i+1,i+1}},$$

$$\text{Far-Eastern: } \cdots (^{\ell_{i,i}+\ell_{i+1,\overline{i+1}}-\ell_{i+1,i+1}}\, \underbrace{()\cdots()}_{\ell_{i+1,i+1}}.$$

Since $\ell_{i,i} > \ell_{i+1,i+1}$, after cancellation these parts of the sequences contain only '(' and the leftmost '(' corresponds to the rightmost $\boxed{i}$ in row i.

Thus if the prefix has an uncanceled '(', then this remains uncanceled in both complete bracketing sequences, and corresponds to the same box for both. If the prefix does not have an uncanceled '(', then in both readings the leftmost uncanceled '(' comes from the rightmost $\boxed{i}$ in row i. Furthermore, the sequences only have an uncanceled ')' if this comes from the prefix, in which case it corresponds to the same box in both. ■

2.2. **Crystal of Kostant partitions.** Here we review the crystal structure on Kostant partitions from [10]. As explained there, this is naturally identified with the crystal of PBW monomials as given in [1, 9] (see also [12]) for the reduced expression

$$w_0 = (s_1 s_2 \cdots s_{n-2} s_{n-1} s_n s_{n-2} \cdots s_1) \cdots (s_{n-2} s_{n-1} s_n s_{n-2}) s_{n-1} s_n.$$

Let $\mathcal{R}$ be the set of symbols $\{(\beta) : \beta \in \Phi^+\}$. Let $\mathrm{Kp}(\infty)$ be the free $\mathbf{Z}_{\geq 0}$-span of $\mathcal{R}$. This is the set of *Kostant partitions*. Elements of $\mathrm{Kp}(\infty)$ are written in the form $\boldsymbol{\alpha} = \sum_{(\beta)\in\mathcal{R}} c_\beta(\beta)$.

Definition 2.12. Consider the following sequences of positive roots depending on $i \in I$ for type B_n or C_n. For $1 \leq i \leq n-1$, define

$$\begin{aligned}
\Phi_i^B = \Phi_i^C &= (\beta_{1,i}, \beta_{1,i-1}, \gamma_{1,i}, \gamma_{1,i+1}, \dots, \beta_{i-1,i}, \beta_{i-1,i-1}, \gamma_{i-1,i}, \gamma_{i-1,i+1}, \beta_{i,i}),\\
\Phi_n^B &= (\beta_{1,n}, \beta_{1,n-1}, \gamma_{1,n}, \beta_{1,n}, \dots, \beta_{n-1,n}, \beta_{n-1,n-1}, \gamma_{n-1,n}, \beta_{n-1,n}, \beta_{n,n}),\\
\Phi_n^C &= (\gamma_{1,1}, \beta_{1,n-1}, \gamma_{1,n}, \gamma_{1,1}, \dots, \gamma_{n-1,n-1}, \beta_{n-1,n-1}, \gamma_{n-1,n}, \gamma_{n-1,n-1}, \gamma_{n,n}).
\end{aligned}$$

Let $\boldsymbol{\alpha} \in \mathrm{Kp}(\infty)$. Define the bracketing sequence $S_i(\boldsymbol{\alpha})$ by replacing the roots in Φ_i^B or Φ_i^C with left and right brackets as follows:

In type B_n and C_n with $1 \leq i < n$, set

$$S_i(\boldsymbol{\alpha}) = \underbrace{)\cdots)}_{c_{\beta_{1,i}}}\underbrace{(\cdots(}_{c_{\beta_{1,i-1}}}\underbrace{)\cdots)}_{c_{\gamma_{1,i}}}\underbrace{(\cdots(}_{c_{\gamma_{1,i+1}}}\ \cdots\ \underbrace{)\cdots)}_{c_{\beta_{i-1,i}}}\underbrace{(\cdots(}_{c_{\beta_{i-1,i-1}}}\underbrace{)\cdots)}_{c_{\gamma_{i-1,i}}}\underbrace{(\cdots(}_{c_{\gamma_{i-1,i+1}}}\underbrace{)\cdots)}_{c_{\beta_{i,i}}}.$$

In type B_n with $i = n$, set

$$S_n(\boldsymbol{\alpha}) = \underbrace{)\cdots)}_{c_{\beta_{1,n}}}\underbrace{(\cdots(}_{2c_{\beta_{1,n-1}}}\underbrace{)\cdots)}_{2c_{\gamma_{1,n}}}\underbrace{(\cdots(}_{c_{\beta_{1,n}}}\ \cdots\ \underbrace{)\cdots)}_{c_{\beta_{n-1,n}}}\underbrace{(\cdots(}_{2c_{\beta_{n-1,n-1}}}\underbrace{)\cdots)}_{2c_{\gamma_{n-1,n}}}\underbrace{(\cdots(}_{c_{\beta_{n-1,n}}}\underbrace{)\cdots)}_{c_{\beta_{n,n}}}.$$

In type C_n with $i = n$, set

$$S_n(\boldsymbol{\alpha}) = \underbrace{)\cdots)}_{c_{\gamma_{1,1}}}\underbrace{(\cdots(}_{c_{\beta_{1,n-1}}}\underbrace{)\cdots)}_{c_{\gamma_{1,n}}}\underbrace{(\cdots(}_{c_{\gamma_{1,1}}}\ \cdots\ \underbrace{)\cdots)}_{c_{\gamma_{n-1,n-1}}}\underbrace{(\cdots(}_{c_{\beta_{n-1,n-1}}}\underbrace{)\cdots)}_{c_{\gamma_{n-1,n}}}\underbrace{(\cdots(}_{c_{\gamma_{n-1,n-1}}}\underbrace{)\cdots)}_{c_{\gamma_{n,n}}}.$$

In each case successively cancel all ()-pairs in $S_i(\boldsymbol{\alpha})$ to obtain a sequence of the form $)\cdots)(\cdots($ which we call the i-signature of $\boldsymbol{\alpha}$ denoted by $S_i^c(\boldsymbol{\alpha})$.

Remark 2.13. Roughly, left brackets correspond to roots $\beta \in \Phi_i$ such that $\beta + \alpha_i$ is a root and right brackets correspond to roots $\beta \in \Phi_i$ such that $\beta - \alpha_i$ is a root (or $\beta = \alpha_i$) except when $i = n$, where some subtleties arise.

Definition 2.14. Let $i \in I$ and $\boldsymbol{\alpha} \in \mathrm{Kp}(\infty)$ with $\boldsymbol{\alpha} = \sum_{(\beta)\in\mathcal{R}} c_\beta(\beta) \in \mathrm{Kp}(\infty)$.

- Define $\mathrm{wt}(\boldsymbol{\alpha}) = -\sum_{\beta\in\Phi^+} c_\beta \beta$.
- Define $\varepsilon_i(\boldsymbol{\alpha})$ = number of uncanceled ')' in $S_i(\boldsymbol{\alpha})$.
- Define $\varphi_i(\boldsymbol{\alpha}) = \varepsilon_i(\boldsymbol{\alpha}) + \langle \alpha_i^\vee, \mathrm{wt}(\boldsymbol{\alpha})\rangle$.

The following two rules hold except in the case where $\mathfrak{g}$ is of type C_n and $i = n$.

