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Itô's Calculus and the Derivation of the Black–Scholes Option-Pricing Model

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Chapter 27 Itô's Calculus: Derivation of the Black–Scholes Option Pricing Model* <P1>

The purpose of this chapter is to develop certain relatively recent mathematical discoveries known generally as **stochastic calculus** (or more specifically as **Itô's Calculus**) and to illustrate their application in the pricing of options. The topic is motivated by a desire to provide an *intuitive* understanding of certain probabilistic methods that have found significant use in financial economics. A rigorous presentation of the same ideas is presented briefly in Malliaris and Brock (1982) and more recently, Chalamandaris and Malliaris (2011).

Itô's Calculus was prompted by purely mathematical questions originating in Wiener's work in 1923 on stochastic integrals and was developed by the Japanese probabilist Kiyosi Itô during 1944–1951. Two decades later economists such as Merton (1973) and Black and Scholes (1973) started using Itô's stochastic differential equation to describe the behavior of asset prices. Because stochastic calculus is now used regularly by financial economists, some attention must be given to its mathematical meaning, its appropriateness in economic modeling, and its applications in economic modeling, and to finance.

27.1 **THE ITÔ PROCESS AND FINANCIAL MODELING**<SI>

Stochastic calculus is the mathematical of random change in continuous time, unlike ordinary calculus, which deals with deterministic change. A key notion in stochastic calculus is the equation:

* This chapter was written by Professor A. G. Malliaris, Loyola University of Chicago.

$$dS(t, w) = \mu[t, S(t, w)]dt + \sigma[t, S(t, w)]dZ(t, w), \quad (27.1)$$

which is analogous to the ordinary differential equation $dS(t)/dt = \mu[t, S(t)]$. This section defines intuitively the Itô equation in Equation (27.1) and discusses its appropriateness to financial modeling.

A stochastic process is an Itô process if the random variable $dS(t, w)$ can be represented by Equation (27.1). The first term, $\mu[t, S(t, w)]dt$, is the expected change in $S(t, w)$ at time t . The second term, $\sigma[t, S(t, w)]dZ(t, w)$, reflects the uncertain term.

The Itô equation is a random equation. The domain of the equation is $[0, \infty) \times \Omega$, with the first argument t denoting time and taking values continuously in the interval $[0, \infty)$, and the second argument w denoting a random element taking values from a random set Ω . The range of the equation is the real numbers or real vectors. For simplicity, only the real numbers, denoted by R , are considered as the range of Equation (27.1). Because time takes values continuously in $[0, \infty)$, the Itô equation is a **continuous-time random equation**.

Although at first a real random variables $S(t, w): [0, \infty) \times \Omega \rightarrow R$ is used — that is, a function having as domain $[0, \infty) \times \Omega$ and as range real numbers — Equation (27.1) expresses not the values of $S(t, w)$, but its infinitesimal differences $dS(t, w)$ as a function of two terms. For example, in finance $S(t, w)$ denotes the price of a stock at time t affected by the state of the economy described by the random element w ; and (27.1) expresses the small changes in the stock price, $dS(t, w)$, at time t affected by

the random element w . The mathematical meaning of this small difference can be explained as follows: $dS(t, w)$ may be viewed as the limit of large finite differences $\Delta S(t, w)$ as Δt approaches zero. Note that $\Delta S(t, w) = S(t + \Delta t, w) - S(t, w)$, where Δt denotes the difference in the change in time. Thus, the Itô equation, expresses random changes in the values of a variable taking place continuously in time.

Moreover, these random changes are given as the sum of two terms. The first term, $\mu[t, S(t, w)]$, is called the **drift component** of the Itô equation, and in finance it is used to compute the instantaneous expected value of the change in the random variable $S(t, w)$. Observe that $\mu[t, S(t, w)]$, as a function used in the computation of a statistical mean, is affected by both time and randomness. If at a given time t the expected change in $dS(t, w)$, expressed as $E[dS(t, w)]$, is desired, this can be answered by computing $E\{\mu[t, S(t, w)] dt\}$.

