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# Exactly solvable systems and the Quantum Hamilton-Jacobi formalism

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## Abstract

We connect Quantum Hamilton-Jacobi Theory with supersymmetric quantum mechanics (SUSYQM). We show that the shape invariance, which is an integrability condition of SUSYQM, translates into fractional linear relations among the quantum momentum functions.

Supersymmetric quantum mechanics (SUSYQM) has provided a powerful tool in analyzing the underlying structure of the Schrödinger equation [1]. In addition to connecting apparently distinct potentials, SUSYQM allows for algebraic solutions to a large class of such potentials: the known shape invariant potentials [2].

Another formulation of quantum mechanics, the Quantum Hamilton-Jacobi (QHJ) formalism, was developed by Leacock and Padgett [3] and independently by Gozzi [4]. In this formalism, which follows classical mechanics closely, one works with the quantum momentum function  $p(x)$ , which is the quantum analog of the classical momentum function  $p_c(x) = \sqrt{E - V}$ . It was shown [3, 4] that the singularity structure of the function  $p(x)$  determines the eigenvalues of the Hamiltonian. Kapoor and his collaborators [5] have shown that the QHJ formalism can be used not only to determine the eigenvalues of the Hamiltonian of the system, but also its eigenfunctions.

In this letter, we connect the Quantum Hamilton-Jacobi formalism to supersymmetry and shape invariance. Using shape invariance, we show that quantum momentum functions

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corresponding to different energies are connected via fractional linear transformations, and we give a general recursion formula for quantum momenta of any energy.

In QHJ formalism the spectrum of a quantum mechanical system is determined by the solution of the equation:

$$-i p'(x, \alpha) + p^2(x, \alpha) = E - V(x, \alpha) \equiv p_c^2(x, \alpha) \quad (1)$$

Here  $\hbar = 1$  and  $2m = 1$  and  $\alpha$  is a parameter characterizing the strength of the potential. This equation is related to the Schrödinger equation

$$-\psi'' + (V(x, \alpha) - E)\psi = 0 \quad (2)$$

via the correspondence

$$p = -i \left( \frac{\psi'}{\psi} \right) \quad \text{whence} \quad \psi(x) \sim e^{i \int p(x) dx} \quad .$$

In SUSYQM, the supersymmetric partner potentials  $V_-$  and respectively  $V_+$  can be written as  $V_-(x, \alpha) = W^2(x, \alpha) - W'(x, \alpha)$ , and  $V_+(x, \alpha) = W^2(x, \alpha) + W'(x, \alpha)$ .  $W(x, \alpha)$  is a real function, called the superpotential [1].  $V_-(x, \alpha)$  is chosen as the potential for a Hamiltonian  $H_-(x, \alpha)$  whose ground state eigenvalue  $E_0^{(-)} = 0$ . The partner Hamiltonian  $H_+(x, \alpha)$  built using the potential  $V_+(x, \alpha)$  has the same set of eigenvalues  $E_n^{(+)} = E_{n+1}^{(-)}$ , except for the groundstate.

We will focus for the moment on  $H_-(x, \alpha)$ . Let us denote the eigenfunctions of  $V_-$  by  $\psi^{(-)}$ , and by  $p^{(-)} \equiv -i \psi^{(-)'} / \psi^{(-)}$  the corresponding quantum momentum. The QHJ equation for the potential  $V_-(x, \alpha)$  can now be written as

$$-i p^{(-)'}(x, \alpha) + p^{(-)2}(x, \alpha) = E^{(-)} - [W^2(x, \alpha) - W'(x, \alpha)] \quad . \quad (3)$$

One can show that for energy  $E^{(-)} = 0$ , eq. (3) has the solution

$$p_{E=0}^{(-)}(x, \alpha) = i W(x, \alpha) \quad . \quad (4)$$

The above equation can be viewed as providing the initial condition on  $p_{E=0}^{(-)}(x, \alpha)$ , which in this case is induced by the presence of supersymmetry.

