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ESTIMATION OF LOCATION PARAMETER WITHIN PRE-SPECIFIED ERROR BOUND WITH SECOND-ORDER EFFICIENT TWO-STAGE PROCEDURE

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This paper develops a general approach for constructing a confidence interval for a parameter of interest with a specified confidence coefficient and a specified width. This is done assuming known a positive lower bound for the unknown nuisance parameter and independence of suitable statistics. Under mild conditions, we develop a modified two-stage procedure which enjoys attractive optimality properties including a second-order efficiency property and asymptotic consistency property. We extend this work for finding a confidence interval for the location parameter of the inverse Gaussian distribution. As an illustration, we developed a modified mean absolute deviation-based procedure in the supplementary section for finding a fixed-width confidence interval for the normal mean.

Key words: Asymptotic efficiency, Taylor's theorem, Two-stage confidence interval procedure.

1. Introduction

It is well known that the length of a $100(1-\alpha)\%$ confidence interval of a parameter decreases if we increase the sample size, but this, however, increases the overall sampling cost. Using a smaller sample might decrease the sampling cost but it might increase the width of the confidence interval. One way to solve this problem is to fix the width of this confidence interval and try to minimise the sample size or in other words the sampling cost. The problem of finding a fixed-width confidence interval for a parameter in the presence of a nuisance parameter cannot be solved with sample size fixed in advance. This problem can only be solved using two-stage or multi-stage sampling methods. Unlike fixed-sample procedures, in sequential or multi-stage procedures, sample sizes are not fixed beforehand. The final sample size depends on the statistical analysis carried out on the already collected observations. A two-stage procedure involves only two steps of sampling observations and is operationally more convenient than multi-stage procedures in many cases. For an extensive review of the literature one may refer to Stein (1945, 1949), Ghosh et al. (1997), Mukhopadhyay and de Silva (2009), and others.

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Under mild restrictions and $0 < \alpha < 1$ fixed, Mukhopadhyay (1982) proposed a generalised version of the two-stage procedure for finding a $100(1-\alpha)\%$ fixed-width $(= 2d)$ confidence interval for a parameter which satisfies the first-order efficiency property. Mukhopadhyay and Duggan (1997) proposed a modified two-stage procedure for finding a fixed-width confidence interval for the mean of the normal distribution based on the sample variance. This procedure satisfies a second-order efficiency property when the data distribution is normal. Chattopadhyay and Mukhopadhyay (2012) proposed a modified two-stage procedure for finding a fixed-width confidence interval for a normal mean based on the sample Gini's mean difference which satisfies a second-order efficiency property when the data distribution is normal with few suspect outliers. Here, we develop a general approach for constructing a fixed-width confidence interval for the location parameter which satisfies attractive asymptotic properties.

The remaining sections are organized as follows. In Section 2, using the asymptotic expression of the percentile points of the pivot, we propose a modified two-stage procedure when a lower bound of the nuisance parameter (i.e. the parameter not of interest) is known. Under mild conditions, we prove the second-order efficiency property of this proposed modified two-stage procedure. In Section 3, we extend this estimation methodology to propose a fixed-precision confidence interval for the inverse Gaussian location parameter assuming that a lower bound for the scale parameter is known. Under mild conditions, this proposed two-stage procedure is also shown to satisfy the second-order efficiency property. Section 4 summarises our concluding thoughts. Section 5 (Appendix) includes detailed proofs of the Theorems stated in Sections 3 and 4.

2. Modified two-stage procedure

Consider the problem of constructing a fixed-width confidence interval for a parameter of interest in the presence of a nuisance parameter. We now formulate the problem along the lines of Mukhopadhyay (1982).

Let $X_1, X_2, \ldots$ be random variables from a continuous distribution with common density function f, with two unknown parameters θ and ξ, such that $(\theta, \xi) \in (\Re \times \Re^+)$. Here θ is the parameter of interest while ξ is the nuisance parameter. Suppose it is reasonable to assume that a lower bound ξ_L, say, of the nuisance parameter ξ is known, such that $0 < \xi_L < \xi$.

Let $U_m = U_m(X_1, \ldots, X_m)$ be an unbiased estimator of θ and $V_m = V_m(X_1, \ldots, X_m)$ be an estimator of ξ. Let us define the following standardised version of the sample location:

$$W_m = \frac{m^\beta (U_m - \theta)}{T_m}, \quad T_m = g(V_m), \tag{2.1}$$

for some $\beta > 0$. Suppose that both U_m and V_m satisfy the following conditions:

(a) For any m $(m \geq 2)$, U_m is independent of $(V_2, \ldots, V_m)$.

