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Nonlinear Bivariate Comovements of Asset Prices: Theory and Tests

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Abstract. Comovements among asset prices have received a lot of attention for several reasons. For example, comovements are important in cross-hedging and cross-speculation; they determine capital allocation both domestically and in international mean-variance portfolios and also, they are useful in investigating the extent of integration among financial markets. In this paper we propose a new methodology for the non-linear modelling of bivariate comovements. Our approach extends the ones presented in the recent literature. In fact, our methodology outlined in three steps, allows the evaluation and the statistical testing of non-linearly driven comovements between two given random variables. Moreover, when such a bivariate dependence relationship is detected, our approach solves for a polynomial approximation. We illustrate our three-steps methodology to the time series of energy related asset prices. Finally, we exploit this dependence relationship and its polynomial approximation to obtain analytical approximations of the Greeks for the European call and put options in terms of an asset whose price comoves with the price of the underlying asset.

Keywords. Comovement, asset prices, bivariate dependence, non-linearity, t -test, polynomial approximation, energy asset, (vanilla) European call and put options, cross-Greeks.

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Nonlinear Bivariate Comovements of Asset Prices: Theory and Tests

1. Introduction

Comovements among asset prices as a topic of research have received a lot of attention for several reasons:

- First, the knowledge of a dependence relationship between the prices of two assets allows one to use publically available information for one asset to deduce forthcoming information for the codependent asset. Moreover, comovement is useful in cross–hedging and cross–speculation.
- Second, the presence of dependence in the form of correlation among the prices of certain assets traded domestically or across different countries is of interest to investors who wish to allocate their capital in mean–variance portfolios since comovements diminish the effectiveness of diversification strategies.
- Third, dependence among globally traded assets may influence the coordination of economic policies;
- Fourth, scholars and policy makers are interested in comovements among asset prices as an indication of the degree of financial integration.
- Finally, comovements of economic variables are the focus of economic analysis in business cycles, global trade, labor economics, regional economics and several other areas.

The increasing interest in the topic of comovements in asset prices has resulted in a large volume of scientific contributions. In the next section we offer a short survey of the more recent contributions. In particular, in most of the papers to be reviewed, the authors follow the well-accepted methodologies based on autoregressive heteroskedastic (ARCH) models, error correction models (ECMs), generalized ARCH (GARCH) models, Granger causality based tests, multivariate cointegrations, structural vector autoregressive (VAR) systems, lag–augmented VAR (LA–VAR) systems, forecast error variance decomposition (VDC) approaches, and vector error–correction models (VECMs).

In this paper, we first, investigate the phenomenon of comovements among asset prices by proposing a new methodology that goes beyond the ones just listed above. In fact, our approach

allows for both the evaluation and statistical testing of non-linearly driven comovements between two given random variables. Moreover, when such a nonlinear bivariate dependence relationship is detected, our approach also gives a polynomial approximation.

In addition, in this paper we also apply our three-step new methodology to the time series of three energy related assets (crude oil, gasoline, and heating oil prices) and use the bivariate dependence relationship and its polynomial approximation in order to obtain analytical approximations of the Greeks for the (vanilla) European call and put options in terms of an asset whose price comoves with the price of the underlying of the investigated option. By so doing, we attain what we call cross-Greeks.

The remainder of this paper is organized as follows. In the next section we present a short review of the recent literature. In section 3, we outline in detail our three-step novel methodology. In section 4, we apply the proposed methodology to the time series of three energy related assets traded in the U.S. In section 5, we present some theoretical results regarding the cross-Greeks and finally, in section 6, we conclude with certain remarks.

2. A short review of the recent literature

In this section we present a short survey of the recent literature about the comovements among asset prices. We emphasize that our survey is brief and selective rather than exhaustive and detailed.

In Eun and Shim (1989) the mechanism of international transmission of stock price movements is investigated by using a nine-market VAR system. In particular, the authors trace out the dynamics of the responses in a given market to the innovations verified in another one. Deb, Trivedi and Varangis (1996) use univariate and multivariate GARCH models to show that the prices of several unrelated markets reveal a persistent tendency to comove, even after accounting for the effects of macroeconomic shocks. Malliaris and Urrutia (1996) identify both short-term and long-term dependence relationships among the prices of six agricultural futures traded at the Chicago Board of Trade by using an error-correction model (ECM). In Hamori and Imamura (2000), the investigation of interdependencies among stock prices is performed by using a LA-VAR system based approach. A significant advantage of such a methodology is the fact that it can be applied regardless of the presence, or lack of cointegration among the considered stock prices. In Algren and Antell (2002) cointegration among stock prices traded in different countries is investigated. In particular, the authors find evidence that the likelihood ratio tests of Johansen are sensitive to the specification of the time lag amplitude in the VAR system.

Some other methodologies that are worth mentioning are the ones able to detect the presence, or lack of common cycles among asset prices. Broome and Morley (2000) use a cointegration technique for testing the presence of long-run common trends among stock prices and the risk free interest rate and perform dependence analyses to investigate the presence and features of short-run common cycles among the same quantities. In Chen and Wun (2004) linear and non-linear Granger causality based tests are used to examine the dynamical dependence relationships between spot and future prices. Finally, in Schich (2004) proper measures of dependence among European stock markets are evaluated by using the multivariate extreme value theory.

3. Our three-step methodology

In this section we present in detail our novel methodology for the non-linear evaluation of bivariate comovements. Since our approach relies on the concept of comonotonicity, before of all we spend some words about this notion. Comonotonicity is one of the strongest measures of dependence existing among random variables. Limiting our interest to the bivariate case, given two random variables $X_1(\cdot)$ and $X_2(\cdot)$, both defined on the same probability space (Ω, F, P) , they are said to be comonotonic if there exists, with probability 1, a subset A of F such that

$$[X_1(\omega_a) - X_1(\omega_b)][X_2(\omega_a) - X_2(\omega_b)] \geq 0 \quad \forall \omega_a, \omega_b \in A \times A.^1$$

In an analogous way, given two random variables $X_1(t)$ and $X_2(t)$, both defined on $[t_0, t_1]$ with $t_0 < t_1$, we say they are codependent if:

$$[X_1(t_3) - X_1(t_2)][X_2(t_3) - X_2(t_2)] \geq 0 \quad \forall t_2, t_3 : t_2 \neq t_3 \wedge t_2, t_3 \in [t_0, t_1].$$

A few remarks about the relationship of codependence are appropriate:

- First, two random variables are codependent if they always vary over the support (time, in our case) in the same direction, besides the quantitative laws describing the dynamic behaviour of each of them;
- Second, codependence as comonotonicity, is an ON/OFF concept. Actually, if there is a unique pair t_2 and t_3 for which $[X_1(t_3) - X_1(t_2)][X_2(t_3) - X_2(t_2)] < 0$, then we say that $X_1(t)$ and $X_2(t)$ are not codependent.

¹ For other equivalent definitions of comonotonicity see Jouini and Napp (2003, 2004), and Wei and Yatracos (2004).