Considering now the supersymmetric partner Hamiltonian  $H_+(x, \alpha)$ , there exists another analogous equation for a partner QMF,  $p^{(+)}(x, \alpha)$  given by

$$-i p^{(+)\prime}(x, \alpha) + p^{(+)\,2}(x, \alpha) = E^{(+)} - [W^2(x, \alpha) + W'(x, \alpha)] . \quad (5)$$

Supersymmetry ensures that these equations lead to the same set of eigenvalues, except for the groundstate. Let us denote by  $E \equiv E^{(-)} = E^{(+)}$ . The corresponding Schrödinger equations are

$$H_{\pm}(x, \alpha) \psi^{(\pm)} = -\psi^{(\pm)\prime\prime} + [W^2(x, \alpha) \pm W'(x, \alpha)] \psi^{(\pm)} = E \psi^{(\pm)} , \quad (6)$$

and the solutions are connected to the QHJ solutions by

$$p^{(\pm)} = -i \left( \frac{\psi^{(\pm)\prime}}{\psi^{(\pm)}} \right) . \quad (7)$$

Defining operators  $A^\dagger = -\frac{d}{dx} + W$  and  $A = \frac{d}{dx} + W$ , one can rewrite the partner Hamiltonians as  $H_+ = AA^\dagger$  and respectively  $H_- = A^\dagger A$ . Furthermore, one finds that  $A^\dagger$  and  $A$  behave as raising and lowering operators between the eigenstates of the partner Hamiltonians. In particular,

$$\begin{aligned} \psi^{(-)} &= C^{(-)} A^\dagger \psi^{(+)} = C^{(-)} (-\psi^{(+)\prime} + W \psi^{(+)}) \\ \psi^{(+)} &= C^{(+)} A \psi^{(-)} = C^{(+)} (\psi^{(-)\prime} + W \psi^{(-)}) , \end{aligned} \quad (8)$$

where  $C^{(-)}$  and  $C^{(+)}$  are normalization constants. To find a relationship between  $p^{(-)}$  and  $p^{(+)}$  we exploit the connection between  $\psi^{(-)}$  and  $\psi^{(+)}$  respectively. Using the equations (6, 7, 8), we obtain

$$\frac{\psi^{(+)\prime}}{\psi^{(-)}} = C^{(+)} (W^2 - E + i W p^{(-)}) \quad (9)$$

$$\frac{\psi^{(-)\prime}}{\psi^{(+)}} = C^{(-)} (-W^2 + E + i W p^{(+)}) . \quad (10)$$

Multiplying eqs. (9) and (10) we get

$$-p^{(+)} p^{(-)} = C^{(-)} C^{(+)} (W^2 - E + i W p^{(-)}) (-W^2 + E + i W p^{(+)}) .$$

To solve the above equation for  $p^{(+)}$  or  $p^{(-)}$ , we evaluate first the product of the normalization constants  $C^{(-)} C^{(+)}$ . Since,  $\psi^{(+)} = C^{(+)} A \psi^{(-)}$ , and  $\psi^{(-)} = C^{(-)} A^\dagger \psi^{(+)}$ , we obtain successively

$$\langle \psi^{(+)} | \psi^{(+)} \rangle = C^{(+)} \langle \psi^{(+)} | A | \psi^{(-)} \rangle = C^{(+)} \langle \psi^{(+)} | A C^{(-)} A^\dagger | \psi^{(+)} \rangle = C^{(+)} C^{(-)} E \langle \psi^{(+)} | \psi^{(+)} \rangle ,$$

where we have used  $AA^\dagger = H_+$  as noted earlier. Thus,  $C^{(+)}C^{(-)} = 1/E$ . This leads to

$$p^{(+)} = \frac{iWp^{(-)} + W^2 - E}{-p^{(-)} + iW}, \quad (11)$$

or,

$$p^{(-)} = \frac{-iWp^{(+)} + W^2 - E}{-p^{(+)} - iW}. \quad (12)$$

It is important to note at this point that both sides of eqs. (11) and (12) are related to the same superpotential  $W(x, \alpha)$  and can be denoted by  $p^{(\pm)}(x, \alpha)$ . Also note that  $p^{(+)}$  in eq. (11) is not defined for the groundstate for which  $-p^{(-)} + iW = 0$ .<sup>5</sup>

We have only applied the conditions of supersymmetry so far. To render a Hamiltonian solvable, the superpotential  $W(x, \alpha)$  needs to satisfy a condition of integrability known as shape invariance. (This additional constraint helps close a potential algebra for the system [6].) We now consider the impact of shape invariance on the Hamiltonian-Jacobi formalism.

