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Abstract

We describe three different methods for generating quasi-exactly solvable potentials, for which a finite number of eigenstates are analytically known. The three methods are respectively based on (i) a polynomial ansatz for wave functions; (ii) point canonical transformations; (iii) supersymmetric quantum mechanics. The methods are rather general and give considerably richer results than those available in the current literature.

In recent years, many authors have discussed examples of potentials for which a finite number of eigenstates are explicitly known, but not the whole spectrum. The first examples of such quasi-exactly solvable (QES) potentials were given by Razavy[1] and others[2, 3]. Subsequently, using a more general approach, many QES potentials were obtained by Turbiner, Ushveridze and Shifman[4, 5, 6] and their connection with finite dimensional group representations was established [7, 8]. Also, Roy and Varshni[9] gave an example of the generation of a new QES potential using supersymmetric quantum mechanics.

In this paper, we develop and discuss three general methods for generating QES potentials. The first method is a systematic extension of the approach of Turbiner[5], which has the distinctive feature of a polynomial ansatz of the same degree for all analytically known eigenstates, in addition to a conventional exponential factor for satisfying boundary conditions. Our extension provides a clearer motivation and consequently yields many three parameter QES potentials which are more general than those available in the literature. The method hinges on the choice of an initial variable $z(x)$ and its use in making a specific ansatz for wave functions.

The second method consists of making point canonical transformations (changes of dependent and independent variables) [10] on known QES potentials in order to obtain new ones [11]. This approach is motivated from the observation that all known exactly solvable problems are inter-related via point canonical transformations.

The third method discussed in this letter applies the techniques of supersymmetric quantum mechanics[12] to calculate the partner potentials of QES potentials with $(n+1)$ known eigenstates. As expected from the level degeneracy theorem for unbroken supersymmetry, the supersymmetric partners are necessarily new quasi-exactly solvable potentials with n known eigenstates.

Method 1: Polynomial Ansatz for Wave Functions:

This method is a generalization of Turbiner's approach [5, 7] for generating QES potentials. We want to solve the Schrödinger equation ($\hbar = 2m = 1$)

$$-\psi''(x) + [V(x) - E]\,\psi(x) = 0 \quad , \tag{1}$$

along with the boundary condition that wave functions for bound states vanish as x approaches the end points of the domain. The initial step in Method 1 for generating quasi-exactly solvable potentials with $(n+1)$ analytically known eigenstates is to choose a non-singular monotonic variable $z(x)$ and write the wave function ψ in the form

$$\psi(x) = p_n(z, a_0, a_1, \ldots, a_{n-1})\ e^{-\int y(x)\,dx}, \tag{2}$$

where

$$p_n(z) = \sum_{j=0}^{n} a_j z^j \quad (a_n = 1) \quad . \tag{3}$$

is a polynomial of degree n in the variable z. [Note the special feature that a polynomial of the same degree n is present in all analytically known wave functions - however, different wave functions will correspond to different values of a_j and consequently will have differing numbers of nodes]. The functional form of $y(x)$ remains to be determined. Substituting for $\psi(x)$ into Eq. (1) yields

$$V(x) - E = \ y^2 - y' + \frac{p_n'' - 2yp_n'}{p_n} \quad . \tag{4}$$

In order that the potential have no poles in the physical domain, we require $p_n'' - 2yp_n'$ to have p_n as a factor. Also, in general, one expects that the energy will be a function of parameters in $y(x)$, as well as the coefficients a_j in p_n. However, we do not want these parameters to show up in the potential, as otherwise a dependence of the potential on energy would creep in. One way to prevent this from happening is to require that $\frac{p_n''-2yp_n'}{p_n}$ be a polynomial of degree one. This condition restricts what the needed functional form for $y(x)$ ought to be. More specifically, we choose $y(x)$ by requiring that the quantity $p_n'' - 2yp_n'$ be a polynomial of degree $(n+1)$ in z.

$$p_n'' - 2yp_n' = \sum_{j=0}^{n+1} b_j z^j \quad . \tag{5}$$

This is equivalent to requiring

$$(z'' - 2yz') \sum_{j=1}^{n} a_j j z^{j-1} + z'^2 \sum_{j=2}^{n} a_j j(j-1) z^{j-2} = \sum_{j=0}^{n+1} b_j z^j \quad . \tag{6}$$