The second term $\sigma[t, S(t, w)] dZ(t, w)$ is itself the product of two factors. Each factor is important and needs special attention. The first factor, $\sigma[t, S(t, w)]$, is used in the calculation of the instantaneous standard derivation of the change in the random variable $S(t, w)$; it is a function of both time t and the range of values taken by $S(t, w)$. When $\sigma[t, S(t, w)]$ is squared to compute the instantaneous variance, it is usually called the **diffusion coefficient**; it measures the variability of $dS(t, w)$ at a given instance in time.

The second factor, $dZ(t, w)$, is called **white noise**; it models financial uncertainty in continuous time. Actually, $dZ(t, w)$ denotes an infinitesimal change in the

Wiener process, $Z(t, w): [0, \infty) \times \Omega \rightarrow R$, a process with increment that are statistically independent and normally distributed with mean zero and variance equal to the increment in time. In other word, for every pair of disjoint time intervals $[t_1, t_2], [t_3, t_4]$ with, say, $t_1 < t_2 \leq t_3 < t_4$, the increments $Z(t_4, w) - Z(t_3, w)$ and $Z(t_2, w) - Z(t_1, w)$ are independent and normally distributed random variables with means:

$$E[Z(t_3, w) - Z(t_4, w)] = E[Z(t_2, w) - Z(t_1, w)] = 0,$$

and respective variances:

$$\begin{aligned} \text{Var}[Z(t_4, w) - Z(t_3, w)] &= t_4 - t_3 \\ \text{Var}[Z(t_2, w) - Z(t_1, w)] &= t_2 - t_1 \end{aligned}$$

By convention it is assumed that at time $t = 0$, the Wiener process is zero — that is, $Z(0, w) = 0$.

The two factors have been described separately; an intuitive explanation of the product of $\sigma[t, S(t, w)]$ and $dZ(t, w)$ is now presented. Because $\sigma[t, S(t, w)]$ measures the instantaneous standard derivation or volatility of $dS(t, w)$ and because $dZ(t, w)$ is an infinitesimal increment (which is, by definition, purely random with mean zero and variance dt), the expression of $\sigma[t, S(t, w)]dZ(t, w)$ is the product of two independent random variables, with

$$E\{\sigma[t, S(t, w)]dZ(t, w)\} = 0 \tag{27.2}$$

$$\begin{aligned} \text{Var}\{\sigma[t, S(t, w)]dZ(t, w)\} &= E\{\sigma[t, S(t, w)]dZ(t, w)\}^2 \\ &= E\{\sigma^2[t, S(t, w)]dt\} \end{aligned} \tag{27.3}$$

Therefore, the product $\sigma[t, S(t, w)]dZ(t, w)$ is a random variable with mean and

variances given by Equations (27.2) and (27.3), which, for a given time t and state of nature w , yields a real number. This number may be either positive or negative depending on the value of $dZ(t, w)$ since $\sigma[t, S(t, w)]$ represents a measure of standard derivation and is always positive. Furthermore, the magnitude of the product $\sigma[t, S(t, w)]dZ(t, w)$ depends on the magnitude of each of the two terms. Indeed, the methodological foundation of the Itô model is that the uncertainty magnitude, $dZ(t, w)$ is multiplied by $\sigma[t, S(t, w)]$ to produce the **total contribution of uncertainty**. Therefore, $\sigma[t, S(t, w)]dZ(t, w)$ describes the total fluctuation produced by volatility as this volatility is aggrandized or reduced by pure randomness.

This analysis explains why uncertainty given by $dZ(t, w)$ is being modeled as a multiplicative factor in the product $\sigma[t, S(t, w)]dZ(t, w)$: to repeat once again, the multiplicative modeling of uncertainty allows the generation of values for $dS(t, w)$ that are above or below the instantaneous expected value, depending on whether uncertainty is positive or negative, respectively. In other words, the product $\sigma[t, S(t, w)]dZ(t, w)$ can be viewed as the total contribution of uncertainty to $dS(t, w)$, with such uncertainty being the product of two factors: the instantaneous standard derivation of changes and the purely random white noise.