Our methodology is articulated in three steps. Before these steps are outlined, we briefly describe each:

- In the first step we propose a simple index able to evaluate any intermediate degree of bivariate dependence from full counterdependence² to full codependence, and we provide some theoretical results about it.
- Next, this simple index provides only a point estimation of the bivariate dependence while in the second step we propose a procedure to test the statistical meaningfulness of the index itself;
- Finally, in the third step we propose an algorithm to provide a polynomial approximation of the unknown bivariate dependence relationship.

3.1 The simple index

Let us start by considering two discrete-time time series, $\{X_1(t), t = t_1, \dots, t_N\}$ and $\{X_2(t), t = t_1, \dots, t_N\}$. The simple index we propose for evaluating the bivariate dependence between the random variables $X_1(t)$ and $X_2(t)$ is defined as follows:

$$\delta_{1,2} = \frac{1}{N-1} \sum_{t=t_2}^{t_N} \Delta(t)_{1,2}, \Delta(t)_{1,2} = \begin{cases} -1 & \text{if } [X_1(t) - X_1(t-1)][X_2(t) - X_2(t-1)] < 0 \\ 1 & \text{if } [X_1(t) - X_1(t-1)][X_2(t) - X_2(t-1)] \geq 0 \end{cases}. \quad (1)$$

Some remarks about this index:

- It is easy to prove that $\delta_{1,2} \in [-1, 1]$. In particular, any two random variables are counterdependent if $\delta_{1,2} = -1$, and are codependent if $\delta_{1,2} = 1$;
- It is also easy to prove that $\delta_{1,2}$ is defined for every pair of discrete-time time series (property of existence), and that $\delta_{1,2} = \delta_{2,1}$ (property of symmetry). Therefore, $\delta_{1,2}$ is a scalar measure of dependence in the sense illustrated in Szego (2005) at section 6;
- The fact that $\delta_{1,2}$ belongs to $[-1, 1]$ makes this index of dependence directly comparable with the well known and widely used Bravais–Pearson linear correlation coefficient $\rho_{1,2}$.³

For the theoretical properties between $\delta_{1,2}$ and $\rho_{1,2}$ we state and prove the proposition below

² Given two random variables $X_1(t)$ and $X_2(t)$, both defined on $[t_0, t_1]$ with $t_0 < t_1$, we say they are counterdependent if $[X_1(t_3) - X_1(t_2)][X_2(t_3) - X_2(t_2)] < 0$ for all t_2 and t_3 such that $t_2 \neq t_3$ and $t_2, t_3 \in [t_0, t_1]$.

³ Also $\rho_{1,2}$ is a scalar measure of dependence in the sense illustrated in [sze] at section 6.

Proposition 1. Let $f(\cdot): \mathbf{R}_+ \rightarrow \mathbf{R}_+$ be the bivariate dependence relationship between $X_1(t)$ and $X_2(t)$, $X_1(t) = f(X_2(t)) + \varepsilon(t)$, where $\varepsilon(t)$ is a standardized error term, and let $f(\cdot)$ be infinitely differentiable in $m_2 = E(X_2(t))$. If

$$\frac{f^{(i)}(m_2)}{i!}(-m_2)^{i-j} = 0 \quad \forall i, j: i = 0, \dots, +\infty \wedge j = 2, \dots, +\infty \wedge i - j \geq 2, \quad (2)$$

where $f^{(i)}(\cdot)$ indicates the i -th derivative of $f(\cdot)$, then the bivariate dependence relationship is affine.

Proof. As $f(\cdot)$ is infinite times derivable in m_2 , we can expand it in Taylor's series about m_2 itself as follows:

$$\begin{aligned} f(X_2(t)) &= \sum_{i=0}^{+\infty} \frac{f^{(i)}(m_2)}{i!} (X_2(t) - m_2)^i \\ &= \sum_{i=0}^{+\infty} \frac{f^{(i)}(m_2)}{i!} \sum_{j=0}^i \binom{i}{j} X_2^{i-j}(t) (-m_2)^j. \end{aligned} \quad (3)$$

After some algebraic manipulations, we can rewrite equation (3) as follows:

$$f(X_2(t)) = \sum_{j=0}^{+\infty} \left[\sum_{i=j}^{+\infty} \frac{f^{(i)}(m_2)}{i!} \binom{i}{i-j} (-m_2)^{i-j} \right] X_2^j(t). \quad (4)$$

Now, by substituting relationship (2) into (4) we obtain the following affine bivariate dependence relationship between $X_1(t)$ and $X_2(t)$:

$$X_1(t) = \sum_{i=0}^{+\infty} \frac{f^{(i)}(m_2)}{i!} \binom{i}{i} (-m_2)^i + \left[\sum_{i=1}^{+\infty} \frac{f^{(i)}(m_2)}{i!} \binom{i}{i-1} (-m_2)^{i-1} \right] X_2(t) + \varepsilon(t). \quad (5)$$

Notice that, if relationship (2) is extended also to $j = 1$, then relationship (5) will become

$$X_1(t) = \sum_{i=0}^{+\infty} \frac{f^{(i)}(m_2)}{i!} \binom{i}{i} (-m_2)^i + \varepsilon(t),$$

i.e. $X_1(t)$ and $X_2(t)$ are independent.

The simple index we proposed here provides only a point estimation of the investigated bivariate dependence. In order to overcome this drawback, in the next subsection we propose a procedure able to statistically test the meaningfulness of the index itself.

3.2 The testing procedure

The intuition of the approach we propose here for testing the statistical meaningfulness of $\delta_{1,2}$ is similar to the one suggested in Kaboudan (2000).

In the remainder of this subsection we present our testing procedure:

- Firstly, we define the index $\delta_{S;1,2}$ as the index (1), but applied to $\{X_1(t), t = t_1, \dots, t_N\}$ and $\{X_2(t), t = t_1, \dots, t_N\}$ once both these time series have been shuffled according to the same independent and identical uniform distribution (notice that, as the shuffling removes any dependence relationship between $X_1(t)$ and $X_2(t)$, $\delta_{S;1,2}$ equals 0);
- Secondly, we define the random variable $\Delta\delta = \delta_{1,2} - \delta_{S;1,2}$, and generate the series $\{\Delta\delta(j), j = 1, \dots, M\}$ by shuffling, for M times, $\{X_1(t), t = t_1, \dots, t_N\}$ and $\{X_2(t), t = t_1, \dots, t_N\}$ as previously described. Notice that, if $X_1(t)$ and $X_2(t)$ were $\delta_{1,2}$ -dependent, then $\Delta\delta$ should be different from 0 ;
- Thirdly, we determine estimations of the sample mean and of the sample standard deviation of $\Delta\delta$, $m_{\Delta\delta}$ and $s_{\Delta\delta}$ respectively, as follows:

$$m_{\Delta\delta} = \frac{1}{M} \sum_{j=1}^M \Delta\delta(j) \text{ and } s_{\Delta\delta} = \sqrt{\frac{1}{M-1} \sum_{j=1}^M (\Delta\delta(j) - m_{\Delta\delta})^2};$$