- Let β be the root corresponding to the rightmost ')' in $S_i^c(\boldsymbol{\alpha})$. Define
$$e_i\boldsymbol{\alpha} = \boldsymbol{\alpha} - (\beta) + (\beta - \alpha_i).$$
Note that if $\beta = \alpha_i$, we interpret (0) as the additive identity in $\mathrm{Kp}(\infty)$. Furthermore, if no such ')' exists, then $e_i\boldsymbol{\alpha} = \mathbf{0}$, where $\mathbf{0}$ is a formal object not contained in $\mathrm{Kp}(\infty)$.
- Let γ denote the root corresponding to the leftmost '(' in $S_i^c(\boldsymbol{\alpha})$. Define,
$$f_i\boldsymbol{\alpha} = \boldsymbol{\alpha} - (\gamma) + (\gamma + \alpha_i).$$
If no such '(' exists, set $f_i\boldsymbol{\alpha} = \boldsymbol{\alpha} + (\alpha_i)$.

If $\mathfrak{g}$ is of type C_n, then e_n and f_n are defined as follows.

- Let β be the root corresponding to the rightmost ')' in $S_n^c(\boldsymbol{\alpha})$. Define $e_n\boldsymbol{\alpha}$ as follows, for $k \in \{1, \dots, n-1\}$. If no such β exists, then $e_n\boldsymbol{\alpha} = \mathbf{0}$.
  (1) If $\beta = \gamma_{k,n}$ and $c_{\gamma_{k,n}} = c_{\beta_{k,n-1}} + 1$, then $e_n\boldsymbol{\alpha} = \boldsymbol{\alpha} - (\beta) + (\beta_{k,n-1})$.
  (2) If $\beta = \gamma_{k,n}$ and $c_{\gamma_{k,n}} > c_{\beta_{k,n-1}} + 1$, then $e_n\boldsymbol{\alpha} = \boldsymbol{\alpha} - 2(\beta) + (\gamma_{k,k})$.
  (3) If $\beta = \gamma_{k,k}$, then $e_n\boldsymbol{\alpha} = \boldsymbol{\alpha} - (\beta) + 2(\beta_{k,n-1})$.
  (4) If $\beta = \gamma_{n,n}$, then $e_n\boldsymbol{\alpha} = \boldsymbol{\alpha} - (\beta)$.
- Let γ denote the root corresponding to the leftmost '(' in $S_n^c(\boldsymbol{\alpha})$. Define $f_n\boldsymbol{\alpha}$ as follows, for $k \in \{1, \dots, n\}$. If no such γ exists, then $f_n\boldsymbol{\alpha} = \boldsymbol{\alpha} + (\gamma_{n,n})$.
  (1) If $\gamma = \beta_{k,n-1}$ and $c_{\gamma_{k,n}} = c_{\beta_{k,n-1}} - 1$, then $f_n\boldsymbol{\alpha} = \boldsymbol{\alpha} - (\gamma) + (\gamma_{k,n})$.
  (2) If $\gamma = \beta_{k,n-1}$ and $c_{\gamma_{k,n}} < c_{\beta_{k,n-1}} - 1$, then $f_n\boldsymbol{\alpha} = \boldsymbol{\alpha} - 2(\gamma) + (\gamma_{k,k})$.
  (3) If $\gamma = \gamma_{k,k}$, then $f_n\boldsymbol{\alpha} = \boldsymbol{\alpha} - (\gamma) + 2(\gamma_{k,n})$.

Example 2.15. Let $\mathrm{Kp}(\infty)$ be of type C_3 and let $\boldsymbol{\alpha} \in \mathrm{Kp}(\infty)$, where
$$\boldsymbol{\alpha} = 4(\beta_{1,2}) + 2(\gamma_{1,3}) + 2(\gamma_{1,1}) + (\gamma_{2,2}) + (\gamma_{2,3}) + (\gamma_{3,3}).$$

We consider the action of f_3, so we must first compute the bracketing sequence:
$$\begin{array}{rccccccccc}
 & c_{\gamma_{1,1}} & c_{\beta_{1,2}} & c_{\gamma_{1,3}} & c_{\gamma_{1,1}} & c_{\gamma_{2,2}} & c_{\beta_{2,2}} & c_{\gamma_{2,3}} & c_{\gamma_{2,2}} & c_{\gamma_{3,3}} \\
S_3(\boldsymbol{\alpha}) = &)) & (({\color{red}((} & {\color{red}))} & {\color{red}((} & {\color{red})} & & {\color{red})} & {\color{red}(} & {\color{red})} \\
S_3^c(\boldsymbol{\alpha}) = &)) & {\color{blue}((} & & & & & & &
\end{array}.$$

Hence $f_3\boldsymbol{\alpha} = 2(\beta_{1,2}) + 2(\gamma_{1,3}) + 3(\gamma_{1,1}) + (\gamma_{2,2}) + (\gamma_{2,3}) + (\gamma_{3,3})$.

Example 2.16. Let $\mathrm{Kp}(\infty)$ be of type C_3 and let $\boldsymbol{\alpha} \in \mathrm{Kp}(\infty)$, where

$$\boldsymbol{\alpha} = 2(\beta_{1,2}) + 2(\gamma_{1,3}) + 3(\gamma_{1,1}) + (\gamma_{2,2}) + (\gamma_{2,3}) + (\gamma_{3,3}).$$

To compute $f_3\boldsymbol{\alpha}$ we first need the relevant bracketing sequence, which is

$$\begin{array}{rccccccccc} & c_{\gamma_{1,1}} & c_{\beta_{1,2}} & c_{\gamma_{1,3}} & c_{\gamma_{1,1}} & c_{\gamma_{2,2}} & c_{\beta_{2,2}} & c_{\gamma_{2,3}} & c_{\gamma_{2,2}} & c_{\gamma_{3,3}} \\ S_3(\boldsymbol{\alpha}) = &))) & ((&)) & (((&) & &) & (&) \\ S_3^c(\boldsymbol{\alpha}) = &))) & & & (& & & & & \end{array}.$$

Hence $f_3\boldsymbol{\alpha} = 2(\beta_{1,2}) + 4(\gamma_{1,3}) + 2(\gamma_{1,1}) + (\gamma_{2,2}) + (\gamma_{2,3}) + (\gamma_{3,3})$.

Proposition 2.17 ([10])**.** *Using the operators defined in Definition 2.14, the set* $\mathrm{Kp}(\infty)$ *is a crystal isomorphic to* $B(\infty)$. ■

3. The isomorphism

Our isomorphism Ψ is given as a reversible algorithm to construct an element of $\mathrm{Kp}(\infty)$ from an element of $\mathcal{T}(\infty)$. We prove that Ψ preserves the crystal structure by using the fact that under the Middle-Eastern reading the bracketing sequence for marginally large tableaux factors by row. Throughout we restrict to types B_n and C_n.