(b) (i) The distribution of $m^\beta \left(U_m - \theta\right) / g(\xi)$ does not depend on m, θ and ξ. Let $k_{\alpha/2}$ denote the upper $100(\alpha/2)\%$ point of the distribution of $m^\beta \left(U_m - \theta\right) / g(\xi)$, that is,

$$F(k_{\alpha/2}) = P\left(\frac{m^\beta \left(U_m - \theta\right)}{g(\xi)} \leq k_{\alpha/2}\right) = 1 - \alpha/2, \quad 0 < \alpha < 1.$$

Since $g(\xi)$ is unknown, we replace $g(\xi)$ with its estimator $T_m = g(V_m)$.

(ii) The distribution of W_m does not involve θ or ξ. Define $b_{m,\alpha/2}$, the upper $100(\alpha/2)\%$ point of W_m, that is,

$$P\left(W_m \leq b_{m,\alpha/2}\right) = 1 - \alpha/2, \quad 0 < \alpha < 1.$$

We take $\theta = 0$ and $g(\xi) = 1$.

Also, in order to test the simple null hypothesis H_0: $\theta = \theta_0$ against an alternative hypothesis H_1: $\theta \neq \theta_0$, suppose that

(c) the acceptance region $U_m \pm m^{-\beta} k_{\alpha/2} g(\xi)$ is uniformly most powerful unbiased (UMPU) for testing H_0 against H_1.

(d) $U_m \pm m^{-\beta} b_{m,\alpha/2} T_m$ is a uniformly most accurate unbiased (UMAU) confidence interval for θ with confidence coefficient $1 - \alpha$.

In addition to the assumptions (a) to (d) defined as in Mukhopadhyay (1982), it is not very restrictive to assume two further conditions:

(e) $E(T_m/g(\xi))^{1/\beta} = c_m$, where c_m is a linear function of negative powers of m, $c_m = \lambda + o(1)$ and λ is any integer.

(f) $E|(T_m/g(\xi))^{1/\beta} - \lambda|^2 = O(m^{-1})$.

Now, let us give a few examples about the constructions in (2.1).

Example 1. Suppose that $(X_1, ..., X_m)$ are i.i.d. random variables drawn from a normal population with common mean μ and variance σ^2. Here, $U_m = \bar{X}_m$ is an unbiased estimator of $\theta = \mu$. As an estimator of $\xi = \sigma^2$, we can take V_m as the sample variance or any unbiased estimator based on Gini's mean difference (GMD), the mean absolute deviation (MAD), the range, etc., with $T_m = g(V_m) = \sqrt{V_m}$. Here, U_m and V_m are independent.

Example 2. Suppose that $(X_1, ..., X_m)$ are i.i.d. random variables drawn from a negative exponential population with location parameter μ and scale σ. Here, $U_m = X_{m(1)}$, the smallest order statistic and $V_m = \sum_{i=1}^{m} |X_i - X_{m(1)}|$ are respectively the estimators of $\theta(= \mu)$ and $\xi(= \sigma)$ and are independent.

Now, W_m can be expressed as the ratio of two independently distributed statistics, Z/Y where (i) $Z \sim F(\cdot)$ is known, (ii) the distribution of $Y = T_m(\equiv g(V_m))$ is unknown. The distribution of W_m does not depend on m, θ and ξ. Suppose, $k_{\alpha/2}$ is the upper $100(\alpha/2)\%$ point of the distribution of $m^\beta \left(U_m - \theta\right)/g(\xi)$ and $b_{m,\alpha/2}$ is the upper $100(\alpha/2)\%$ point of the distribution of W_m. Then,

$$F(k_{\alpha/2}) = P\left(\frac{m^\beta \left(U_m - \theta\right)}{g(\xi)} \leq k_{\alpha/2}\right) = 1 - \alpha/2, \quad 0 < \alpha < 1.$$

Using Mukhopadhyay (1982), the proposed $100(1 - \alpha)\%$ fixed-width (2d, say) confidence interval for the parameter θ can be obtained by using an optimal sample size given by

$$C = \left(\frac{k_{\alpha/2}\xi_L}{d}\right)^{1/\beta},$$

where $\xi_L > 0$ is a lower bound for ξ.