For $n = 1$, the z'^2 term is absent, and it is sufficient to choose $y(x)$ such that

$$z'' - 2yz' = q_0 + q_1 z + q_2 z^2 \quad . \tag{7}$$

Although the three constants q_0, q_1, q_2 in $y(x)$ are arbitrary, their range of values is usually restricted by the requirement that the bound state wave function $\psi(x)$ vanish at the end points. For $n > 1$, the z'^2 term in Eq. (6) does contribute. Therefore, one needs the additional necessary constraint that the choice of initial variable $z(x)$ be such that

$$z'^2 = B_0 + B_1 z + B_2 z^2 + B_3 z^3 \quad . \tag{8}$$

This above constraint ensures that $p_n'' - 2yp_n'$ is a polynomial of order of $(n+1)$. It also constrains the choices allowed for $z(x)$, which now must have the form

$$\int \frac{dz}{\sqrt{B_0 + B_1 z + B_2 z^2 + B_3 z^3}} = x. \tag{9}$$

The variables $z = x^{\pm 2}, \cos\, x, \cosh\, x, e^{\pm x}, \mathrm{sech}^2\, x, \cdots$, etc. used in references [1, 4, 5, 7], are all special cases of the above Eq. (9). Dividing Eq. (5) by Eq. (3) gives

$$\frac{p_n'' - 2yp_n'}{p_n} = b_{n+1} z + (b_n - a_{n-1} b_{n+1}) + R_n(z) \quad , \tag{10}$$

where the remainder polynomial is

$$R_n(z) = \sum_{j=0}^{n-1} z^j f_n(j) \quad ; \quad f_n(j) \equiv b_j - a_{j-1} b_{n+1} - a_j b_n + a_j a_{n-1} b_{n+1} \quad . \tag{11}$$

Clearly, the requirement of no poles in the potential $V(x)$ implies

$$f_n(j) = 0 \quad , \quad j = 0, 1, \ldots, n-1 \quad . \tag{12}$$

These n constraint equations can be solved to determine the constants $a_0, a_1, \ldots, a_{n-1}$. There are n solutions for any constant a_j, since it satisfies an equation of degree n. The potential $V(x)$ and energies E are given by

$$V(x) = y^2 - y' + b_{n+1} z(x) \quad ; \quad E = -b_n + a_{n-1} b_{n+1} \quad . \tag{13}$$

The potential $V(x)$ depends only on the parameters q_0, q_1, q_2 appearing in y and $b_{n+1} = nq_2 + n(n-1)B_3$. Also, since $b_n = nq_1 + (n-1)q_2 a_{n-1} + n(n-1)B_2 + (n-1)(n-2)B_3 a_{n-1}$, a useful, alternative form for the energies is $E = -nq_1 - n(n-1)B_2 + a_{n-1}[q_2 + 2(n-1)B_3]$. Clearly, there are n energy eigenvalues corresponding to the n values of the constant a_{n-1}.

For concreteness, let us fully describe the important special case corresponding to the value $n = 1$. Here, we are constructing QES potentials with two analytically known eigenstates. The wave functions for these two states are both of the form given in Eqs. (2) and (3).

$$\psi(x) = p_1(z, a_0)\, e^{-\int y\, dx}, \quad p_1(z) = a_0 + z(x) \quad . \tag{14}$$

The quantity $y(x)$ is given by Eq. (7). For this case, the constraints given by Eqs. (12) read

$$f_1(0) \equiv q_0 - a_0 q_1 + a_0^2 q_2 = 0 \quad . \tag{15}$$

This quadratic equation for a_0 has two solutions. The QES potential is $V(x) = y^2 - y' + q_2 z(x)$, and it has two analytically known eigenvalues

$$E = a_0 q_2 - q_1 = -\frac{q_1}{2} \pm \frac{1}{2}\sqrt{q_1^2 - 4q_0 q_2} \quad , \tag{16}$$

with corresponding eigenfunctions given by Eq. (14).

To illustrate the above-described polynomial ansatz method for generating QES potentials with $(n+1)$ known eigenstates, we give three explicit examples with different choices of the initial variable $z(x)$. These examples have been specifically chosen, since the potentials they generate have not been previously discussed in the literature.

Example 1: $z = x^\epsilon, n = 1$ [2 known eigenstates].

