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Can Newton's Third Law Be "Derived" from the Second?

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Newton's laws[1] have engendered much discussion over several centuries.[2,3] Today, the internet is awash with a plethora of information on this topic. We find many references to Newton's laws, often discussions of various types of misunderstandings and ways to explain them. Here we present an intriguing example that shows an assumption hidden in Newton's third law that is often overlooked. As is well known, the first law defines an inertial frame of reference and the second law determines the acceleration of a particle in such a frame due to an external force. The third law describes forces exerted on each other in a two-particle system, and allows us to extend the second law to a system of particles. Students are often taught that the three laws are independent. Here we present an example that challenges this assumption. At first glance, it seems to show that, at least for a special case, the third law follows from the second law. However, a careful examination of the assumptions demonstrates that is not quite the case. Ultimately, the example does illustrate the significance of the concept of mass in linking Newton's dynamical principles.

Motion of an inhomogeneous block under a constant force

Consider a block of mass $m_1 + m_2$ that is made by welding together a block of gold (mass m_1) and a block of copper (mass m_2). This block, as shown in Fig 1, is pulled with a constant force $\boldsymbol{F}$.

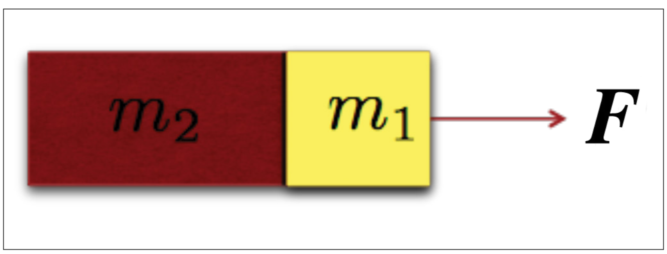

Fig. 1. A block of mass $m_1 + m_2$ made up of two metals.

By the second law of motion, the acceleration $\boldsymbol{a}$ of this block is given by

$$\boldsymbol{F} = (m_1 + m_2)\,\boldsymbol{a}. \tag{1}$$

We will now analyze each of these two conjoined parts as independent subsystems with their own force diagrams given in Figs. 2(a) and (b). **Here we use the notation that $\boldsymbol{F}_{A/B}$ represents a force exerted on A by B.** To keep our force diagrams simple, we assume that this experiment is being carried out in space, in a region with negligible gravity.[4]

We want to emphasize that at this point we do not assume

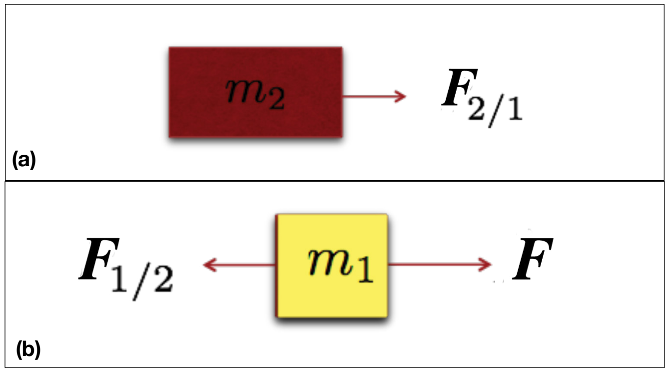

Fig. 2. Free-body diagrams for (a) copper (m_2) and (b) gold (m_1) blocks.

any relationship between forces $\boldsymbol{F}_{1/2}$ and $\boldsymbol{F}_{2/1}$. Applying Newton's second law on these subsystems, we get

$$\boldsymbol{F} + \boldsymbol{F}_{1/2} = m_1\,\boldsymbol{a}, \tag{2}$$

and

$$\boldsymbol{F}_{2/1} = m_2\,\boldsymbol{a}. \tag{3}$$

Adding Eqs. (2) and (3), we get

$$\boldsymbol{F} + \boldsymbol{F}_{1/2} + \boldsymbol{F}_{2/1} = (m_1 + m_2)\,\boldsymbol{a}\,. \tag{4}$$

However, from Eq. (1), we have $\boldsymbol{F} = (m_1 + m_2)\,\boldsymbol{a}$. Thus, combining Eqs. (1) and (4), we reach the following surprising result:

$$\boldsymbol{F}_{1/2} + \boldsymbol{F}_{2/1} = 0. \tag{5}$$

But that is the statement of the third law! We seem to have derived the third law by making multiple uses of the second law. So, where did we go wrong? Actually, we are not wrong. Newton's second law alone did not bring us to Eq. (5). Along the way, in Eq. (1), we assumed that the mass of the composite object was equal to the sum of its components. In other words, we assumed that mass has an additive property. That was the extra input required for this derivation.[5] However, during Newton's time the additivity of mass was well accepted, and hence this "derivation" of the equality and oppositeness of the action-reaction forces for this particular setting was indeed possible. Since conservation of mass is well accepted by our students in introductory physics, this example could be used to motivate the introduction of Newton's third law.