Using the moments of T_m and generalising Mukhopadhyay and Chattopadhyay (2012), the percentile points of the distribution of W_m can be expressed as follows

$$b_{m,\alpha/2} = k_{\alpha/2} + \frac{b_1}{m} + \frac{b_2}{m^2} + \frac{b_3}{m^3} + \frac{b_4}{m^4} + \frac{b_5}{m^5} + O(m^{-6}). \tag{2.2}$$

We note that the conditions (a)-(d) are given by Mukhopadhyay (1982). Along the lines of Mukhopadhyay and Duggan (1997) and Chattopadhyay and Mukhopadhyay (2012), we define

$$m \equiv m(d) = \max\left\{m_0, \left\langle \left(k_{\alpha/2}\xi_L/d\right)^{1/\beta}\right\rangle + 1\right\}, \tag{2.3}$$

with $m_0 \geq 2$. Observe that the restriction on m_0 is such that we have a minimum number of observations to compute the required moments. We begin with pilot observations $X_1, ..., X_m$ and define the final sample size

$$Q \equiv Q(d) = \max\left\{m, \left\langle \left(b_{m,\alpha/2}T_m/d\right)^{1/\beta}\right\rangle + 1\right\}. \tag{2.4}$$

If $Q = m$, no further observations are observed beyond the pilot sample, but if $Q > m$, we collect $Q - m$ additional observations in the second stage. Finally, based on the combined data $X_1, ..., X_Q$ from both the stages, we construct a fixed-width confidence interval J_Q for θ for a given fixed-width $2d(>0)$ and confidence coefficient at least $1-\alpha$, with α pre-specified. The form of J_Q may be determined for specific distributions. For instance, in the case of Example 1, $J_Q = [\bar{X}_Q - d, \bar{X}_Q + d]$ and in case of Example 2, $J_Q = [X_{Q(1)} - 2d, X_{Q(1)}]$ as in Mukhopadhyay (1982).

Lemma 1. *For the modified two-stage procedure from* (2.3)–(2.4)*, assuming that conditions (a)–(f) are satisfied, for all* $(\theta,\xi) \in \Re \times \Re^+$, $0 < \xi_L < \xi$ *and* $0 < \alpha < 1$*, we have*

$$P_{\theta,\xi}\left(Q = m\right) = O(m^{-1}),$$

for any fixed $d(>0)$.

Proof. Please refer to the Appendix. ∎

Theorem 1. *For the modified two-stage procedure from* (2.3)–(2.4)*, assuming conditions (a)–(f) are satisfied, for all* $(\theta,\xi) \in \Re \times \Re^+$, $0 < \xi_L < \xi$ *and* $0 < \alpha < 1$*, we have*

(i) $P_{\theta,\xi}\left\{\theta \in J_Q\right\} \geq 1-\alpha$*, for any fixed d [exact consistency];*

(ii) $Q/C \xrightarrow{P} 1$ *as* $d \to 0$;

(iii) $P_{\theta,\xi}\left\{\theta \in J_Q\right\} \to 1-\alpha$ *as* $d \to 0$ *[asymptotic consistency];*

(iv) $E_{\theta,\xi}\left[Q/C\right] \to 1$ *as* $d \to 0$ *[first-order efficiency];*

(v) $E_{\theta,\xi}\left[Q - C\right]$ *is bounded as* $d \to 0$ *[second-order efficiency].*

Proof. Please refer to the Appendix. ∎

The reader may refer to the supplementary section for a discussion of the modified mean absolute deviation-based procedure. This is useful for finding a fixed-width confidence interval for the mean of a normal population and also satisfies the second-order asymptotic efficiency property.

3. An extension: inverse Gaussian location parameter

Suppose that $(X_1, ..., X_n)$ are i.i.d. random variables from an inverse Gaussian distribution with mean parameter $\theta(>0)$ and shape parameter $\xi(>0)$, i.e. $X_i \sim IG(\theta, \xi)$. Here, $\bar{X}_n$ and $V_n = (n-1)^{-1}\sum_{i=1}^{n}(X_i^{-1} - \bar{X}_n^{-1})$ are unbiased and consistent estimators of θ and ξ^{-1} respectively. For a known value of ξ, the $100(1-\alpha)\%$ confidence interval for θ is