Shape invariance allows one to find the solution of  $H_+$  or  $H_-$  by algebraic means. If the partner potentials can be related by a single ‘shift’ of parameters  $\alpha$ :  $V^{(+)}(x, \alpha_i) = V^{(-)}(x, \alpha_{i+1}) + R(\alpha_i)$ , where  $R(\alpha_i)$  is a constant, then  $\psi_n^{(+)}(x, \alpha_i)$  may be related to  $\psi_n^{(-)}(x, \alpha_{i+1})$  [2]. But since  $\psi_n^{(-)}(x, \alpha_{i+1})$  is already related to  $\psi_{n-1}^{(+)}(x, \alpha_{i+1})$  by operators  $A$  and  $A^\dagger$ , this connects  $\psi_n^{(+)}(x, \alpha_i)$  with  $\psi_{n-1}^{(+)}(x, \alpha_{i+1})$ , and so forth. That is, a simple parameter shift allows for construction of the entire ladder of eigenstates  $\psi_n^{(-)}$  or  $\psi_n^{(+)}$ . Furthermore,  $E_n = \sum_{i=0}^{n-1} R(\alpha_i)$ . We shall employ a similar technique to construct the ladders of QMF’s  $p^{(-)}$ . Let us replace the subscript  $n$ , which was a label for energy  $E$ , with  $E$  itself. Subscript  $n - 1$  is replaced by  $E - R(\alpha_i)$ . Thus, shape invariance identifies  $\psi_E^{(+)}(x, \alpha_i)$  with  $\psi_{E-R(\alpha_i)}^{(-)}(x, \alpha_{i+1})$ , which in QHJ becomes a relationship between quantum momentum functions. From eq. (7), one finds  $p_{E-R(\alpha_i)}^{(-)}(x, \alpha_{i+1}) = p_E^{(+)}(x, \alpha_i)$ . Substituting this relation in eq. (11), we get the following recursion relation for  $p_E^{(-)}(x, \alpha_i)$ :

$$p_{E-R(\alpha_i)}^{(-)}(x, \alpha_{i+1}) = \frac{iW(x, \alpha_i)p_E^{(-)}(x, \alpha_i) + W^2(x, \alpha_i) - E(\alpha_i)}{-p_E^{(-)}(x, \alpha_i) + iW(x, \alpha_i)} \quad (13)$$

We invert this relation to determine  $p_E^{(-)}(x, \alpha_i)$ , which is given by:

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<sup>5</sup>This is due to the unbroken nature of supersymmetry which implies that there is no normalizable  $\psi^{(+)}$  at zero energy.

$$p_E^{(-)}(x, \alpha_i) = \frac{i W(x, \alpha_i) p_{E-R(\alpha_i)}^{(-)}(x, \alpha_{i+1}) - W^2(x, \alpha_i) + E(\alpha_i)}{p_{E-R(\alpha_i)}^{(-)}(x, \alpha_{i+1}) + i W(x, \alpha_i)}. \quad (14)$$

This recursion relation, along with the initial condition given by eq. (4) determines all functions  $p_E^{(-)}(x, \alpha_i)$ . Thus, eq. (13) is the shape invariance integrability relation for Quantum Hamilton Jacobi formalism.