Theorem 3.1. *Define* $\Psi\colon \mathcal{T}(\infty) \longrightarrow \mathrm{Kp}(\infty)$ *by the following process. Fix* $T \in \mathcal{T}(\infty)$ *and let* $R_1, \dots, R_n$ *denote the rows of* T *starting at the top. Set* $\Psi(T) = \sum_{j=1}^n \Psi(R_j)$*, where* $\Psi(R_j)$ *is defined as follows. If* T *is of type* B_n*:*

(1) *each pair* $\left(\boxed{n}, \boxed{\overline{n}}\right)$ *maps to* $2(\beta_{j,n})$*;*
(2) *each* $\boxed{0}$ *maps to* $(\beta_{j,n})$*;*
(3) *if* $j = n$*, then each* $\boxed{\overline{n}}$ *maps to* $2(\beta_{n,n})$*.*

If T *is of type* C_n*:*

(4) *each pair* $\left(\boxed{n}, \boxed{\overline{n}}\right)$ *maps to* $(\gamma_{j,j})$*;*
(5) *if* $j = n$*, then each* $\boxed{\overline{j}}$ *maps to* $(\gamma_{n,n})$*.*

For all remaining boxes:

(6) $\boxed{\overline{j}}$ *maps to* $(\beta_{j,j}) + (\gamma_{j,j+1})$*;*
(7) *each pair* $\left(\boxed{k}, \boxed{\overline{k}}\right)$*, where* $j < k < n$*, maps to* $(\beta_{j,k}) + (\gamma_{j,k+1})$*;*
(8) *each unpaired* $\boxed{k}$ *maps to* $(\beta_{j,k-1})$*, for* $k \in \{j+1, \dots, n\}$*;*
(9) *each unpaired* $\boxed{\overline{k}}$ *maps to* $(\gamma_{j,k})$*, for* $\overline{k} \in \{\overline{n}, \dots, \overline{j+1}\}$*.*

Then Ψ *is a crystal isomorphism.*

The proof of Theorem 3.1 will occupy the rest of this section.

Example 3.2. Let T be the marginally large tableau of type B_3 from Example 2.8. By Theorem 3.1,

$$\Psi(T) = 2(\beta_{1,1}) + (\beta_{1,2}) + (\beta_{1,3}) + 2(\gamma_{1,3}) + 2(\gamma_{1,2}) + 3(\beta_{2,2}) + (\beta_{2,3}) + 2(\gamma_{2,3}) + 4(\beta_{1,3}).$$

Then

$$\begin{array}{rcccccccc} & c_{\beta_{1,3}} & 2c_{\beta_{1,2}} & 2c_{\gamma_{1,3}} & c_{\beta_{1,3}} & c_{\beta_{2,3}} & 2c_{\beta_{2,2}} & 2c_{\gamma_{2,3}} & c_{\beta_{3,3}} \\ S_3(\Psi(T)) = &) & ((&)))) & (& (& ((((((&)))) &)))) \\ S_3^c(\Psi(T)) = &) & &)) & & & & & \end{array},$$

so $f_3\Psi(T) = \Psi(T) + (\beta_{3,3})$, which agrees with

$$\Psi(f_3T) = 2(\beta_{1,1}) + (\beta_{1,2}) + (\beta_{1,3}) + 2(\gamma_{1,3}) + 2(\gamma_{1,2}) + 3(\beta_{2,2}) + (\beta_{2,3}) + 2(\gamma_{2,3}) + 5(\beta_{1,3}).$$

Example 3.3. Consider type C_3 and

$$T = \begin{array}{l} \boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{2}\boxed{2}\boxed{3}\boxed{3}\boxed{3}\boxed{3}\boxed{\overline{3}}\boxed{\overline{3}}\boxed{\overline{2}}\boxed{\overline{2}} \\ \boxed{2}\boxed{2}\boxed{2}\boxed{3}\boxed{\overline{3}}\boxed{\overline{3}} \\ \boxed{3}\boxed{\overline{3}} \end{array}.$$

Then

$$\begin{array}{rl} \mathrm{read}_{\mathrm{ME}}(T) = & \overline{2}\ \overline{2}\ \overline{3}\ \overline{3}\ 3\ 3\ 3\ 3\ 2\ 2\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ \overline{3}\ \overline{3}\ 3\ 2\ 2\ 2\ \overline{3}\ 3 \\ \mathrm{br}_3(T) = & \quad)\ )\ (\ (\ (\ (\qquad\qquad\qquad\quad)\ )\ (\qquad\quad)\ (\\ \mathrm{br}_3^c(T) = & \quad)\ )\ (\ (\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad (\end{array},$$

so

$$f_3T = \begin{array}{l} \boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{2}\boxed{2}\boxed{3}\boxed{3}\boxed{3}\boxed{\overline{3}}\boxed{\overline{3}}\boxed{\overline{3}}\boxed{\overline{2}}\boxed{\overline{2}} \\ \boxed{2}\boxed{2}\boxed{2}\boxed{3}\boxed{\overline{3}}\boxed{\overline{3}} \\ \boxed{3}\boxed{\overline{3}} \end{array}.$$

We now apply the isomorphism from Theorem 3.1 to T and f_3T to get

$$\begin{aligned} \Psi(T) &= 4(\beta_{1,2}) + 2(\gamma_{1,1}) + 2(\gamma_{1,3}) + (\gamma_{2,2}) + (\gamma_{2,3}) + (\gamma_{3,3}), \text{ and} \\ \Psi(f_3T) &= 2(\beta_{1,2}) + 3(\gamma_{1,1}) + 2(\gamma_{1,3}) + (\gamma_{2,2}) + (\gamma_{2,3}) + (\gamma_{3,3}). \end{aligned}$$

Note that these are the same Kostant partitions as in Example 2.15. Hence

$$f_3\Psi(T) = \Psi(T) - 2(\beta_{1,2}) + (\gamma_{1,1}) = \Psi(f_3T).$$

Denote by $e_i^{\mathcal{T}}$ and $f_i^{\mathcal{T}}$ the Kashiwara operators on $\mathcal{T}(\infty)$ from Definition 2.7, and e_i^{Kp} and f_i^{Kp} as the operators on $\mathrm{Kp}(\infty)$ from Definition 2.14.

Lemma 3.4. *Fix $i \in I \setminus \{n\}$ and a row index j. Let $T \in \mathcal{T}(\infty)$ be such that the only unshaded boxes occur in row j. If the leftmost '(' in $\mathrm{br}_i^c(T)$ comes from R_j, then $f_i^{\mathrm{Kp}}\Psi(T) = \Psi(f_i^{\mathcal{T}}T)$.*

Proof. First consider $i \in \{1, \dots, n-1\}$ and row R_j for $j < i$. We are only interested in boxes which give rise to brackets in $\mathrm{br}_i(R_j)$ or $S_i\big(\Psi(R_j)\big)$. Following Definition 2.7 these boxes are the pairs $\left(\boxed{i-1}, \boxed{\overline{i-1}}\right)$ and the $\boxed{i}$, $\boxed{i+1}$, $\boxed{\overline{i+1}}$, and $\boxed{\overline{\imath}}$.

A pair $\left(\boxed{i-1}, \boxed{\overline{i-1}}\right)$ corresponds to no brackets in $\mathrm{br}_i(R_j)$, and to $(\beta_{j,i-1})$, $(\gamma_{j,i})$ in $\Psi(R_j)$, corresponding to a canceling pair of brackets in $S_i\big(\Psi(R_j)\big)$. So the statement is true if and only if it is true with these removed. Thus we can assume R_j has no such pairs.