- Fourthly, recalling from basic statistics that

$$\frac{m_{\Delta\delta} - \Delta\delta}{s_{\Delta\delta}/\sqrt{M-1}} \xrightarrow{d} N(0,1) \text{ as } M \rightarrow +\infty,$$

for M large enough we can perform the following bilateral t -test:

$$\begin{cases} H_0 : m_{\Delta\delta} = 0, \text{ i.e. } X_1(t) \text{ and } X_2(t) \text{ are independent} \\ H_1 : m_{\Delta\delta} \neq 0, \text{ i.e. } X_1(t) \text{ and } X_2(t) \text{ are } \delta_{1,2} \text{-dependent} \end{cases}; \quad (6)$$

in particular, the acceptance interval for the null hypothesis is $\left(-s_{\Delta\delta}t_{\alpha/2}/\sqrt{M-1}, s_{\Delta\delta}t_{\alpha/2}/\sqrt{M-1}\right)$, where $t_{\alpha/2}$ is the value taken by a t -distributed random variable in correspondence of a pre-established confidence interval α for given degrees of freedom;

- Finally, if the null hypothesis is rejected, then we perform two more unilateral t -tests in order to verify whether the $\delta_{1,2}$ -dependence between $X_1(t)$ and $X_2(t)$ is negative or positive. In particular, both such tests differ from the one introduced in the previous point only in the alternative hypothesis, which is $H_1 : m_{\Delta\delta} < 0$ in the negative $\delta_{1,2}$ -dependence case, and is $H_1 : m_{\Delta\delta} > 0$ in the positive $\delta_{1,2}$ -dependence case.

Notice that, in order to reduce the amplitude of the acceptance intervals, *i.e.* to reduce $s_{\Delta\delta}/\sqrt{M-1}$ to $s_{\Delta\delta}/(c\sqrt{M-1})$, with $c > 1$, one has to increase M to $\lceil c^2(M-1)+1 \rceil$.⁴ Because of that, profitable applications of our methodology could be sometimes time-consuming.

3.3 The polynomial approximation

If at the end of the testing procedure the null hypothesis has been rejected in favour of the negative/positive $\delta_{1,2}$ -dependence between $X_1(t)$ and $X_2(t)$, then we begin to model analytically the unknown bivariate dependence relationship $X_1(t) = f(X_2(t)) + \varepsilon(t)$. In particular, we search for a polynomial approximation of $f(\cdot)$ which is a properly truncated version of equation (4), *i.e.*

$$f(X_2(t)) = \sum_{j=0}^K a_j X_2^j(t) + r(K). \quad (7)$$

where K is the truncation order of the Taylor's series (4), $a_j = \sum_{i=j}^K \frac{f^{(i)}(m_2)}{i!} \binom{i}{i-j} (-m_2)^{i-j}$, and $r(K)$ is a suitable remainder function.

Of course, in this approach a crucial role is played by K . In order to detect its “optimal” value, we propose an algorithm whose search procedure is based on a standard cross-validation technique, as suggested in Poggio and Smale (2003) in section 4:

- In particular, we begin by considering as a starting data set D the discrete-time bivariate time series $\{(X_1(t), X_2(t)), t = t_1, \dots, t_N\}$;
- Sccondly, we suitably split D into two data subsets, the learning one D_L and the validation one D_V , such that $D_L \cup D_V = D$ and $D_L \cap D_V = \emptyset$;⁵

⁴ $\lceil \cdot \rceil$ is the minimal integer which exceeds the value taken by the expression inside the notation itself.

⁵ The way in which to suitably split D is made clear in subsection 4.2.

- Thirdly, we consider a finite series of polynomials of the form in (7) with $K = 0, \dots, \overline{K}$, where \overline{K} is a pre-established integer value;
- Fourthly, for each of the polynomials considered in the previous point we estimate the parameters a_0, \dots, a_K via ordinary least square regression by using the data subset D_L , and evaluate the index $\delta_{1,2}$ between $\hat{X}_1(t) = \sum_{j=0}^K \hat{a}_j X_2^j(t)$ and $X_2(t)$ by using the data subset D_V ; ⁶
- Finally, we choose as “best” approximating polynomial the one associated with the highest absolute value of $\delta_{1,2}$.

Notice that identifying the “optimal” approximating polynomial is accomplished by using a cross-validation approach. That is, we perform ordinary least squares by using the learning data subset and evaluate the validation criterion $|\delta_{1,2}|$ by using the validation data subset.

4. Applications to energy asset prices time series

In this section we give the empirical results of the three-step methodology to the time series of three energy related prices

In general terms, for each empirical computation we do the following:

- We start by considering the discrete-time bivariate time series $\{(X_1(t), X_2(t)), t = t_1, \dots, t_N\}$;
- We split the chronologically the last 10 per cent of the realizations of the time series introduced in the previous point in order to utilize them as forecasting data subset D_F at the end of the application for performing a simple out-of-sample check. We use the remaining 90 percent of the discrete-time bivariate time series as the starting data set D ;
- We split D into the learning data subset D_L (the chronologically first 70 per cent of its realizations) and the validation data subset D_V (the chronologically last 30 per cent of its realizations); ⁷
- Finally, we apply our methodology by using D_L and D_V .

4.1 The data

⁶ $\hat{\cdot}$ indicates the estimator of the quantity below.

⁷ The percentages we set for D_L and D_V are the ones usually utilized in several empirical works using cross-validation techniques as in Belcaro, Canestrelli and Corazza(1996) and their references.

Each discrete–time univariate time series we utilize here contains 2,026 daily spot closing prices for three energy assets traded in U.S.A.: crude oil, gasoline, and heating oil. Such prices have been collected from January 3, 1994 to February 6, 2002. In the remainder of this subsection and in the next one, we refer to these time series respectively as $\{X_{CO}(t), t = t_1, \dots, t_{2,026}\}$, $\{X_G(t), t = t_1, \dots, t_{2,026}\}$, and $\{X_{HO}(t), t = t_1, \dots, t_{2,026}\}$. Notice that, given the percentages indicated earlier for the data subsets used in each application, the cardinalities of these same data subsets are: $|D_L| = 1,277$, $|D_V| = 547$, and $|D_F| = 202$.

The discrete–time bivariate time series whose non–linear comovements we investigate here, come from the simple discrete–time univariate time series listed as: $\{(X_{CO}(t), X_G(t)), t = t_1, \dots, t_{2,026}\}$, $\{(X_{CO}(t), X_{HO}(t)), t = t_1, \dots, t_{2,026}\}$, $\{(X_G(t), X_{CO}(t)), t = t_1, \dots, t_{2,026}\}$, $\{(X_G(t), X_{HO}(t)), t = t_1, \dots, t_{2,026}\}$, $\{(X_{HO}(t), X_{CO}(t)), t = t_1, \dots, t_{2,026}\}$, and $\{(X_{HO}(t), X_G(t)), t = t_1, \dots, t_{2,026}\}$.

Table 1 reports some standard descriptive statistics for each of the discrete–time univariate time series.

<**Table 1** from here beyond (if possible, approximately here)>

4.2 The results

The exposition of the results from the application of our three–step methodology to the discrete–time bivariate time series is organized in two tables, and in six figures.