To provide a concrete example, let us determine the quantum momentum function related to the first non-zero eigenstate of the system,  $E = R(\alpha_1)$ . Thus, in eq. (14), let us substitute  $E - R(\alpha_2) = 0$ . Then  $p_{E-R(\alpha_2)}^{(-)} = p_0^{(-)} = iW$

$$\begin{aligned} -i p_{R(\alpha_1)}^{(-)}(x, \alpha_1) &= \frac{W(x, \alpha_2) \cdot W(x, \alpha_1) + W^2(x, \alpha_1) - R(\alpha_1)}{W(x, \alpha_1) + W(x, \alpha_2)}, \\ &= W(x, \alpha_1) - \frac{R(\alpha_1)}{W(x, \alpha_1) + W(x, \alpha_2)}. \end{aligned} \quad (15)$$

Therefore, starting from  $p_0^{(-)}(x, \alpha_2) = W(x, \alpha_2)$ , we have derived the higher level QMF, i.e.,  $p_{R(\alpha_1)}^{(-)}(x, \alpha_1)$ .

This procedure can be iterated to generate  $p_E^{(-)}(x, \alpha_1)$  for any eigenvalue  $E$ . Using the recursion formula (13) we can write

$$p_{E-R(\alpha_1)}^{(-)}(x, \alpha_2) = \frac{i W(x, \alpha_1) p_E^{(-)}(x, \alpha_1) + W^2(x, \alpha_1) - E}{-p_E^{(-)}(x, \alpha_1) + i W(x, \alpha_1)} \quad (16)$$

and respectively

$$p_{E'-R(\alpha_2)}^{(-)}(x, \alpha_3) = \frac{i W(x, \alpha_2) p_{E'}^{(-)}(x, \alpha_2) + W^2(x, \alpha_2) - E'}{-p_{E'}^{(-)}(x, \alpha_2) + i W(x, \alpha_2)}. \quad (17)$$

Now, let us consider the energy  $E' = E - R(\alpha_1)$ ; then eq. (17) becomes

$$p_{E-R(\alpha_2)-R(\alpha_1)}^{(-)}(x, \alpha_3) = \frac{i W(x, \alpha_2) p_{E-R(\alpha_1)}^{(-)}(x, \alpha_2) + W^2(x, \alpha_2) - E + R(\alpha_1)}{-p_{E-R(\alpha_1)}^{(-)}(x, \alpha_2) + i W(x, \alpha_2)}. \quad (18)$$

Plugging (16) into (18) we obtain

$$p_E^{(-)}(x, \alpha_1) = \frac{A_3 p_{E-R(\alpha_2)-R(\alpha_1)}^{(-)}(x, \alpha_3) + B_3}{C_3 p_{E-R(\alpha_2)-R(\alpha_1)}^{(-)}(x, \alpha_3) + D_3} \quad (19)$$

where

$$\begin{aligned}
A_3 &= E - R(\alpha_1) - W(x, \alpha_2)W(x, \alpha_1) - W^2(x, \alpha_2) , \\
B_3 &= iW(x, \alpha_2) (W^2(x, \alpha_1) - E) + iW(x, \alpha_1) (W^2(x, \alpha_2) - E + R(\alpha_1)) , \\
C_3 &= -iW(x, \alpha_2) - iW(x, \alpha_1) , \\
D_3 &= E - W(x, \alpha_2)W(x, \alpha_1) - W^2(x, \alpha_1) .
\end{aligned}$$

We note that the general recursion equation (13) is a fractional linear transformation in  $p_E^{(-)}$ . A fractional linear transformation [7] has the general form

$$f(z) = \frac{az + b}{cz + d} . \quad (20)$$

The composition of two fractional linear transformations may be tedious to compute. A short-cut is provided by the map

$$\frac{az + b}{cz + d} \mapsto \begin{bmatrix} a & b \\ c & d \end{bmatrix} . \quad (21)$$

It is easy to check that the function composition corresponds to matrix multiplication. That is, if  $f_1$  and  $f_2$  are two transformations given by

$$f_1(z) = \frac{a_1z + b_1}{c_1z + d_1} , \quad f_2(z) = \frac{a_2z + b_2}{c_2z + d_2} , \quad (22)$$

then

$$(f_2 \circ f_1)(z) = \frac{az + b}{cz + d} , \quad (23)$$

where the coefficients  $a, b, c$  and  $d$  are given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \cdot \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} . \quad (24)$$