Now, assume row j of T has p, q, r, and s boxes of $\overline{\imath+1}$, $i+1$, i, and $\bar\imath$ respectively:

$$R_j = \underbrace{\boxed{i}\boxed{\cdots}\boxed{i}}_{r}\underbrace{\boxed{i+1}\boxed{\cdots}\boxed{i+1}}_{q}\boxed{\cdots}\underbrace{\boxed{\overline{\imath+1}}\boxed{\cdots}\boxed{\overline{\imath+1}}}_{p}\underbrace{\boxed{\bar\imath}\boxed{\cdots}\boxed{\bar\imath}}_{s}$$

Define $\Psi(R_j) = \sum_{(\beta)\in\mathcal{R}} c_\beta(\beta)$. The general bracketing sequences for both are

$$\mathrm{br}_i(R_j) =)^s\ (^p\ )^q\ (^r\ , \qquad \text{and} \qquad S_i\big(\Psi(R_j)\big) =)^{c_{\beta_{j,i}}}(^{c_{\beta_{j,i-1}}})^{c_{\gamma_{j,i}}}(^{c_{\gamma_{j,i+1}}}.$$

Case 1: $p > q$, $r > s$ and $1 \le j < i < n$. By the definition of Ψ,

$$\begin{aligned}\Psi(R_j) &= q(\beta_{j,i+1} + \gamma_{j,i+2}) + (p-q)(\gamma_{j,i+1}) + (r-s)(\beta_{j,i-1}) + s(\beta_{j,i} + \gamma_{j,i+1})\\ &= (r-s)(\beta_{j,i-1}) + s(\beta_{j,i}) + q(\beta_{j,i+1}) + (s+p-q)(\gamma_{j,i+1}) + q(\gamma_{j,i+2}).\end{aligned}$$

Calculating the action of $f_i^{\mathcal{T}}$ on R_j gives

$$f_i^{\mathcal{T}} R_j = \underbrace{\boxed{i}\boxed{\cdots}\boxed{i}}_{r}\underbrace{\boxed{i+1}\boxed{\cdots}\boxed{i+1}}_{q}\boxed{\cdots}\underbrace{\boxed{\overline{\imath+1}}\boxed{\cdots}\boxed{\overline{\imath+1}}}_{p-1}\underbrace{\boxed{\bar\imath}\boxed{\cdots}\boxed{\bar\imath}}_{s+1}\,.$$

Then

$$\begin{aligned}\Psi(f_i^{\mathcal{T}} R_j) &= q(\beta_{j,i+1} + \gamma_{j,i+2}) + (p-q-1)(\gamma_{j,i+1}) + (r-s-1)(\beta_{j,i-1}) + (s+1)(\beta_{j,i} + \gamma_{j,i+1})\\ &= (r-s-1)(\beta_{j,i-1}) + (s+1)(\beta_{j,i}) + q(\beta_{j,i+1}) + (s+p-q)(\gamma_{j,i+1}) + q(\gamma_{j,i+2}).\end{aligned}$$

We now apply the operator f_i^{Kp} to $\Psi(R_j)$ to show equivalence. In $S_i^c\big(\Psi(R_j)\big)$ the leftmost '(' corresponds to $\beta_{j,i-1}$ so

$$\begin{aligned}f_i^{\mathrm{Kp}}\Psi(R_j) &= (r-s-1)(\beta_{j,i-1}) + (s+1)(\beta_{j,i}) + q(\beta_{j,i+1}) + q(\gamma_{j,i+2}) + (s+p-q)(\gamma_{j,i+1})\\ &= \Psi(f_i^{\mathcal{T}} R_j).\end{aligned}$$

Case 2: $p > q$, $r \le s$, and $1 \le j < i < n$. By the definition of Ψ,

$$\begin{aligned}\Psi(R_j) &= q(\beta_{j,i+1} + \gamma_{j,i+2}) + (p-q)(\gamma_{j,i+1}) + (s-r)(\gamma_{j,i}) + r(\beta_{j,i} + \gamma_{j,i+1})\\ &= r(\beta_{j,i}) + q(\beta_{j,i+1}) + q(\gamma_{j,i+2}) + (r+p-q)(\gamma_{j,i+1}) + (s-r)(\gamma_{j,i}).\end{aligned}$$

By the definition of $f_i^{\mathcal{T}}$, we have

$$f_i^{\mathcal{T}} R_j = \underbrace{\boxed{i}\boxed{\cdots}\boxed{i}}_{r}\underbrace{\boxed{i+1}\boxed{\cdots}\boxed{i+1}}_{q}\boxed{\cdots}\underbrace{\boxed{\overline{\imath+1}}\boxed{\cdots}\boxed{\overline{\imath+1}}}_{p-1}\underbrace{\boxed{\bar\imath}\boxed{\cdots}\boxed{\bar\imath}}_{s+1}\,.$$

Then

$$\begin{aligned}\Psi(f_i^{\mathcal{T}} R_j) &= q(\beta_{j,i+1} + \gamma_{j,i+2}) + (p-q-1)(\gamma_{j,i+1}) + (s-r+1)(\gamma_{j,i}) + r(\beta_{j,i} + \gamma_{j,i+1})\\ &= r(\beta_{j,i}) + q(\beta_{j,i+1}) + q(\gamma_{j,i+2}) + (r+p-q-1)(\gamma_{j,i+1}) + (s-r+1)(\gamma_{j,i}).\end{aligned}$$

On the other hand, in $S_i^c\big(\Psi(R_j)\big)$ the leftmost '(' corresponds to $\gamma_{j,i+1}$ so

$$\begin{aligned} f_i^{\mathrm{Kp}}\Psi(R_j) &= r(\beta_{j,i}) + q(\beta_{j,i+1}) + q(\gamma_{j,i+2}) + (r+p-q-1)(\gamma_{j,i+1}) + (s-r+1)(\gamma_{j,i}) \\ &= \Psi(f_i^{\mathcal{T}} R_j). \end{aligned}$$

Furthermore,

$$\mathrm{br}_i(R_j) =)^s\ (^{p-q}\ (^r \qquad \text{and} \qquad S_i\big(\Psi(R_j)\big) =)^r\ )^{s-r}\ (^{r+p-q},$$

so both $\mathrm{br}_i^c(R_j)$ and $S_i^c\big(\Psi(R_j)\big)$ have s ')' and $r+p-q$ '('.