The columns of **Table 2** are described as follows:

- The first column indicates the two random variables that specify the discrete–time bivariate time series which has investigated;
- The second column reports the value of the simple index $\delta_{i,j}$, with $i, j \in \{CO, G, HO\}$ and $i \neq j$, evaluated on the learning data subset D_L (see, for more details, subsection 3.1);
- The third column provides the response of the bilateral t –test (6):⁸ label “A” or label “R” for, respectively, the acceptance or the rejection of the null hypothesis (see, for more details, subsection 3.2);

⁸ In performing this bilateral test we set $M = 100$ and $\alpha = 5\%$.

- If the null hypothesis of the bilateral t -test (6) is rejected, then the fourth column gives the response of the check, based on two more unilateral t -test,⁹ whether the $\delta_{i,j}$ -dependence, with $i, j \in \{CO, G, HO\}$ and $i \neq j$, between the two investigated univariate time series is negative or positive: label “ N ” or label “ P ” respectively (see, for more details, again subsection 3.2);
- The fifth column gives the value of the Bravais–Pearson linear correlation coefficient $\rho_{i,j}$, with $i, j \in \{CO, G, HO\}$ and $i \neq j$, evaluated on the learning data subset D_L (we report the value of this coefficient for possible comparisons).

Finally, we recall that the property of symmetry holds for the simple index (1), *i.e.* $\delta_{i,j} = \delta_{j,i}$ for all i and j such that $i, j \in \{CO, G, HO\}$.

<**Table 2** from here beyond (if possible, approximately here)>

A few remarks about the results reported in **Table 2**:

- The fact that $\delta_{i,j}$ is statistically significantly different from 0 for all i and j such that $i, j \in \{CO, G, HO\}$ and $i \neq j$ (see jointly the second and the third column of **Table 2**) indicates the existence of a bivariate dependence relationship between $X_i(t)$ and $X_j(t)$ for all the considered i and j ;
- Recalling that the Bravais–Pearson coefficient measures only the linear correlation, the fact that $\delta_{i,j}$ is significantly different from $\rho_{i,j}$ for all i and j such that $i, j \in \{CO, G, HO\}$ and $i \neq j$ (see jointly the second and the fifth column of **Table 2**) offers evidence of non-linearity in the bivariate dependence relationships;
- The fact that $\delta_{i,j}$ and $\rho_{i,j}$ are both positive for all i and j such that $i, j \in \{CO, G, HO\}$ and $i \neq j$ (see jointly the second and the fifth column of **Table 2** again) can be interpreted as an indicator of the positiveness of the dependence between $X_i(t)$ and $X_j(t)$ for all the considered i and j .

We next turn to **Table 3** and describe its columns:

- The first column indicates the two random variables that specify the discrete-time bivariate time series which is investigated;

⁹ Also in performing these unilateral tests we set $M = 100$ and $\alpha = 5\%$.

- The second column provides the estimation of the “best” polynomial approximation of the unknown bivariate dependence relationship between $X_i(t)$ and $X_j(t)$ for all i and j such that $i, j \in \{CO, G, HO\}$ and $i \neq j$ (see, for more details, subsection 3.3).

<Table 3 from here beyond (if possible, approximately here)>

Some remarks about the results reported in **Table 3**:

- The fact that the degree of the “best” polynomial approximation is greater than 1 in a significant percentage of the considered cases confirms the presence of non-linearities in some of the investigated bivariate dependence relationships;
- With specific regard to the fifth polynomial approximation, the fact that the coefficients associated to the highest powers of $X_{CO}(t)$ are evidently close to 0, *i.e.* the fact that their “explanatory contributions” are probably negligible, *i.e.* the fact that the degree of the approximating polynomial is probably unnecessarily high, can be interpreted as a symptom of the need that the validation procedure we propose and use here has to be probably a little bit refined.

Finally, at the end of this section we utilize all the polynomial approximations reported in **Table 3** applying each of them to the corresponding data subset D_F . By so doing, we provide an out-of-sample visual check (see **Figure 1** to **Figure 3**) of the goodness of our three-step methodology.

<Figure 1 from here beyond (if possible, approximately here)>

<Figure 2 from here beyond (if possible, approximately here)>

<Figure 2 from here beyond (if possible, approximately here)>

Notice that, although in the graph on the right of **Figure 3** the polynomial approximation of $X_{HO}(t)$ in D_F is generated by the approximating polynomial whose degree is probably unnecessarily high, only the estimation $\hat{X}_{HO}(21)$ is evidently poor. We can interpret it as an indication of the robustness of our three-step methodology.

5. Cross-Greeks

In this section we present some theoretical results concerning a possible utilization of the proposed polynomial approximation of the vicariate dependence relationship in one of the research field in which the co movements among asset prices show to play a role of evident importance, *i.e.* the research field of option contracts. In particular, given two assets whose prices are $X_1(t)$ and $X_2(t)$ respectively, both defined on $[t_0, t_1]$ with $t_0 < t_1$, and given the polynomial approximation of their vicariate dependence relationship $X_1(X_2(t)) = \sum_{i=0}^K a_i X_2^i(t)$, with $K \in \mathbf{N}^0$ and $a_i \in \mathbf{R}$, we provide the analytical approximations in terms of $X_2(t)$ the Greeks of the (vanilla) European call and put options for which the price of the underlying is $X_1(t)$. Notice that such results, beyond their theoretical significance, also appear of some operative interest, like, for instance, in the case of definition of strategies of cross-hedging, cross-speculation, and so on (see, for instance, the last section of Malliaris and Urrutia (1996)).

Before to present our theoretical results, we need to specify our notation in order to formulate and prove such results:

- we denote the l -the derivative of $X_1(X_2(t))$ by

$$X_1^{(l)}(X_2(t)) = \frac{d^l}{dX_2^l(t)} X_1(X_2(t)) = \sum_{i=l}^K a_i \prod_{j=0}^{l-1} (i-j) X_2^{i-l}, \quad K \in \mathbf{N}^0 \wedge a_i \in \mathbf{R}$$

- we denote the function of the cumulative probability distribution of a standard normally distributed random variable and its first derivative, respectively by

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{t^2}{2}} dt \quad \text{and} \quad \Phi^{(1)}(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}};$$

- we denote the strike of the considered option, the volatility of the underlying, the time until the expiration of the considered option and the (continuously compounded) risk free interest rate of return, respectively by

$$X, \sigma, \tau, \text{ and } r.$$

5.1 Cross-Greeks for the European call option

In this subsection we provide the theoretical results regarding the cross-Greeks of the (vanilla) European call option.

Proposition 2. *Let the usual hypotheses concerning the Black–and–Scholes environment hold, and let $X_1(t)$ and $X_2(t)$ be the prices of two assets, both defined on $[t_0, t_1]$ with $t_0 < t_1$. If*

$$X_1(X_2(t)) = \sum_{i=0}^K a_i X_2^i(t), \quad K \in \mathbf{N}^0 \wedge a_i \in \mathbf{R} \quad (8)$$

then

$$\begin{aligned} \text{cross} - \text{delta}_{\text{call}} &= \Phi(d_1^*) X_1^{(1)}(X_2(t)), \\ \text{cross} - \text{gamma}_{\text{call}} &= \frac{\Phi^{(1)}(d_1^*)}{X_1(X_2(t)) \sigma \sqrt{\tau}} [X_1^{(1)}(X_2(t))]^2 + \Phi(d_1^*) X_1^{(2)}(X_2(t)), \\ \text{cross} - \text{vega}_{\text{call}} &= X_1(X_2(t)) \sqrt{\tau} \Phi^{(1)}(d_1^*),^{10} \\ \text{cross} - \text{theta}_{\text{call}} &= \frac{X_1(X_2(t)) \sigma}{2\sqrt{\tau}} \Phi^{(1)}(d_1^*) + X r e^{-r\tau} \Phi(d_2^*) \text{ and} \\ \text{cross} - \text{rho}_{\text{call}} &= X \tau e^{-r\tau} \Phi(d_2^*), \end{aligned}$$

where $d_1^* = \frac{\log(X_1(X_2(t))/X) + r\tau + \sigma^2\tau/2}{\sigma\sqrt{\tau}}$ and $d_2^* = d_1^* - \sigma\sqrt{\tau}$.