$$\left(\frac{n\xi\bar{X}_n}{n\xi + z_{\alpha/2}\sqrt{n\xi\bar{X}_n}}, \frac{n\xi\bar{X}_n}{n\xi - z_{\alpha/2}\sqrt{n\xi\bar{X}_n}}\right),$$

where, $z_{\alpha/2}$ is the $100(1-\alpha/2)^{\text{th}}$ percentile of the standard normal distribution. Using Mukhopadhyay (1982) and Arefi et al. (2008), we write,

$$P\left(\frac{n\xi\bar{X}_n}{n\xi + z_{\alpha/2}\sqrt{n\xi\bar{X}_n}} < \theta < \frac{n\xi\bar{X}_n}{n\xi - z_{\alpha/2}\sqrt{n\xi\bar{X}_n}}\right) \geq 1-\alpha,$$
$$P\left(-z_{\alpha/2} < \sqrt{n\xi\bar{X}_n}\left(\frac{1}{\bar{X}_n} - \frac{1}{\theta}\right) < z_{\alpha/2}\right) \geq 1-\alpha.$$

It is important to note that the inverse Gaussian random variables are non-negative with probability one. If one considers a fixed-width confidence interval for its mean parameter, along the lines of Section 2 with a fixed $d(>0)$ apriori, the lower bound of such a confidence interval given by a two-stage or multi-stage procedure can be negative with positive probability, however small d may be (apriori). In such a scenario, the whole two-stage or multi-stage methodology will fall apart. Similar arguments will hold true for a fixed-width confidence interval for the reciprocal of the mean parameter. For such a case, we refer to the confidence interval in equation (3.2) of Mukhopadhyay (1982), which introduces the idea of "proportional closeness" in a loss function. Along similar lines, the optimal sample size required to obtain such a confidence interval is $C_1 = z_{\alpha/2}^2/(d^2\xi)$, for fixed 'precision' $d > 0$.

Provided we know the upper bound of the shape parameter, say ξ_U, the pilot sample size can be defined as in Section 2 by

$$m_1 = \max\left\{m_0, \left\langle\left(z_{\alpha/2}^2/(d^2\xi_U)\right)\right\rangle + 1\right\}, \tag{3.1}$$

with $m_0 \geq 2$. We begin with pilot observations $X_1, ..., X_{m_1}$ and then define the final sample size as

$$N \equiv N(d) = \max\left\{m_1, \left\langle\left(t_{m_1-1,\alpha/2}^2 V_{m_1}/d^2\right)\right\rangle + 1\right\}, \tag{3.2}$$

where $t_{m_1-1,\alpha/2}$ is the $100(1-\alpha/2)^{\text{th}}$ percentile of the student's t distribution. Based on the final sample size, N, the $100(1-\alpha)\%$ confidence interval for θ will be

$$J_N = \left(\frac{\bar{X}_N}{1 + d\sqrt{\bar{X}_N}}, \frac{\bar{X}_N}{1 - d\sqrt{\bar{X}_N}} \right).$$

Here, $d(>0)$ is not the margin of error for this confidence interval for the parameter θ. As $d \to 0$, the final sample size will increase, which decreases the width of the confidence interval, and thus the precision of the confidence interval increases, keeping everything else the same. Thus for fixed $d > 0$, this is a fixed precision confidence interval for θ. This is very similar to the fixed accuracy confidence interval. For more details, one may refer to Mukhopadhyay and Banerjee (2014) and Banerjee and Mukhopadhyay (2016). This type of interval would not be symmetric around $\bar{X}_N$ for the unknown parameter θ, but it would be so around $\bar{X}_N^{-1}$ for the unknown parameter θ^{-1}.

Lemma 2. *For the modified two-stage procedure from* (3.1)–(3.2)*, for all* $(\theta, \xi) \in \Re^+ \times \Re^+$, $0 < \xi < \xi_U$ *and* $0 < \alpha < 1$, *we have*

$$P_{\theta,\xi}\,(N = m_1) = O(m_1^{-1}).$$

Proof. Please refer to the Appendix. ■

Theorem 2. *For the modified two-stage methodology from* (3.1)–(3.2)*, for all* $(\theta, \xi) \in \Re^+ \times \Re^+$, $0 < \xi < \xi_U$ *and* $0 < \alpha < 1$, *we have*

(i) $P_{\theta,\xi}\,\{\theta \in J_N\} \geq 1 - \alpha$ *, for any fixed d [exact consistency];*

(ii) $P_{\theta,\xi}\,\{\theta \in J_N\} \to 1 - \alpha$ *as* $d \to 0$ *[asymptotic consistency];*