Case 3: $p \le q$, $r > s$, and $1 \le j < i < n$. By the definition of Ψ,

$$\begin{aligned} \Psi(R_j) &= p(\beta_{j,i+1} + \gamma_{j,i+2}) + (q-p)(\beta_{j,i}) + (r-s)(\beta_{j,i-1}) + s(\beta_{j,i} + \gamma_{j,i+1}) \\ &= (r-s)(\beta_{j,i-1}) + (s+q-p)(\beta_{j,i}) + p(\beta_{j,i+1}) + p(\gamma_{j,i+2}) + s(\gamma_{j,i+1}). \end{aligned}$$

By the definition of $f_i^{\mathcal{T}}$, we have

$$f_i^{\mathcal{T}} R_j = \underbrace{\boxed{i}\boxed{\cdots}\boxed{i}}_{r-1}\underbrace{\boxed{i+1}\boxed{\cdots}\boxed{i+1}}_{q+1}\boxed{\cdots}\underbrace{\boxed{\overline{i+1}}\boxed{\cdots}\boxed{\overline{i+1}}}_{p}\underbrace{\boxed{\bar{\imath}}\boxed{\cdots}\boxed{\bar{\imath}}}_{s}\,.$$

Then

$$\begin{aligned} \Psi(f_i^{\mathcal{T}} R_j) &= p(\beta_{j,i+1} + \gamma_{j,i+2}) + (q-p+1)(\beta_{j,i}) + (r-s-1)(\beta_{j,i-1}) + s(\beta_{j,i} + \gamma_{j,i+1}) \\ &= (r-s-1)(\beta_{j,i-1}) + (s+q-p+1)(\beta_{j,i}) + p(\beta_{j,i+1}) + p(\gamma_{j,i+2}) + s(\gamma_{j,i+1}). \end{aligned}$$

On the other hand, in $S_i^c\big(\Psi(R_j)\big)$ the leftmost '(' corresponds to $\beta_{j,i-1}$ so

$$\begin{aligned} f_i^{\mathrm{Kp}}\Psi(R_j) &= (r-s-1)(\beta_{j,i-1}) + (s+q-p+1)(\beta_{j,i}) + p(\beta_{j,i+1}) + p(\gamma_{j,i+2}) + s(\gamma_{j,i+1}) \\ &= \Psi(f_i^{\mathcal{T}} R_j). \end{aligned}$$

Case 4: $p \le q$, $r \le s$, and $1 \le j < i < n$. By the definition of Ψ,

$$\begin{aligned} \Psi(R_j) &= p(\beta_{j,i+1} + \gamma_{j,i+2}) + (q-p)(\beta_{j,i}) + (s-r)(\gamma_{j,i}) + r(\beta_{j,i} + \gamma_{j,i+1}) \\ &= (r+q-p)(\beta_{j,i}) + p(\beta_{j,i+1}) + p(\gamma_{j,i+2}) + r(\gamma_{j,i+1}) + (s-r)(\gamma_{j,i}). \end{aligned}$$

If $r = 0$, then f_i will act on the rightmost $\boxed{i}$ in R_i of T (see Case 6 for details on this situation). When $r > 0$, by the definition of $f_i^{\mathcal{T}}$, we have

$$f_i^{\mathcal{T}} R_j = \underbrace{\boxed{i}\boxed{\cdots}\boxed{i}}_{r-1}\underbrace{\boxed{i+1}\boxed{\cdots}\boxed{i+1}}_{q+1}\boxed{\cdots}\underbrace{\boxed{\overline{i+1}}\boxed{\cdots}\boxed{\overline{i+1}}}_{p}\underbrace{\boxed{\bar{\imath}}\boxed{\cdots}\boxed{\bar{\imath}}}_{s}\,.$$

Then

$$\begin{aligned} \Psi(f_i^{\mathcal{T}}(R_j) &= p(\beta_{j,i+1} + \gamma_{j,i+2}) + (q-p+1)(\beta_{j,i}) + (s-r+1)(\gamma j, i) + (r-1)(\beta_{j,i} + \gamma_{j,i+1}) \\ &= (r+q-p)(\beta_{j,i}) + p(\beta_{j,i+1}) + p(\gamma_{j,i+2}) + (r-1)(\gamma_{j,i+1}) + (s-r+1)(\gamma_{j,i}). \end{aligned}$$

On the other hand, in $S_i^c\big(\Psi(R_j)\big)$ the leftmost '(' corresponds to $\gamma_{j,i+1}$ so

$$\begin{aligned} f_i^{\mathrm{Kp}}\Psi(R_j) &= (r+q-p)(\beta_{j,i}) + p(\beta_{j,i+1}) + p(\gamma_{j,i+2}) + (r-1)(\gamma_{j,i+1}) + (s-r+1)(\gamma_{j,i}) \\ &= \Psi(f_i^{\mathcal{T}} R_j). \end{aligned}$$

We now establish the result for $i \in \{1,\dots,n-1\}$ and row $j = i$. The general bracketing sequences for both are given here:

$$\mathrm{br}_i(R_i) = {)^s}\ {(^p}\ {)^q}\ {(^r}\ , \qquad\qquad S_i\big(\Psi(R_i)\big) = {)^{c_{\beta_{i,i}}}}.$$

Following Definition 2.14, since there is no '(' in $S_i\big(\Psi(R_i)\big)$ the action of f_i^{Kp} will always be to add $(\beta_{i,i})$.

Case 5: $p > q$, and $1 \le j = i < n$. If $p > q$, then the leftmost '(' comes from an $\boxed{\overline{i+1}}$ so $f_i^{\mathcal{T}} R_i$ sends an $\boxed{\overline{i+1}}$ to a $\boxed{\overline{i}}$. Since $p > q$ this does not change the number of $\big(\boxed{i+1}, \boxed{\overline{i+1}}\big)$ pairs, so $\Psi(f_i^{\mathcal{T}} R_i) = \Psi(R_i) + (\beta_{i,i}) = f_i^{\mathrm{Kp}}\Psi(R_i)$.

Case 6: $p \le q$, and $1 \le j = i < n$. If $p \le q$, then the leftmost '(' comes from a $\boxed{i}$ so $f_i^{\mathcal{T}} R_i$ sends an $\boxed{i}$ to an $\boxed{i+1}$. Since $p \le q$ this does not change the number of $\big(\boxed{i+1}, \boxed{\overline{i+1}}\big)$ pairs, so $\Psi(f_i^{\mathcal{T}} R_i) = \Psi(R_i) + (\beta_{i,i}) = f_i^{\mathrm{Kp}}\Psi(R_i)$. ∎

Lemma 3.5. *Fix a row index $j \in I$. Let $T \in \mathcal{T}(\infty)$ be such that the only unshaded boxes occur in row j. If the leftmost '(' in $\mathrm{br}_n^c(T)$ comes from R_j, then $f_n^{\mathrm{Kp}}\Psi(T) = \Psi(f_n^{\mathcal{T}} T)$.*

Proof. Consider f_n and T to be of type B_n. We need only consider the $\boxed{n}$, $\boxed{0}$, and $\boxed{\overline{n}}$ boxes, since a pair $\big(\boxed{n-1}, \boxed{\overline{n-1}}\big)$ corresponds to no brackets in $\mathrm{br}_n(R_j)$, and to $(\beta_{j,n-1})$, $(\gamma_{j,n})$ in $\Psi(R_j)$, which gives a canceling pair of brackets in $S_n\big(\Psi(R_j)\big)$. Assume row j of T has p $\boxed{\overline{n}}$ boxes, z $\boxed{0}$ boxes, and q $\boxed{n}$ boxes:

$$R_j = \underbrace{\boxed{n}\boxed{\cdots}\boxed{n}}_{q}\underbrace{\boxed{0}}_{z}\underbrace{\boxed{\overline{n}}\boxed{\cdots}\boxed{\overline{n}}}_{p}\,.$$

The bracketing sequences are:

$$\mathrm{br}_n(R_j) = {)^{2p}}\ {)^{z}}\ {(^{z}}\ {(^{2q}}\ , \qquad\qquad S_n\big(\Psi(T)\big) = {)^{c_{\beta_{j,n}}}}\,{(^{2c_{\beta_{j,n-1}}}}\,{)^{2c_{\gamma_{j,n}}}}\,{(^{c_{\beta_{j,n}}}}.$$

Case 1: $p \ge q$, $z = 0$, and $1 \le j < n$.