Proof. As the usual hypotheses concerning the Black–and–Scholes environment hold, we can attain the usual Black–and–Scholes valuation formula for the (vanilla) European call option for which the price of the underlying is $X_1(t)$, defined on $[t_0, t_1]$ with $t_0 < t_1$, i.e.

$$c_1(t) = X_1(t) \Phi(d_1) - X e^{-r\tau} \Phi(d_2), \quad (9)$$

where $d_1 = \frac{\log(X_1(t)/X) + r\tau + \sigma^2\tau/2}{\sigma\sqrt{\tau}}$ and $d_2 = d_1 - \sigma\sqrt{\tau}$.

Then, by substituting relationship (8) into (9) we obtain the following approximation in terms of $X_2(t)$ of the valuation formula (9):

$$c_1^*(t) = X_1(X_2(t)) \Phi(d_1^*) - X e^{-r\tau} \Phi(d_2^*), \quad (10)$$

¹⁰ Notice that vega is not a letter of the Greek alphabet.

At this point, by determining in the ways which follow the first and (when necessary) the second order partial derivatives of relationship (10) with respect to $X_2(t)$, σ , τ and r respectively, we obtain the investigated cross-Greeks:

$$\begin{aligned} cross - delta_{call} &= \frac{\partial}{\partial X_2(t)} c_1^*(t) = X_1^{(1)}(X_2(t))\Phi(d_1^*) + X_1(X_2(t))\Phi^{(1)}(d_1^*)\frac{\partial}{\partial X_2(t)} d_1^* - \\ &\quad - X e^{-r\tau}\Phi^{(1)}(d_2^*)\frac{\partial}{\partial X_2(t)} d_2^*; \end{aligned} \quad (11)$$

now, noting that $\frac{\partial}{\partial X_2(t)} d_1^* = \frac{\partial}{\partial X_2(t)} d_2^* = \frac{X_1^{(1)}(X_2(t))}{X_1(X_2(t))\sigma\sqrt{\tau}}$, by substituting this relationship and the expression of $\Phi^{(1)}(\cdot)$ into (11), after some algebraic manipulations, we obtain that $cross - delta_{call} = \Phi(d_1^*)X_1^{(1)}(X_2(t))$;

$$\begin{aligned} cross - gamma_{call} &= \frac{\partial^2}{\partial X_2^2(t)} c_1^*(t) = \frac{\partial}{\partial X_2(t)} cross - delta_{call} = \Phi^{(1)}(d_1^*)\frac{\partial}{\partial X_2(t)} d_1^* \cdot \\ &\quad \cdot X_1^{(1)}(X_2(t)) + \Phi(d_1^*)X_1^{(2)}(X_2(t)); \end{aligned} \quad (12)$$

now, noting that $\frac{\partial}{\partial X_2(t)} d_1^* = \frac{X_1^{(1)}(X_2(t))}{X_1(X_2(t))\sigma\sqrt{\tau}}$, by substituting this relationship into (12), after some algebraic manipulations, we obtain that $cross - gamma_{call} = \frac{\Phi^{(1)}(d_1^*)}{X_1(X_2(t))\sigma\sqrt{\tau}} [X_1^{(1)}(X_2(t))]^2 + \Phi(d_1^*) \cdot X_1^{(2)}(X_2(t))$;

$$cross - vega_{call} = \frac{\partial}{\partial \sigma} c_1^*(t) = X_1(X_2(t))\Phi^{(1)}(d_1^*)\frac{\partial}{\partial \sigma} d_1^* - X e^{-r\tau}\Phi^{(1)}(d_2^*)\frac{\partial}{\partial \sigma} d_2^*; \quad (13)$$

now, noting that $\Phi^{(1)}(d_2^*) = \Phi^{(1)}(d_1^*)e^{r\tau} \frac{X_1(X_2(t))}{X}$ and that $\frac{\partial}{\partial \sigma} d_2^* = \frac{\partial}{\partial \sigma} d_1^* - \sqrt{\tau}$, by substituting these relationships into (13), after some algebraic manipulations, we obtain that $cross - vega_{call} = X_1(X_2(t))\sqrt{\tau}\Phi^{(1)}(d_1^*)$;

$$\begin{aligned} cross - theta_{call} &= \frac{\partial}{\partial \tau} c_1^*(t) = X_1(X_2(t))\Phi^{(1)}(d_1^*)\frac{\partial}{\partial \tau} d_1^* - X e^{-r\tau}(-r)\Phi(d_2^*) - \\ &\quad - X e^{-r\tau}\Phi^{(1)}(d_2^*)\frac{\partial}{\partial \tau} d_2^*; \end{aligned} \quad (14)$$

now, noting that $\Phi^{(1)}(d_2^*) = \Phi^{(1)}(d_1^*)e^{r\tau} \frac{X_1(X_2(t))}{X}$ and that $\frac{\partial}{\partial \tau} d_2^* = \frac{\partial}{\partial \tau} d_1^* - \frac{\sigma}{2\sqrt{\tau}}$, by substituting these relationships into (14), after some algebraic manipulations, we obtain that

$$cross - theta_{call} = \frac{X_1(X_2(t))\sigma}{2\sqrt{\tau}} \Phi^{(1)}(d_1^*) + Xre^{-r\tau} \Phi(d_2^*);$$

$$\begin{aligned} cross - rho_{call} &= \frac{\partial}{\partial r} c_1^*(t) = X_1(X_2(t)) \Phi^{(1)}(d_1^*) \frac{\partial}{\partial r} d_1^* - Xe^{-r\tau} (-\tau) \Phi(d_2^*) - \\ &\quad - Xe^{-r\tau} \Phi^{(1)}(d_2^*) \frac{\partial}{\partial r} d_2^*; \end{aligned} \quad (15)$$

now, noting that $\Phi^{(1)}(d_2^*) = \Phi^{(1)}(d_1^*)e^{r\tau} \frac{X_1(X_2(t))}{X}$ and that $\frac{\partial}{\partial r} d_2^* = \frac{\partial}{\partial r} d_1^*$, by substituting these relationships into (15), after some algebraic manipulations, we obtain that

$$cross - rho_{call} = X\tau e^{-r\tau} \Phi(d_2^*).$$

5.2 Cross-Greeks for the European put option

In this subsection we provide the theoretical results concerning the cross-Greeks of the (vanilla) European put option.