We consider the polynomial $p_1 = z + a_0$. Eq. (7) then gives

$$y = \frac{\epsilon - 1}{2x} - \frac{1}{2\epsilon}\left(q_2 x^{\epsilon+1} + q_1 x + q_0 x^{-\epsilon+1}\right) \quad .$$

The QES potential with two known eigenstates is

$$\begin{aligned} V_2(x) \;&=\; \frac{(\epsilon+1)(\epsilon-1)}{4x^2} - \frac{1}{2\epsilon}\left[q_1(\epsilon-2) - 2q_2(\epsilon+1)x^\epsilon + 2q_0(\epsilon-1)x^{-\epsilon}\right] \\ &+\frac{1}{4\epsilon^2}\left[2q_0q_1x^{2-\epsilon} + 2q_1q_2x^{2+\epsilon} + (q_1^2 + 2q_0q_2)x^2 + q_0^2x^{2-2\epsilon} + q_2^2x^{2+2\epsilon}\right] \quad . \end{aligned} \tag{17}$$

The range of the potential is the half-line $0 < x < \infty$ [unless $\epsilon = 1$, in which case the range is the full line]. The known energy eigenvalues are given by Eq. (16), and the eigenfunctions obtained using Eq. (14) are:

$$\psi = (x^\epsilon + a_0)\exp\left[\frac{1-\epsilon}{2}\log x + \frac{1}{2\epsilon}\left(\frac{q_2}{2+\epsilon}x^{2+\epsilon} + \frac{q_0}{2-\epsilon}x^{2-\epsilon} + \frac{1}{2}q_1x^2\right)\right] \quad \text{for } \epsilon \neq \pm 2. \tag{18}$$

The arbitrary constants are constrained by requiring that ψ satisfies boundary conditions. For instance, for any choice $\epsilon > 2$, one needs $q_0 > 0$, $q_2 < 0$. As an explicit example, consider $\epsilon = 3, q_0 = 1, q_2 = -1, q_1 = 0$. This gives $a_0 = \mp 1$, eigenenergies $E = \pm 1$ and eigenfunctions

$$\frac{(x^3 \mp 1)}{x}\exp[-\frac{1}{6}(\frac{x^5}{5} + x^{-1})].$$

Note that the polynomial $x^3 + a_0$ has no zero for the ground state, but one for the first excited state.

Example 2: $z = \cosh x$, [$(n+1)$ known eigenstates].

Since this choice of variable satisfies Eq. (8), one can use it to generate QES potentials with an arbitrary number of known eigenstates. [The variable $\cosh x$ also appears in Table 4.1 of Ref. [12], and corresponds to the exactly solvable generalized Pöschl-Teller potential $V(x) = A^2 + (B^2 + A^2 + A)\text{cosech}^2 r - B(2A+1)\coth r\, \text{cosech}\ r$ with eigenenergies $E_n = A^2 - (A-n)^2$]. First we generate a quasi-exactly solvable potential with two eigenstates by taking the polynomial $p_1(x) = \cosh x + a_0$. This gives the following form for the function $y(x)$:

$$y(x) = \alpha \sinh x - \beta \text{cosech} x - \gamma \coth x, \tag{19}$$

where α, β, γ are arbitrary constants. As before, a_0 obeys a quadratic equation

$$2\alpha a_0^2 + (1+2\gamma)\, a_0 - 2(\alpha+\beta) = 0. \tag{20}$$

Its two roots are given by

$$a_0 = \frac{-(1+2\gamma) \pm \sqrt{(1+2\gamma)^2 + 16\alpha(\alpha+\beta)}}{4\alpha}. \tag{21}$$

These roots give rise to different number of nodes in the polynomial p_1 and hence in the wave functions ψ:

$$\psi \propto (\cosh x + a_0) e^{-\alpha \cosh x} \left(\cosh \frac{x}{2}\right)^{\gamma-\beta} \left(\sinh \frac{x}{2}\right)^{\gamma+\beta}.$$

The requirement that ψ vanishes at $x = \infty$ and $x = 0$ implies that $\alpha > 0$ and $\gamma + \beta > 0$. The potential is

$$V_2(x) = \left(\gamma^2 - \gamma + \beta^2\right) \coth^2 x - \alpha\,(2\gamma+3) \cosh x + \alpha^2 \sinh^2 x + \beta(2\gamma-1) \coth x\ \text{cosech}\ x. \tag{22}$$

It is independent of a_0 and singular at the origin, where it diverges as $\frac{(\gamma+\beta)(\gamma+\beta-1)}{x^2}$. The two lowest energy eigenvalues are given by

$$E = \left(2\alpha\beta + \beta^2 - 3\gamma - 2\alpha a_0 - 1\right), \tag{23}$$

where, the solutions a_0 are given in Eq. (21).