By the definition of Ψ,

$$\Psi(R_j) = 2q(\beta_{j,n}) + (p-q)(\gamma_{j,n}).$$

If $q = 0$, then f_n will act on the $\boxed{n}$ in R_n of T (see Case 5 for more details in this situation). If $q > 0$ then, by the definition of $f_n^{\mathcal{T}}$,

$$f_n^{\mathcal{T}} R_j = \underbrace{\boxed{n}\boxed{\cdots}\boxed{n}}_{q-1}\underbrace{\boxed{0}}_{1}\underbrace{\boxed{\overline{n}}\boxed{\cdots}\boxed{\overline{n}}}_{p}\,.$$

Since $p \geq q$ there is one less $\left(\boxed{n},\boxed{\overline{n}}\right)$, one more $\boxed{0}$, and one more unpaired $\boxed{\overline{n}}$, so

$$\begin{aligned}\Psi(f_n^{\mathcal{T}} R_j) &= 2(q-1)(\beta_{j,n}) + (\beta_{j,n}) + (p-q+1)(\gamma_{j,n}),\\ &= (2q-1)(\beta_{j,n}) + (p-q+1)(\gamma_{j,n}).\end{aligned}$$

On the other hand, in $S_n^{c}\big(\Psi(R_j)\big)$ the leftmost '(' corresponds to $\beta_{j,n}$ so

$$f_n^{\mathrm{Kp}}\Psi(R_j) = (2q-1)(\beta_{j,n}) + (p-q+1)(\gamma_{j,n}) = \Psi(f_n^{\mathcal{T}} R_j).$$

Case 2: $p < q$, $z = 0$, and $1 \leq j < n$. By the definition of Ψ,

$$\Psi(R_j) = (q-p)(\beta_{j,n-1}) + 2p(\beta_{j,n}).$$

By the definition of $f_n^{\mathcal{T}}$, we have

$$f_n^{\mathcal{T}} R_j = \boxed{\underbrace{n \mid \cdots \mid n}_{q-1} \mid \underbrace{0}_{1} \mid \underbrace{\overline{n} \mid \cdots \mid \overline{n}}_{p}}\,.$$

The number of $\left(\boxed{n},\boxed{\overline{n}}\right)$ pairs is unchanged, there is one less unpaired $\boxed{n}$ and one more $\boxed{0}$, so

$$\begin{aligned}\Psi(f_n^{\mathcal{T}} R_j) &= (q-p-1)(\beta_{j,n-1}) + (\beta_{j,n}) + 2(p)(\beta_{j,n})\\ &= (q-p-1)(\beta_{j,n-1}) + (2p+1)(\beta_{j,n}).\end{aligned}$$

On the other hand, in $S_n^{c}\big(\Psi(R_j)\big)$ the leftmost '(' corresponds to $\beta_{j,n-1}$ so

$$f_n^{\mathrm{Kp}}\Psi(R_j) = (q-p-1)(\beta_{j,n-1}) + (2p+1)(\beta_{j,j}) = \Psi(f_n^{\mathcal{T}} R_j).$$

Case 3: $p \geq q$, $z = 1$, and $1 \leq j < n$. By the definition of Ψ,

$$\Psi(R_j) = (2q+1)(\beta_{j,n}) + (p-q)(\gamma_{j,n}).$$

By the definition of $f_n^{\mathcal{T}}$, we have

$$f_n^{\mathcal{T}} R_j = \boxed{\underbrace{n \mid \cdots \mid n}_{q} \mid \underbrace{0}_{0} \mid \underbrace{\overline{n} \mid \cdots \mid \overline{n}}_{p+1}}\,.$$

There is one less $\boxed{0}$ and one more unpaired $\boxed{\overline{n}}$, so

$$\Psi(f_n^{\mathcal{T}} R_j) = 2q(\beta_{j,n}) + (p-q+1)(\gamma_{j,n}).$$

On the other hand, in $S_n^{c}\big(\Psi(R_j)\big)$ the leftmost '(' corresponds to $\beta_{j,n}$ so

$$f_n^{\mathrm{Kp}}\Psi(R_j) = 2q(\beta_{j,n}) + (p-q+1)(\gamma_{j,n}) = \Psi(f_n^{\mathcal{T}} R_j).$$

Case 4: $p < q$, $z = 1$ and $1 \leq j < n$. By the definition of Ψ,

$$\Psi(R_j) = (q-p)(\beta_{j,n-1}) + (2p+1)(\beta_{j,n}).$$

By the definition of $f_n^{\mathcal{T}}$, we have

$$f_n^{\mathcal{T}} R_j = \underbrace{\boxed{n}\boxed{\cdots}\boxed{n}}_{q}\underbrace{\boxed{0}}_{0}\underbrace{\boxed{\overline{n}}\boxed{\cdots}\boxed{\overline{n}}}_{p+1}.$$

There is one less $\boxed{0}$, one less unpaired $\boxed{n}$, and one more $\left(\boxed{n},\boxed{\overline{n}}\right)$ pair, so

$$\Psi(f_n^{\mathcal{T}} R_j) = (q-p-1)(\beta_{j,n-1}) + (2p+2)(\beta_{j,n})$$

On the other hand, in $S_n^c\big(\Psi(R_j)\big)$ the leftmost '(' corresponds to $\beta_{j,n-1}$ so

$$f_n^{\mathrm{Kp}}\Psi(R_j) = (q-p-1)(\beta_{j,n-1}) + (2p+2)(\beta_{j,n}) = \Psi(f_n^{\mathcal{T}} R_j).$$

Case 5: $j = n$. The bracketing sequences are

$$\mathrm{br}_n(R_n) =)^{2p}\ )^{z}\ (^{z}\ (^{2q}, \qquad S_n\big(\Psi(R_n)\big) =)^{c_{\beta_{n,n}}}.$$

Since there is no '(' in $S_n\big(\Psi(R_n)\big)$, f_n^{Kp} will add $(\beta_{n,n})$ to $\Psi(R_n)$.