Proposition 3. *Let the usual hypotheses concerning the Black-and-Scholes environment hold, and let $X_1(t)$ and $X_2(t)$ be the prices of two assets, both defined on $[t_0, t_1]$ with $t_0 < t_1$. If*

$$X_1(X_2(t)) = \sum_{i=0}^K a_i X_2^i(t), \quad K \in \mathbf{N}^0 \wedge a_i \in \mathbf{R} \quad (16)$$

then

$$\begin{aligned} cross - delta_{put} &= [\Phi(d_1^*) - 1] X_1^{(1)}(X_2(t)), \\ cross - gamma_{put} &= \frac{\Phi^{(1)}(d_1^*)}{X_1(X_2(t))\sigma\sqrt{\tau}} [X_1^{(1)}(X_2(t))]^2 + [\Phi(d_1^*) - 1] X_1^{(2)}(X_2(t)), \\ cross - vega_{put} &= X_1(X_2(t))\sqrt{\tau} \Phi^{(1)}(d_1^*), \\ cross - theta_{put} &= \frac{X_1(X_2(t))\sigma}{2\sqrt{\tau}} \Phi^{(1)}(d_1^*) + Xre^{-r\tau} [\Phi(d_2^*) - 1] \text{ and} \\ cross - rho_{put} &= X\tau e^{-r\tau} [\Phi(d_2^*) - 1], \end{aligned}$$

where $d_1^* = \frac{\log(X_1(X_2(t))/X) + r\tau + \sigma^2\tau/2}{\sigma\sqrt{\tau}}$ and $d_2^* = d_1^* - \sigma\sqrt{\tau}$.

Proof. As the usual hypotheses concerning the Black–and–Scholes environment hold, we can attain the usual Black–and–Scholes valuation formula (9) and the usual put–call parity relationship between the prices of a (vanilla) European put option and of a (vanilla) call option for both of which the price of the underlying is $X_1(t)$, defined on $[t_0, t_1]$ with $t_0 < t_1$, i.e.

$$p_1(t) = c_1(t) + Xe^{-r\tau} - X_1(t). \quad (17)$$

Then, by substituting relationships (9) and (16) into (17) we obtain the following approximation in terms of $X_2(t)$ of the put–call parity relationship

$$p_1^*(t) = c_1^*(t) + Xe^{-r\tau} - X_1(X_2(t)). \quad (18)$$

At this point, by determining in the ways which follow the first and (when necessary) the second order partial derivatives of relationship (18) with respect to $X_2(t)$, σ , τ and r respectively, we obtain the investigated cross–Greeks:

$$\begin{aligned} cross - delta_{put} &= \frac{\partial}{\partial X_2(t)} p_1^*(t) = \frac{\partial}{\partial X_2(t)} c_1^*(t) + \frac{\partial}{\partial X_2(t)} Xe^{-r\tau} - \frac{\partial}{\partial X_2(t)} X_1(X_2(t)) = \\ &= cross - delta_{call} - X_1^{(1)}(X_2(t)); \end{aligned} \quad (19)$$

now, by substituting the expression of $cross - delta_{call}$ into (19), after some algebraic manipulations, we obtain that $cross - delta_{put} = [\Phi(d_1^*) - 1]X_1^{(1)}(X_2(t))$;

$$\begin{aligned} cross - gamma_{put} &= \frac{\partial^2}{\partial X_2^2(t)} p_1^*(t) = \frac{\partial^2}{\partial X_2^2(t)} c_1^*(t) + \frac{\partial^2}{\partial X_2^2(t)} Xe^{-r\tau} - \frac{\partial^2}{\partial X_2^2(t)} X_1(X_2(t)) = \\ &= cross - gamma_{call} - X_1^{(2)}(X_2(t)); \end{aligned} \quad (20)$$

now, by substituting the expression of $cross - gamma_{call}$ into (20), after some algebraic manipulations, we obtain that $cross - gamma_{put} = \frac{\Phi^{(1)}(d_1^*)}{X_1(X_2(t))\sigma\sqrt{\tau}} [X_1^{(1)}(X_2(t))]^2 + [\Phi(d_1^*) - 1] \cdot X_1^{(2)}(X_2(t))$;

$$\begin{aligned}
cross-vega_{put} &= \frac{\partial}{\partial \sigma} p_1^*(t) = \frac{\partial}{\partial \sigma} c_1^*(t) + \frac{\partial}{\partial \sigma} X e^{-r\tau} - \frac{\partial}{\partial \sigma} X_1(X_2(t)), \\
&= cross-vega_{call} = X_1(X_2(t)) \sqrt{\tau} \Phi^{(1)}(d_1^*), \\
cross-theta_{put} &= \frac{\partial}{\partial \tau} p_1^*(t) = \frac{\partial}{\partial \tau} c_1^*(t) + \frac{\partial}{\partial \tau} X e^{-r\tau} - \frac{\partial}{\partial \tau} X_1(X_2(t)) = \\
&= cross-theta_{call} + X e^{-r\tau} (-r);
\end{aligned} \tag{21}$$

now, by substituting the expression of $cross-theta_{call}$ into (21), after some algebraic

manipulations, we obtain that $cross-theta_{put} = \frac{X_1(X_2(t))\sigma}{2\sqrt{\tau}} \Phi^{(1)}(d_1^*) + X r e^{-r\tau} [\Phi(d_2^*) - 1]$;

$$\begin{aligned}
cross-rho_{put} &= \frac{\partial}{\partial r} p_1^*(t) = \frac{\partial}{\partial r} c_1^*(t) + \frac{\partial}{\partial r} X e^{-r\tau} - \frac{\partial}{\partial r} X_1(X_2(t)) = \\
&= cross-rho_{call} + X e^{-r\tau} (-\tau);
\end{aligned} \tag{22}$$

now, by substituting the expression of $cross-rho_{call}$ into (22), after some algebraic manipulations,

we obtain that $cross-rho_{put} = X \tau e^{-r\tau} [\Phi(d_2^*) - 1]$.

6. Final remarks for future research

We conclude this paper by presenting few remarks for possible extensions of our work:

- First, the approximating fifth order polynomial reported in **Table 3** is probably unnecessarily high. Some aspects of the validation procedure, like, for instance, the determination of the validation data subset or the specification of the validation criterion need to be carefully verified by means of further applications of our three-step methodology and, on the basis of the information obtained further refinements are desirable.
- Second, in subsection **4.2** we provide an out-of-sample check which is only visual. It is probably suitable to develop it in a more formal way (like, for instance, the one given by a set of proper indices) in order to get more objective validation information;
- Finally, we have shown that our approach offers opportunities for possible generalizations. In fact, our three-step methodology can be developed in order to analyze, beyond time no-lagged bivariate dependence relationships like $X_1(t) = f(X_2(t)) + \varepsilon(t)$, also time lagged bivariate dependence relationships like, for instance, $X_1(t) = f(X_2(t), X_2(t-1), \dots, X_2(t-T)) + \varepsilon(t)$, with $T \in \mathbf{N}^0$, time no-lagged multivariate dependence relationships like, for instance,

$X_1(t) = f(X_2(t), X_3(t), \dots, X_I(t)) + \varepsilon(t)$, with $I \in \mathbf{N}^0$, and time lagged multivariate dependence relationships like, for instance, $X_1(t) = f(X_2(t), X_2(t-1), \dots, X_2(t-T_2), X_3(t), X_3(t-1), \dots, X_3(t-T_3), \dots, X_I(t), X_I(t-1), \dots, X_I(t-T_I))$, with $I, T_1, \dots, T_I \in \mathbf{N}^0$. Moreover, also our theoretical results concerning the cross-Greeks can be generalized in order to take into account, beyond “standard” underlyings of the investigated (vanilla) European options, and also underlyings like, for instance, currencies and futures contracts.