(iii) $N/C_1 \xrightarrow{P} 1$ *as* $d \to 0$;

(iv) $E_{\theta,\xi}\,[N/C_1] \to 1$ *as* $d \to 0$ *[first-order efficiency];*

(v) $E_{\theta,\xi}\,[N - C_1]$ *is bounded as* $d \to 0$ *[second-order efficiency].*

Proof. Please refer to the Appendix. ■

4. Concluding remarks

Here we propose a modified two-stage procedure for constructing a $100(1-\alpha)\%$ confidence interval for the location parameter under the mild assumptions of a known positive lower bound for the unknown nuisance parameter. We have shown that the difference of the average final sample size and the optimal sample size is bounded and also that the ratio of the average final sample size and the optimal sample size approaches 1 asymptotically. Additionally, we have shown that the modified two-stage procedure attains the required coverage probability. While deducing the above optimality results, we note that the distributional assumption of the estimator of the nuisance parameter(s) is (are) not required, only moment assumptions are necessary. Based on these results, we conclude that our proposed two-stage procedure can efficiently construct a $100(1-\alpha)\%$ fixed-width confidence interval for the location parameter. An illustration of our modified two-stage procedure to construct a fixed-width confidence interval for the normal mean under suspect outliers, using the mean absolute deviation as an estimator of the population standard deviation, can be found in the Supplementary Materials section.

5. Appendix

5.1 Proof of Lemma 1

Here we have,

$$P_{\theta,\xi}\left(Q=m\right)=P_{\theta,\xi}\left(T_m<m^{\beta}d/b_{m,\alpha/2}\right)=P_{\theta,\xi}\left(\left(\frac{T_m}{g(\xi)}\right)^{1/\beta}-\lambda<h_m\right), \tag{5.1}$$

where

$$h_m=\tfrac{c_m}{g^{1/\beta}(\xi)}\left(\tfrac{m^{\beta}d}{b_{m,\alpha/2}}\right)^{1/\beta}-\lambda.$$

However, we observe

$$c_m=\lambda+o(1),\quad m^{\beta}d=k_{\alpha/2}g(\xi_L)+o(1), \tag{5.2}$$

and using (2.2), $b_{m,\alpha/2}=k_{\alpha/2}+o(1)$.Thus we have

$$h_m=\lambda\left(\left(g(\xi_L)g^{-1}(\xi)\right)^{1/\beta}-1\right)+o(1). \tag{5.3}$$

For significantly large m, that is for significantly small $d(>0)$, we can claim

$$h_m<\lambda\left(\left(g(\xi_L)g^{-1}(\xi)\right)^{1/\beta}-1\right).$$

Observe that the upper bound for h_m is negative. Hence, from (5.1)–(5.3) we can conclude that for large m,

$$\begin{aligned}P_{\theta,\xi}\left(Q=m\right)&=P_{\theta,\xi}\left(\left(\frac{T_m}{g(\xi)}\right)^{1/\beta}-\lambda<h_m\right)\\&\le P_{\theta,\xi}\left(\left|\left(\frac{T_m}{g(\xi)}\right)^{1/\beta}-\lambda\right|>\frac{1}{2}\lambda\left(\left(\frac{g(\xi_L)}{g(\xi)}\right)^{1/\beta}-1\right)\right)\\&\le\left\{\frac{1}{2}\lambda\left(\left(\frac{g(\xi_L)}{g(\xi)}\right)^{1/\beta}-1\right)\right\}^{-2}E_{\theta,\xi}\left|\left(\frac{T_m}{g(\xi)}\right)^{1/\beta}-\lambda\right|^2.\end{aligned}$$

Finally, using condition (f), we have that

$$P_{\theta,\xi}\left(Q=m\right)=O(m^{-1}).$$

Our proof of this lemma is now complete. ■

5.2 Proof of Theorem 1

Part (i) has already been proved. See Mukhopadhyay (1982).

To verify part (ii), observe from Lemma 1 that $I(Q = m) \xrightarrow{P} 0$ as $d \to 0$ where the random variable $I(Q = m)$ takes the value 1 only if $Q = m$ and is 0 otherwise. Now, dividing both sides of (2.4) by C and taking limits as $d \to 0$, part (ii) would follow. Next, parts (iii)-(iv) are routine.