If $z = 1$, then the leftmost '(' in $\mathrm{br}_n(R_n)$ comes from the $\boxed{0}$ so $f_n^{\mathcal{T}}(R_i)$ sends the $\boxed{0}$ to $\boxed{\overline{n}}$. According to Ψ we then have that

$$\Psi(f_n^{\mathcal{T}} R_n) = \Psi(R_n) + (\beta_{n,n}) = f_n^{\mathrm{Kp}}\Psi(R_n).$$

If $z = 0$, then the leftmost '(' comes from an $\boxed{n}$ so $f_n^{\mathcal{T}} R_n$ sends an $\boxed{n}$ to an $\boxed{0}$. Again

$$\Psi(f_n^{\mathcal{T}} R_n) = \Psi(R_n) + (\beta_{n,n}) = f_n^{\mathrm{Kp}}\Psi(R_n).$$

Now, consider T to be of type C_n. Assume row j of T has p $\boxed{\overline{n}}$ boxes and q $\boxed{n}$ boxes,

$$R_j = \underbrace{\boxed{n}\boxed{\cdots}\boxed{n}}_{q}\underbrace{\boxed{\overline{n}}\boxed{\cdots}\boxed{\overline{n}}}_{p}.$$

The bracketing sequences are:

$$\mathrm{br}_n(R_j) =)^{p}\ (^{q}\ , \qquad S_n\big(\Psi(R_j)\big) =)^{c_{\gamma_{j,j}}}\ (^{c_{\beta_{j,n-1}}}\ )^{c_{\gamma_{j,n}}}\ (^{c_{\gamma_{j,j}}}.$$

Case 6: $p \geq q$, and $1 \leq j < n$. By the definition of Ψ,

$$\Psi(R_j) = q(\gamma_{j,j}) + (p-q)(\gamma_{j,n}).$$

If $q = 0$ then f_n will act on the $\boxed{n}$ in R_n of T (see Case 9 for more details in this situation). When $q > 0$ by the definition of $f_n^{\mathcal{T}}$ we have

$$f_n^{\mathcal{T}} R_j = \underbrace{\boxed{n}\boxed{\cdots}\boxed{n}}_{q-1}\underbrace{\boxed{\overline{n}}\boxed{\cdots}\boxed{\overline{n}}}_{p+1}.$$

There are two more unpaired $\boxed{\overline{n}}$ and one less $\left(\boxed{n},\boxed{\overline{n}}\right)$, so

$$\Psi(f_n^{\mathcal{T}} R_j) = (q-1)(\gamma_{j,j}) + (p-q+2)(\gamma_{j,n}).$$

On the other hand, in $S_n^c\big(\Psi(R_j)\big)$ the leftmost '(' corresponds to $\gamma_{j,j}$ so

$$f_n^{\mathrm{Kp}}\Psi(R_j) = (q-1)(\gamma_{j,j}) + (p-q+2)(\gamma_{j,n}) = \Psi(f_n^{\mathcal{T}} R_j).$$

Case 7: $q > p+1$, and $1 \leq j < n$. By the definition of Ψ,

$$\Psi(R_j) = (q-p)(\beta_{j,n-1}) + p(\gamma_{j,j}).$$

By the definition of $f_n^{\mathcal{T}}$, we have

$$f_n^{\mathcal{T}} R_j = \underbrace{\boxed{n}\boxed{\cdots}\boxed{n}}_{q-1}\underbrace{\boxed{\overline{n}}\boxed{\cdots}\boxed{\overline{n}}}_{p+1}.$$

There is one more $\left(\boxed{n},\boxed{\overline{n}}\right)$ pair and two less unpaired $\boxed{n}$, so

$$\Psi(f_n^{\mathcal{T}} R_j) = (q-p-2)(\beta_{j,n-1}) + (p+1)(\gamma_{j,j}).$$

On the other hand, in $S_n^c\big(\Psi(R_j)\big)$ the leftmost '(' corresponds to $\beta_{j,n-1}$ so

$$f_n^{\mathrm{Kp}}\Psi(R_j) = (q-p-2)(\beta_{j,n-1}) + (p+1)(\gamma_{j,j}) = \Psi(f_n^{\mathcal{T}} R_j).$$

Case 8: $q = p+1$, and $j < n$. By the definition of Ψ,

$$\Psi(R_j) = (q-p)(\beta_{j,n-1}) + p(\gamma_{j,j}).$$

By the definition of $f_n^{\mathcal{T}}$, we have

$$f_n^{\mathcal{T}} R_j = \underbrace{\boxed{n}\boxed{\cdots}\boxed{n}}_{q-1}\underbrace{\boxed{\overline{n}}\boxed{\cdots}\boxed{\overline{n}}}_{p+1}.$$

Since $q-p=1$ the number of $\left(\boxed{n},\boxed{\overline{n}}\right)$ pairs is unchanged. There is one less $\boxed{n}$ and one more $\boxed{\overline{n}}$, so

$$\Psi(f_n^{\mathcal{T}} R_j) = (q-p-1)(\beta_{j,n-1}) + (\gamma_{j,n}) + p(\gamma_{j,j}).$$

On the other hand, in $S_n^c\big(\Psi(R_j)\big)$ the leftmost '(' corresponds to $\beta_{j,n-1}$ so

$$f_n^{\mathrm{Kp}}\Psi(R_j) = (q-p-1)(\beta_{j,n-1}) + (\gamma_{j,n}) + p(\gamma_{j,j}) = \Psi(f_n^{\mathcal{T}} R_j).$$

Case 9: $j = n$. The only positive root that can be in $\Psi(R_n)$ is $(\gamma_{n,n})$, so there is no '(' in $S_n\big(\Psi(R_n)\big)$ and, by Definition 2.14, f_n^{Kp} adds a $(\gamma_{n,n})$. The leftmost '(' in $\mathrm{br}_n(T)$ comes from an $\boxed{n}$, so $f_n^{\mathcal{T}}$ sends an $\boxed{n}$ to an $\boxed{\overline{n}}$. Hence $\Psi(f_n^{\mathcal{T}} R_n) = \Psi(R_n) + (\gamma_{n,n}) = f_n^{\mathrm{Kp}}\Psi(R_n)$. ∎

Proof of Theorem 3.1. It suffices to show that for all i we have $f_i^{\mathrm{Kp}}\Psi(T) = \Psi(f_i^{\mathcal{T}} T)$. By the definition of the bracketing sequences and of Ψ, we have

$$\mathrm{br}_i(T) \text{ factors as } \mathrm{br}_i(R_1)\mathrm{br}_i(R_2)\cdots\mathrm{br}_i(R_n), \text{ and}$$

$$S_i\big(\Psi(T)\big) \text{ factors as } S_i\big(\Psi(R_1)\big)S_i\big(\Psi(R_2)\big)\cdots S_i\big(\Psi(R_n)\big).$$

Suppose that the leftmost '(' in $\mathrm{br}_i^c(T)$ comes from row R_j. There will always be an uncanceled bracket coming from row i so we may assume $j \le i$. By applying Lemma 3.4 or Lemma 3.5 to each R_j, the leftmost '(' in $S_i\big(\Psi(T)\big)$ comes from $S_i\big(\Psi(R_j)\big)$, and also $\Psi\big(f_i^{\mathcal{T}} R_j\big) = f_i^{\mathrm{Kp}}\Psi(R_j)$. The result follows. ■

4. Stack notation

This work extends results from [3, 11] in types A_n and D_n to types B_n and C_n. The type A_n result can be described using the multisegments from [6, 8, 13] which are a diagrammatic notation that makes the crystal structure apparent. In [11] this was extended to type D_n by introducing a *stack* notation for Kostant partitions in which the crystal structure can easily be read off. We now define a similar *stack* notation for types B_n and C_n.

In type B_n we associate positive roots to "stacks" with

$$\beta_{j,k} = \begin{array}{c} k \\ \vdots \\ j \end{array}, \qquad \gamma_{\ell,m} = \begin{array}{c} m \\ \vdots \\ n{-}1 \\ n\ n \\ n{-}1 \\ \vdots \\ \ell \end{array},$$

for $1 \le j \le k \le n$ and $1 \le \ell < m \le n$.