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Table 1.

Energy asset: crude oil							
Set	<i>Minimum</i>	<i>Maximum</i>	Mean	Median	Stan. dev.	Skewness	Kurtosis
D_L	10.8000	26.6100	18.4145	18.4100	3.2108	0.0862	-0.1969
D_V	11.3600	37.0000	25.6444	26.6700	5.9660	-0.5178	-0.5487
D_F	17.4800	32.3000	24.2956	26.0200	3.8651	-0.2215	-1.4887
D	10.8000	37.0000	20.9529	19.7600	5.3571	0.6041	-0.2614
Energy asset: gasoline							
Set	<i>Minimum</i>	<i>Maximum</i>	Mean	Median	Stan. dev.	Skewness	Kurtosis
D_L	0.2907	0.7639	0.5339	0.5319	0.0925	-0.1089	-0.3639
D_V	0.2930	1.1093	0.7330	0.7514	0.1775	-0.4763	-0.5422
D_F	0.4622	1.0120	0.6860	0.6679	0.1527	0.5196	-0.7952
D	0.2907	1.1093	0.6028	0.5750	0.1564	0.6909	0.0246
Energy asset: heating oil							
Set	<i>Minimum</i>	<i>Maximum</i>	Mean	Median	Stan. dev.	Skewness	Kurtosis
D_L	0.3008	0.7990	0.5121	0.5010	0.0964	0.3598	0.2283
D_V	0.2842	1.1052	0.7046	0.7389	0.2019	-0.1637	-0.8480
D_F	0.4691	0.8512	0.6626	0.6881	0.1020	-0.2607	-1.3261
D	0.2842	1.1052	0.5791	0.5350	0.1602	0.8369	0.3973

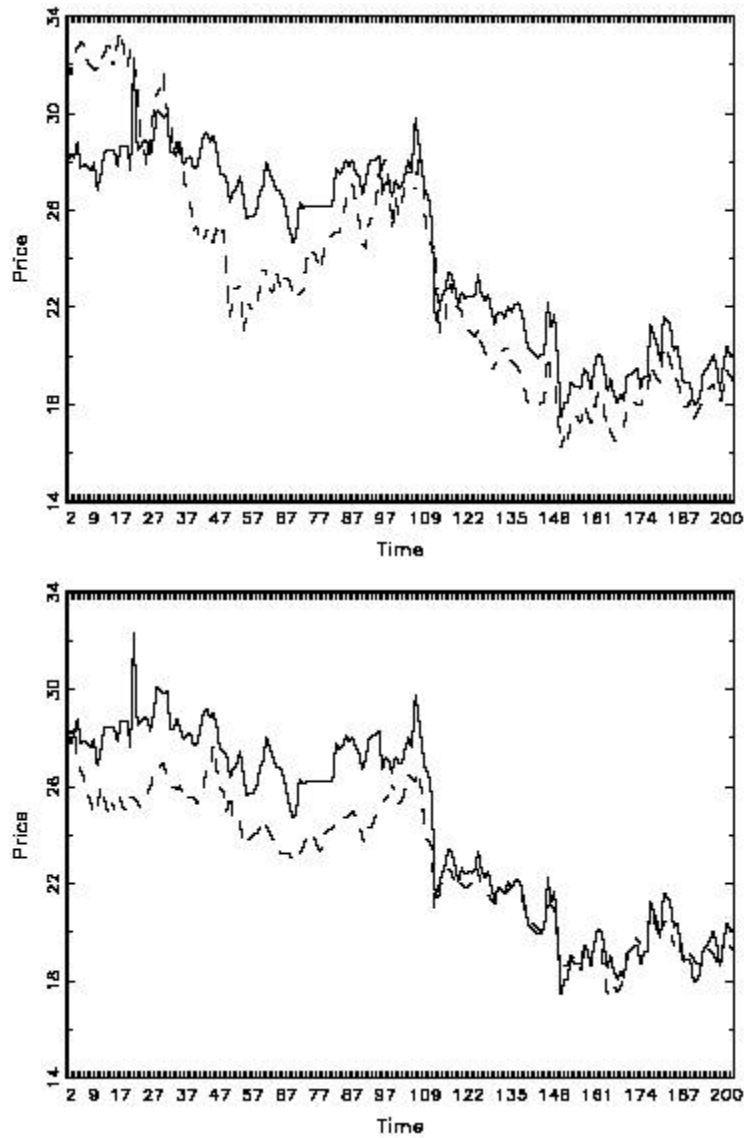
Table 2.

Random variables	$\delta_{i,j}$	Bilateral t -test	Check on the $\delta_{1,2}$ -dep.	$\rho_{i,j}$
$X_{CO}(t), X_G(t)$	0.35407	R	P	0.40761
$X_{CO}(t), X_{HO}(t)$	0.39259	R	P	0.49802
$X_G(t), X_{HO}(t)$	0.48642	R	P	0.58000

Table 3.