From Eq. (13), and using $b_{n+1} = nq_2$, we can get a potential with $n+1$ states

$$V_{n+1} = \left(\gamma^2 - \gamma + \beta^2\right) \coth^2 x - \alpha\,(2\gamma + 2n + 1) \cosh x + \alpha^2 \sinh^2 x + \beta(2\gamma-1) \coth x\ \text{cosech}\ x. \tag{24}$$

The energy eigenvalues are given by Eq. (13), where a_{n-1} satisfies an algebraic equation of degree $n+1$. In particular, for $n=2$, the equation for a_1 ia a cubic:

$$-2\alpha^2 a_1^3 - \alpha(7+6\gamma)a_1^2 + 2(4\alpha^2 + 4\alpha\beta - 5\gamma - 2\gamma^2 - 3)a_1 + 4(\alpha + 2\beta + 2\alpha\gamma + 2\beta\gamma) = 0.$$

Note that a special case of the QES potential of Eq. (24) corresponding to $\beta = 0, \gamma = 0, 1$ was considered by Razavy[1]. The two choices of γ give the even and odd eigenstates respectively. For these situations, the range of the potential is $-\infty < x < \infty$, since there is no singularity at $x = 0$. Also, in the limit $\alpha \to 0$, this model goes to the generalized Pöschl-Teller potential.

Example 3: $z = \mathrm{sech}^2 x$.

This is another example where we can get a quasi-exactly solvable potential with more than two known states. The constants B_j in Eq. (8) are $B_0 = B_1 = 0, B_2 = 4, B_3 = -4$. The function $y(x)$ is

$$y = \left(3 + \frac{q_2}{2}\right) \mathrm{cosech}\, 2x + \left(\frac{q_1}{4} - 1\right) \coth x + \frac{q_0}{4} \cosh^2 x \coth x,$$

and the potential is given by Eq. (13) with $b_{n+1} = nq_2 - 4n(n-1)$. A special case of this potential was discussed by Turbiner in Ref. [5]. For $n = 1$, the two lowest energy eigenvalues are given by $E = -q_1 + q_2\, a_0$, and corresponding eigenfunctions are

$$\psi = \left(\mathrm{sech}^2 x + a_0\right) (\cosh\, x)^{\frac{(q_2+6)}{4}} (\sinh\, x)^{\frac{-(q_2+q_1+q_0+2)}{4}} \exp\left[-\frac{q_0 \cosh\, 2x}{16}\right].$$

Method 2: Point Canonical Transformations:

Point canonical transformations (PCT) are known [10] to transform the Schrödinger equation for a potential $V(x)$ into the Schrödinger equation for a new potential $\tilde{V}(\xi)$. In this section, we employ PCT to generate new quasi-exactly solvable models starting from a known one. In a point canonical transformation, one changes from the independent variable x to the variable ξ, where $x = f(\xi)$ and the wave function is transformed by: $\psi(\alpha_i; x) \longrightarrow \left(\frac{df}{d\xi}\right)^{1/2} \tilde{\psi}\,(\alpha_i; \xi)$. The new potential and energy (denoted with tilde on the top) are then given by:

$$\tilde{V} - \tilde{E} = f'^2 \left[V(\alpha_i; f(\xi)) - E(\alpha_i)\right] + \frac{1}{2}\left(\frac{3f''^2}{2f'^2} - \frac{f'''}{f'}\right) \tag{25}$$

The constant term on the right hand side represents the energy of the new potential. It should be emphasized that the above-described technique is quite general. We will illustrate it with a specific example.