For part (v), to show the second-order efficiency, it is enough to show that the difference $E_{\theta,\xi}\,[Q - C]$ is bounded. Using (2.4), we can write

$$\left(b_{m,\alpha/2}^{1/\beta} - k_{\alpha/2}^{1/\beta}\right) g^{1/\beta}(\xi) d^{-1/\beta} \le E_{\theta,\xi}\,[Q - C] \le m P_{\theta,\xi}\,(Q = m) + \left(b_{m,\alpha/2}^{1/\beta} - k_{\alpha/2}^{1/\beta}\right) g^{1/\beta}(\xi) d^{-1/\beta} + 1.$$

Now,

$$\left(b_{m,\alpha/2}^{1/\beta} - k_{\alpha/2}^{1/\beta}\right) g^{1/\beta}(\xi) d^{-1/\beta} = \frac{b_1}{\beta} k_{\alpha/2}^{-1} \left(g(\xi)/g(\xi_L)\right)^{1/\beta} d^{-1/\beta} + o\,(d)\,. \tag{5.4}$$

Part (v) obviously follows from Lemma 1 and equation (5.4). ■

5.3 Proof of Lemma 2

Using Mukhopadhyay (1982) and Arefi et al. (2008),

$$\bar{X}_{m_1} \sim IG(\theta, n\xi) \text{ and } (m_1 - 1)\xi V_{m_1} \sim \chi^2_{m_1-1}. \tag{5.5}$$

Here we have

$$P_{\theta,\xi}\,(N = m_1) = P_{\theta,\xi}\left(V_{m_1} < m_1 d^2 / t^2_{m_1-1,\alpha/2}\right) = P_{\theta,\sigma}\left(V_{m_1}\xi - 1 < h_{m_1}\right),$$

where $h_{m_1} = m_1 d^2 \xi / t^2_{m_1-1,\alpha/2} - 1$. However, we observe

$$m_1 d^2 = z^2_{\alpha/2}/\xi_U + o(1),$$

and using (2.2),

$$t^2_{m_1-1,\alpha/2} = z^2_{\alpha/2} + o(1).$$

Thus we have,

$$h_{m_1} = (\xi/\xi_U - 1) + o(1). \tag{5.6}$$

From (5.5)–(5.6) we thus conclude that for large m_1 and for any arbitrary but fixed $d(>0)$:

$$
\begin{aligned}
P_{\theta,\xi}\left(N = m_1\right) &= P_{\theta,\xi}\left(V_{m_1}\xi - 1 < h_{m_1}\right) \\
&\leq P_{\theta,\xi}\left(\left|V_{m_1}\xi - 1\right| < \frac{1}{2}\left(1 - \frac{\xi}{\xi_U}\right)\right) \\
&\leq E\left[V_{m_1}\xi - 1\right]^2\left(1 - \frac{\xi}{\xi_U}\right)^{-2} = O(m_1^{-1}).
\end{aligned}
$$

Our proof of this lemma is now complete. ■

5.4 Proof of Theorem 2

Part (i) has already been proved. See Mukhopadhyay (1982). Part (ii) is routine.

To verify part (iii), observe from Lemma 2 that $I(N = m_1) \xrightarrow{P} 0$ as $d \to 0$ where the random variable $I(N = m_1)$ takes the value 1 only if $N = m_1$ and is 0 otherwise. Also, V_{m_1} is a consistent estimator of ξ^{-1}. Now, if we divide both sides of (3.2) by C_1, we observe that both sides approach 1 and thus part (iii) follows. Next, part (iv) is routine.

For part (v), in order to show second-order efficiency, it is enough to show that the difference $E_{\theta,\xi}\left[N - C_1\right]$ is bounded. Using equation (3.2), we can write

$$
\left(t^2_{m_1-1,\alpha/2} - z^2_{\alpha/2}\right)\frac{1}{d^2\xi} \leq E_{\theta,\xi}\left[N - C_1\right] \leq m_1 P_{\theta,\xi}\left(N = m_1\right) + \left(t^2_{m_1-1,\alpha/2} - z^2_{\alpha/2}\right)\frac{1}{d^2\xi} + 1.
$$

Now,

$$
\left(t^2_{m_1-1,\alpha/2} - z^2_{\alpha/2}\right)\frac{1}{d^2\xi} = \frac{(z^2_{\alpha/2}+1)\xi_U}{2\xi} + o(d). \tag{5.7}
$$

Part (v) obviously follows from Lemma 2 and equation (5.7). ■

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