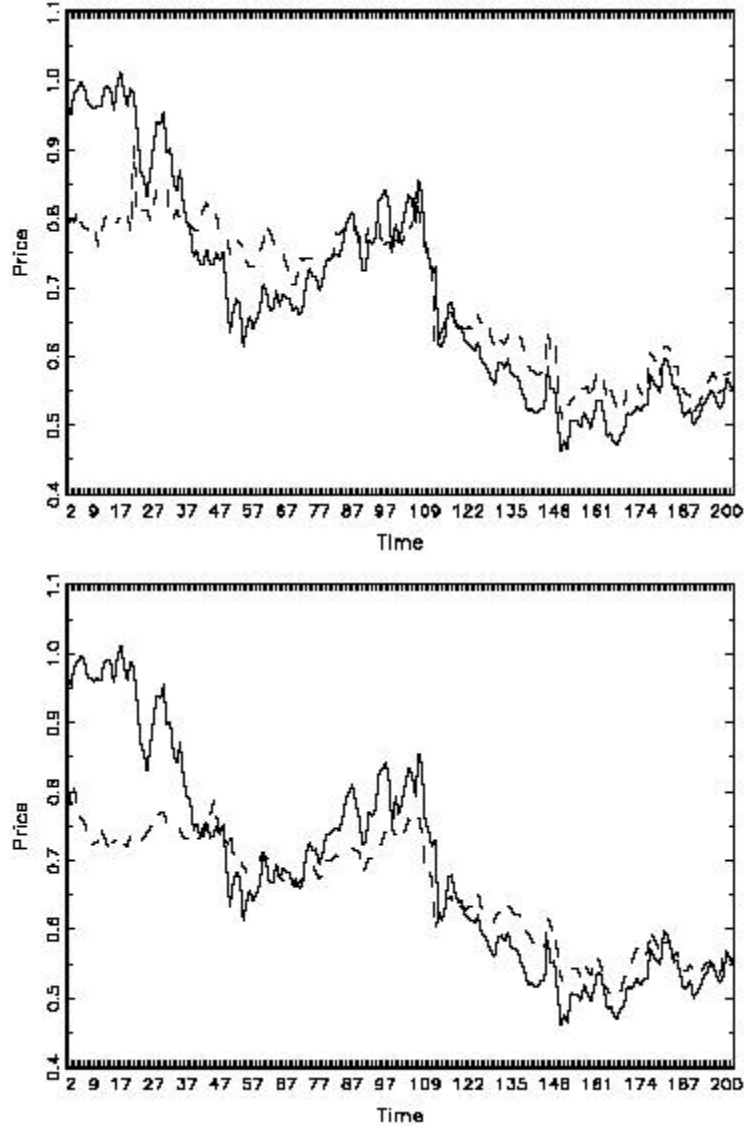
Random variables	Polynomial approximation
$X_{CO}(t), X_G(t)$	$\hat{X}_{CO}(t) = 1.68731 + 31.36198 X_G(t)$
$X_{CO}(t), X_{HO}(t)$	$\hat{X}_{CO}(t) = -9.38380 + 97.98516 X_{HO}(t) - 116.50908 X_{HO}^2(t) + 63.03937 X_{HO}^3(t)$
$X_G(t), X_{CO}(t)$	$\hat{X}_G(t) = 0.03803 + 0.02687 X_{CO}(t)$
$X_G(t), X_{HO}(t)$	$\hat{X}_G(t) = 0.12943 + 0.79407 X_{HO}(t)$
$X_{HO}(t), X_{CO}(t)$	$\hat{X}_{HO}(t) = -3.80625 + 0.87743 X_{CO}(t) - 0.06879 X_{CO}^2(t) + 0.00240 X_{CO}^3(t) - 0.00003 X_{CO}^4(t)$
$X_{HO}(t), X_G(t)$	$\hat{X}_{HO}(t) = -0.19987 + 2.37713 X_G(t) - 2.98411 X_G^2(t) + 1.89758 X_G^3(t)$

Figure 1.



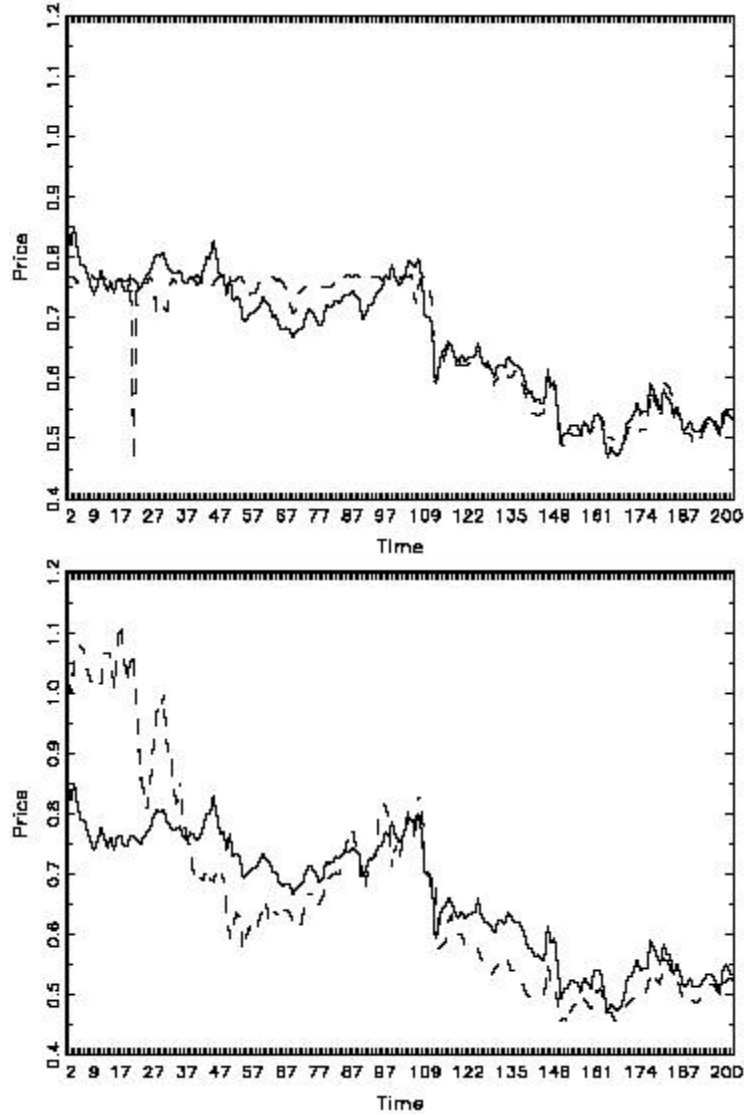
In both the graphs, the continuous uneven line represents the behaviour of $X_{CO}(t)$ in the out-of-sample data subset D_F . In the graph on the right, the dotted uneven line represents the behaviour in D_F of the polynomial approximation of $X_{CO}(t)$ in terms of $X_G(t)$, i.e. $\hat{X}_{CO}(t) = 1.68731 + 31.36198X_G(t)$. In the graph on the left, the dotted uneven line represents the behaviour in D_F of the polynomial approximation of $X_{CO}(t)$ in terms of $X_{HO}(t)$, i.e. $\hat{X}_{CO}(t) = -9.38380 + 97.98516X_{HO}(t) - 116.50908X_{HO}^2(t) + 63.03937X_{HO}^3(t)$.

Figure 2.



In both the graphs, the continuous uneven line represents the behaviour of $X_G(t)$ in the out-of-sample data subset D_F . In the graph on the right, the dotted uneven line represents the behaviour in D_F of the polynomial approximation of $X_G(t)$ in terms of $X_{CO}(t)$, i.e. $\hat{X}_G(t) = 0.03803 + 0.02687X_{CO}(t)$. In the graph on the left, the dotted uneven line represents the behaviour in D_F of the polynomial approximation of $X_G(t)$ in terms of $X_{HO}(t)$, i.e. $\hat{X}_G(t) = 0.12943 + 0.79407X_{HO}(t)$.

Figure 3.



In both the graphs, the continuous uneven line represents the behaviour of $X_{HO}(t)$ in the out-of-sample data subset D_F . In the graph on the right, the dotted uneven line represents the behaviour in D_F of the polynomial approximation of $X_{HO}(t)$ in terms of $X_{CO}(t)$, i.e. $\hat{X}_{HO}(t) = -3.80625 + 0.87743X_{CO}(t) - 0.06879X_{CO}^2(t) + 0.00240X_{CO}^3(t) - 0.00003X_{CO}^4(t)$. In the graph on the left, the dotted uneven line represents the behaviour in D_F of the polynomial approximation of $X_{HO}(t)$ in terms of $X_G(t)$, i.e. $\hat{X}_{HO}(t) = -0.19987 + 2.37713X_G(t) - 2.98411X_G^2(t) + 1.89758X_G^3(t)$.