It is important to add that while the third law has a much larger range of validity,[6] this derivation is only valid when two interacting objects move as one, i.e., they have a common ve-

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locity and a common acceleration. The motion of two masses moving with a common velocity described above could be viewed as a limiting case of a perfectly inelastic collision between two objects[7] where Newton's third law leads to the conservation of momentum:

$$m_1 \boldsymbol{v}_1 + m_2 \boldsymbol{v}_2 = m_{12} \boldsymbol{v}_{12}\,, \tag{6}$$

where subscripts "1" and "2" refer to masses and velocities of two objects before collision and the subscript "12" refers to the composite object after a perfect inelastic collision. For the case considered above, both masses had the same velocities at all times. Hence, substituting $\boldsymbol{v}_1 = \boldsymbol{v}_2 = \boldsymbol{v}_{12}$, we see that the momentum conservation reduces to the additivity condition for the masses[8]:

$$m_1 + m_2 = m_{12}\,. \tag{7}$$

Thus, by assuming the additivity of masses, we had actually assured that momentum was conserved for the motion described above. And, it is well known that law of conservation of momentum is equivalent to the third law of motion.

Mach has written extensively on the role mass plays in Newton's laws. In particular, he argues that the second law does not completely describe all properties of mass.[3,5] Additivity of masses in the context of Newton's laws has been discussed in literature before.[9] However, we think that this is a rather simple example that clearly shows that additive property of mass does not follow from the second law, and has to be considered an additional assumption.

Conclusion

We show that Newton's second law of motion and additivity of masses lead to Newton's third law of motion, albeit for a very restricted class of motion. Since additivity of masses is a very well accepted notion, this discussion could be used as a prelude to the introduction of the third law in its general form.

References

1. Interestingly, even though we call them Newton's laws, Newton himself generously gave credit for the first and the second laws to Galileo. Please see p. 113 in Isaac Newton, *The Principia: Mathematical Principles of Natural Philosophy*, a new translation by I. Bernard Cohen and Anne Whitman, assisted by Julia Budenz (University of California Press, 1999), ISBN: 9780520088160.
2. "Newton's Laws of Motion," in W. Thomson (Lord Kelvin) and P. G. Tait, *Treatise on Natural Philosophy*, Vol. I (1867), Sect. 242; and Benjamin Crowell, *Newtonian Physics* (2000).
3. Ernst Mach, *The Science Of Mechanics,* 4th ed. (The Open Court Publishing Company, Chicago, 1919).
4. This removes any need of introducing a normal force or weight in the force diagrams.
5. Newton wrote, "The quantity of matter is the measure of the same, arising from its density and bulk conjointly." Mach criticized this definition as being circular. M. Strauss, the author of *Modern Physics and its Philosophy* (Springer-Verlag, 1972), states on p. 125 that Newton's above definition of mass could be dropped only if the additivity is introduced as an axiom. At least for homogeneous materials, the above definition involving densities implies the additivity of mass (to be able to compare densities).
6. It works well for most forces encountered in an introductory physics course. It however fails for forces between two charged particles in relative motion unless electromagnetic fields are brought into the analysis. See David J. Griffiths, *Introduction to Electrodynamics*, 4th ed. (Pearson 2012), Sect. 8.2.1.
7. If $\boldsymbol{v}_1$ were strictly equal to $\boldsymbol{v}_2$, a "collision" would not happen. This collision is to be seen more as the limiting case of $\boldsymbol{v}_1 - \boldsymbol{v}_2 = \varepsilon$, with $|\varepsilon| \to 0$.
8. A very similar argument can be found in the following reference, where the author uses accelerations instead of velocities: K. J. McQueen, "Mass additivity and a priori entailment," *Synthese* **192**, 1373–1392 (2015).
9. N. Feather, "Additivity of mass in Newtonian mechanics," *Am. J. Phys.* **34**, 511–516 (June 1966).

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