Example 4:

As a starting point, we choose the new QES potential $V_2(x)$ obtained in Eq. (22) of Example 2. The change of variables $x = i\xi$ converts hypergeometric functions into trigonometric functions. The new potential for $0 < \xi < \pi$ is:

$$\tilde{V}(\xi) = \left(\gamma^2 - \gamma + \beta^2\right) \cot^2 \xi + \alpha\,(2\gamma + 3) \cos \xi + \alpha^2 \sin^2 \xi + \beta(2\gamma - 1) \cot \xi \text{cosec}\xi. \quad (26)$$

The two known eigenvalues of this potential $\tilde{E}$ are given by $-E$, with E given by Eq. (23). The corresponding wave functions are given by

$$\psi \propto (\cos \xi + a_0) e^{-\alpha \cos \xi} \left(\cos \frac{\xi}{2}\right)^{\gamma-\beta} \left(\sin \frac{\xi}{2}\right)^{\gamma+\beta} .$$

The constraints $\gamma - \beta > 0, \gamma + \beta > 0$ ensure that ψ vanishes at the end points.

Alternatively, if we make the change of variables $x = \coth^{-1}[i \sinh \xi]$, then the new potential is

$$\begin{aligned} \tilde{V} \quad &= \quad (3\alpha - \beta + 2\alpha\gamma + 2\beta\gamma) \tanh \xi - \left(2\alpha^2 - \beta^2 + E + \gamma - \gamma^2\right) \tanh^2 \xi \\ &\quad - (3\alpha + 2\alpha\gamma) \tanh^3 \xi + \alpha^2 \tanh^4 \xi. \end{aligned} \quad (27)$$

This is a well defined QES potential over the range $-\infty < \xi < \infty$. The two lowest eigenvalues are $\tilde{E} = -\alpha^2 - E$, where E is given by Eq. (23) and the eigenfunctions are

$$\psi \propto (\tanh z + a_0) e^{-\alpha \tanh z} (\tanh z)^{\gamma+\beta} (\text{sech} z)^{\gamma-\beta-\frac{1}{2}} . \quad (28)$$

Method 3: Supersymmetric Quantum Mechanics:

If one has a potential $V_-(x)$ whose lowest eigenstate has energy $E_0^{(-)} = 0$ and wave function $\psi_0(x)$, then its supersymmetric partner potential $V_+(x)$ is given by [12]

$$V_+(x) = V_-(x) - 2\frac{d}{dx}\left(\frac{\psi_0'}{\psi_0}\right) . \quad (29)$$

The degeneracy theorem for unbroken supersymmetry states that V_+ and V_- have the same energy levels (except for the zero energy ground state) and simply related eigenfunctions:

$$E_{k-1}^{(+)} = E_k^{(-)} \; ; \; \psi_{k-1}^{(+)} = E_k^{(-)^{-\frac{1}{2}}} \left(\frac{d}{dx} - \frac{\psi_0'}{\psi_0} \right) \psi_k^{(-)} \; ; \; k = 1, 2, \ldots \tag{30}$$

Clearly, if one applies the above results to a QES potential with $(n+1)$ known eigenstates, one gets a new QES potential with n known eigenstates. For the simplest case of $n = 1$ and $V_-(x) = x^6 - 7x^2 + 2\sqrt{2}$, this procedure was discussed in Ref. [9].

As a general illustration, we want to apply the supersymmetry approach to a QES potential obtained by Method 1 using a polynomial ansatz for the known wave functions. Since Eq. (2) gives

$$\psi_0 \propto p_0 e^{-\int y(x) \, dx},$$

the superpotential $-\psi_0'/\psi_0$ appearing in Eqs. (29) and (30) is just $y(x) - p_0'/p_0$, where the parameters a_j appearing in the polynomial p_0 correspond to the ground state. The supersymmetric partner potential is

$$V_+(x) = V_-(x) + 2y'(x) - \frac{2(p_0 p_0'' - p_0'^2)}{p_0^2} \tag{31}$$

and the lowest n eigenfunctions are

$$\psi_{k-1}^{(+)} \propto \left(\frac{p_k'}{p_k} - \frac{p_0'}{p_0} \right) p_k \; e^{-\int y(x) dx} \; , \; k = 1, 2, \ldots, n \; . \tag{32}$$

where p_k is the polynomial $p(z)$ with parameters a_j corresponding to the k^{th} state of $V_-(x)$. Note that the same procedure of generating a supersymmetric partner potential can again be repeated by starting from the potential $V_+(x)$. Also, the standard methods of supersymmetric quantum mechanics [12] can be used to generate multi-parameter isospectral potential families of $V_-(x)$, which are of course new QES potentials.

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