For any transformation  $f$  of form (20) there exist an inverse transformation  $f^{-1}$  if and only if  $ad - bc \neq 0$ . In this case the matrix associated to  $f$  is nonsingular and its inverse gives the coefficients of  $f^{-1}$ . The invertible linear fractional transformations form a group isomorphic to  $PGL(2, \mathbb{C})$ , the projective group of  $2 \times 2$  matrices with complex entries. Projective implies simply that the matrices

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$$

are considered identical, because they correspond to the same linear fractional transformation.

Now, let us associate to transformation (16) the matrix

$$m_1 = \begin{bmatrix} iW(x, \alpha_1) & W^2(x, \alpha_1) - E \\ -1 & iW(x, \alpha_1) \end{bmatrix}. \quad (25)$$

Similarly, we associate to transformation (17) the matrix

$$m_2 = \begin{bmatrix} iW(x, \alpha_2) & W^2(x, \alpha_2) - E + R(\alpha_1) \\ -1 & iW(x, \alpha_2) \end{bmatrix}. \quad (26)$$

Then the coefficients  $A_3, B_3, C_3, D_3$  of eq. (19) are simply obtained from

$$\begin{bmatrix} A_3 & B_3 \\ C_3 & D_3 \end{bmatrix} = m_2 \cdot m_1. \quad (27)$$

Using the fractional linear transformation property, we can easily generalize this result. We have

$$p_E^{(-)}(x, \alpha_1) = \frac{A_{n+1} p_{E - \sum_{i=1}^n R(\alpha_i)}^{(-)}(x, \alpha_{n+1}) + B_{n+1}}{C_{n+1} p_{E - \sum_{i=1}^n R(\alpha_i)}^{(-)}(x, \alpha_{n+1}) + D_{n+1}}, \quad (28)$$

where

$$\begin{bmatrix} A_{n+1} & B_{n+1} \\ C_{n+1} & D_{n+1} \end{bmatrix} = m_n \cdot m_{n-1} \cdots m_1 \quad (29)$$

and

$$m_k = \begin{bmatrix} iW(x, \alpha_k) & W^2(x, \alpha_k) - E + \sum_{j=1}^{k-1} R(\alpha_j) \\ -1 & iW(x, \alpha_k) \end{bmatrix}, \quad k = 1, 2, \dots, n. \quad (30)$$

Note that the determinant of  $m_k$  is

$$\det(m_k) = E - \sum_{j=1}^{k-1} R(\alpha_j), \quad (31)$$

therefore, for those values of energy where  $E = \sum_{j=1}^{k-1} R(\alpha_j)$  the matrix  $m_k$  is singular. This property is going to be connected with the structure of poles of the corresponding  $p_E^{(-)}(x, \alpha)$ , and thus with obtaining of the eigenvalues of the Hamiltonian through the behavior of the QMF's in the complex plane. We shall comment on these results in a subsequent publication.



We have connected Quantum Hamilton-Jacobi Theory with supersymmetric quantum mechanics, and have shown that the quantum momenta of supersymmetric partner potentials are connected via linear fractional transformations. Then, by making use of the matrix representation of the linear fractional transformations, we have derived specific quantum momentum recursion relations for any shape-invariant potential. This connection of supersymmetric QHJ with the underlying group theory should provide a deeper understanding of the properties of the QMF's, in much the same way that our earlier work [6] demonstrated the group structure which underlies the family of shape invariant potentials.

Still to be explored are the specifics of the pole structure of the QMF's, and the constraints that they place on calculations involving superpotentials. We are also investigating the relation between QHJ and supersymmetry for the Dirac Equation. Work in this area on QHJ has produced promising results [8]. The application to supersymmetry should provide new methods for obtaining algebraically the energies and the components of the Dirac spinors for those cases where the potentials are shape invariant.

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