In type C_n we associate positive roots to "stacks" with

$$\beta_{j,k} = \begin{array}{c} k \\ \vdots \\ j \end{array}, \qquad \gamma_{\ell,m} = \begin{array}{c} m \\ \vdots \\ n{-}1 \\ n \\ n{-}1 \\ \vdots \\ \ell \end{array}, \qquad \gamma_{h,h} = \begin{array}{c} n \\ n{-}1\ n{-}1 \\ \vdots \\ h\ h \end{array},$$

for $1 \le j \le k < n$, $1 \le \ell < m \le n$, and $1 \le h \le n$.

Then the sequences of roots Φ_i from Definition 2.12 are exactly those positive roots where we can either add or remove an i from the top of the corresponding stack and still have either a valid stack, an empty stack, or in type C_n with $i = n$ where we have two valid stacks side by side. Once the stacks are ordered as in Definition 2.12, the bracketing sequence is created by placing a left bracket for each i that can be added to the top of a stack, and a right bracket for each i that can be removed from the top. Note that if both happen then the root corresponding to the stack appears twice in Definition 2.12, in which case the ')' is placed over the left copy and the '(' over the right copy. If there is a leftmost uncanceled '(' the crystal operator f_i adds an i to the top of the corresponding stack (or, in the case of $i = n$ in type C_n, may combine two stacks together and attach an n at the top). Otherwise f_i creates a new stack consisting of just i.

Remark 4.1. Being able to add or remove an i from the top of a stack is different from being able to add or remove an α_i from the corresponding root. For instance, in type B_3, if $\beta = \alpha_1+\alpha_2+2\alpha_3$, then $\beta-\alpha_1$ is a root, but there is no 1 at the top of the stack corresponding to β, so β is not in Φ_1^B. Similarly, in type C_3, although $\substack{2\\3\\2\\1}$ is a stack, $\alpha_1+2\alpha_2+\alpha_3$ is not in Φ_1^C because the stack for $2\alpha_1+2\alpha_2+\alpha_3$ is $\begin{smallmatrix}&3&\\2&&2\\1&&1\end{smallmatrix}$, not $\substack{1\\2\\3\\2\\1}$.

Example 4.2. Consider type C_3 and $\boldsymbol{\alpha} \in \mathrm{Kp}(\infty)$ given in stack notation by

$$\boldsymbol{\alpha} = \substack{2\\1}\ \substack{2\\1}\ \substack{2\\1}\ \substack{3\\2\\1}\ \substack{3\\2\\1}\ \begin{smallmatrix}&3&\\2&&2\\1&&1\end{smallmatrix}\ \begin{smallmatrix}&3&\\2&&2\\1&&1\end{smallmatrix}\ \begin{smallmatrix}&3&\\2&&2\end{smallmatrix}\ \substack{2\\3}\ 3\,.$$

The corresponding 3-signature is

$$\begin{array}{rccccccccccccc}
 & \begin{smallmatrix}&3&\\2&&2\\1&&1\end{smallmatrix} & \begin{smallmatrix}&3&\\2&&2\\1&&1\end{smallmatrix} & \substack{2\\1} & \substack{2\\1} & \substack{2\\1} & \substack{3\\2\\1} & \substack{3\\2\\1} & \begin{smallmatrix}&3&\\2&&2\\1&&1\end{smallmatrix} & \begin{smallmatrix}&3&\\2&&2\\1&&1\end{smallmatrix} & \begin{smallmatrix}&3&\\2&&2\end{smallmatrix} & \substack{3\\2} & \begin{smallmatrix}&3&\\2&&2\end{smallmatrix} & 3 \\
S_3(\boldsymbol{\alpha}) = &) &) & (& (& (&) &) & (& (&) &) & (&) \\
S_3^c(\boldsymbol{\alpha}) = &) &) & (& & & & & & & & & &
\end{array}.$$

Thus the action of f_3 on $\boldsymbol{\alpha}$ adds a 3 to top of a $\substack{2\\1}$. This gives

$$f_3\boldsymbol{\alpha} = \substack{2\\1}\ \substack{2\\1}\ \substack{3\\2\\1}\ \substack{3\\2\\1}\ \substack{3\\2\\1}\ \begin{smallmatrix}&3&\\2&&2\\1&&1\end{smallmatrix}\ \begin{smallmatrix}&3&\\2&&2\\1&&1\end{smallmatrix}\ \begin{smallmatrix}&3&\\2&&2\end{smallmatrix}\ \substack{2\\3}\ 3\,.$$

Example 4.3. Consider type C_3 and $\boldsymbol{\alpha}$ as in Example 2.16. In stack notation,

$$\boldsymbol{\alpha} = \substack{2\\1}\ \substack{2\\1}\ \substack{3\\2\\1}\ \substack{3\\2\\1}\ \begin{smallmatrix}&3&\\2&&2\\1&&1\end{smallmatrix}\ \begin{smallmatrix}&3&\\2&&2\\1&&1\end{smallmatrix}\ \begin{smallmatrix}&3&\\2&&2\\1&&1\end{smallmatrix}\ \begin{smallmatrix}&3&\\2&&2\end{smallmatrix}\ \substack{3\\2}\ 3\,.$$

Recalculating the 3-signature using stack notation gives

$$\begin{array}{rcccccccccccccc}
 & \begin{smallmatrix}&3&\\2&&2\\1&&1\end{smallmatrix} & \begin{smallmatrix}&3&\\2&&2\\1&&1\end{smallmatrix} & \begin{smallmatrix}&3&\\2&&2\\1&&1\end{smallmatrix} & \substack{2\\1} & \substack{2\\1} & \substack{3\\2\\1} & \substack{3\\2\\1} & \begin{smallmatrix}&3&\\2&&2\\1&&1\end{smallmatrix} & \begin{smallmatrix}&3&\\2&&2\\1&&1\end{smallmatrix} & \begin{smallmatrix}&3&\\2&&2\\1&&1\end{smallmatrix} & \begin{smallmatrix}&3&\\2&&2\end{smallmatrix} & \substack{3\\2} & \begin{smallmatrix}&3&\\2&&2\end{smallmatrix} & 3 \\
S_3(\boldsymbol{\alpha}) = &) &) &) & (& (&) &) & (& (& (&) &) & (&) \\
S_3^c(\boldsymbol{\alpha}) = &) &) &) & & & & & (& & & & & &
\end{array}.$$

Since the leftmost '(' comes from a $\begin{smallmatrix}&3&\\2&&2\\1&&1\end{smallmatrix}$, we should add a 3 to the top of this stack, which gives $\begin{smallmatrix}3&3\\2&2\\1&1\end{smallmatrix}$. That is not the stack of a single root, but should be thought of as two copies of $\substack{3\\2\\1}$, which is the stack of a root. The result is

$$f_3\boldsymbol{\alpha} = \substack{2\\1}\ \substack{2\\1}\ \substack{3\\2\\1}\ \substack{3\\2\\1}\ \substack{3\\2\\1}\ \substack{3\\2\\1}\ \begin{smallmatrix}&3&\\2&&2\\1&&1\end{smallmatrix}\ \begin{smallmatrix}&3&\\2&&2\\1&&1\end{smallmatrix}\ \begin{smallmatrix}&3&\\2&&2\end{smallmatrix}\ \substack{3\\2}\ 3\ .$$

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