The conceptual components of the Itô process can now be collected to offer a more general interpretation of its meaning in finance. If at a given time t the possible future change in the price of an asset during the next trading interval is being evaluated, this change can be decomposed into two nonoverlapping components: the *expected* change and the *unexpected* change. The expected change, $E[dS(t, w)]$, is described

by $E\{\sigma[t, S(t, w)] dt\}$ and the unexpected change is given by $E\{\sigma[t, S(t, w)] dZ(t, w)\}$. As already noted, this unexpected change depends on the asset's volatility and pure randomness, and because uncertainty cannot be anticipated, the unexpected change is zero. This holds because $\sigma[t, S(t, w)]$ and $dZ(t, w)$ are independent random variables; and therefore, from Equation (27.2):

$$E\{\sigma[t, S(t, w)] dZ(t, w)\} = E\{\sigma[t, S(t, w)]\} E[dZ(t, w)] = 0.$$

Because by definition $E[dZ(t, w)] = 0$. Thus, Equation (27.1), which was developed by mathematicians, captures the spirit of financial modeling admirably because it is an equation that involves three important concepts in finance: *mean, standard derivation, and randomness*.

However, Equation (27.1) captures the spirit of finance not only because it involves means, standard derivations, and randomness but, more important, because it also expresses key methodological elements of modern financial theory. In an elegant paper, Merton (1982) identified several foundational notions of the appropriateness of the Itô equation in modern finance.

- (1) Itô's equation allows uncertainty not to disappear even as trading intervals become extremely short. In the real world of financial markets, uncertainty evolves continuously, and $dZ(t, w)$ captures this uncertainty because the value of $dZ(\Delta t, w)$ is not zero even as Δt becomes small. On the other hand, as trading intervals increase uncertainty increase because by definition

$$\text{Var}[Z(t + \Delta t, w) - Z(t, w)] = \Delta t.$$

- (2) Itô's equation incorporates uncertainty for all times. This means that uncertainty is present at all trading periods.

- (3) The rate of change described by $\mu[t, S(t, w)] dt$ is finite, and uncertainty does not cause the term $\mu[t, S(t, w)] dZ(t, w)$ to become unbounded. These notions of the Itô equation are consistent with real-world observations of finite means, finite variances, and uncertainty (which evolve nicely by obtaining continuously finite values).
- (4) Finally, all that counts in an Itô equation is the present time t . In other words, the past and future are independent. This notion expresses mathematically the concept of economic markets efficiency. That is, knowledge of past price behavior does not allow for above-average returns in the future.

Sample Problems 27.1 and 27.2 provide illustrations of the concepts discussed.

Sample Problem 27.1

Consider the following example of an Itô process describing the price of given stock.

$$dS(t, w) = 0.001 S(t, w) dt + 0.025 S(t, w) dZ(t, w).$$

If both sides are divided by $S(t, w)$ we get:

$$\frac{dS(t, w)}{S(t, w)} = 0.001 dt + 0.025 dZ(t, w), \quad (27.4)$$

which gives the proportional change in the price of the stock. Assume that dt equals one trading period, such as one day. Using appropriate coefficients in Equation (27.4), dt could be denoted as one second. In Equation (27.4), the expected daily proportional change is given by

$$E \left[\frac{dS(t, w)}{S(t, w)} \right] = E [0.001 dt + 0.025 dZ(t, w)] = 0.001, \quad (27.5)$$

which means that at any given trading day, the price of the stock is expected to change by 0.1%. The standard derivation of the proportional change is $\sigma = 0.025$

or 2.5 %.

What actually occurs at a given trading period depends on the evolution of the uncertainty as modeled by the normally distributed random variable $dZ(t, w)$, with mean 0 and variance 1. From the properties of the normal distribution, it can be deduced that although Equation (27.5) says that the expected daily proportional change is 0.001 and there is a 68.26% probability that the daily proportional change will be between $0.001 \pm (1) (0.025)$, or a 95.46% probability that it will be between $0.001 \pm (2) (0.025)$, or a 99.74% probability that it will be between $0.001 \pm (3) (0.025)$.

Sample Problem 27.2

This second example shows how the Itô equation combines the notion of a trading interval dt , the expected change in the price of the stock μ , the volatility of the stock σ , and pure uncertainty $dZ(t, w)$ to describe changes in the price of an asset. The same example will be used later to show the behavior of the price of the stock $S(t, w)$. Equation (27.5) illustrates only approximately the infinitesimal proportional change in the price of the stock and in order to obtain the solution of the stochastic differential equation in (27.5), we need the following useful lemma.

27.2 ITÔ LEMMA<SI>

The preceding two sections dealt with the Itô process, both its intuitive mathematical meaning and its financial interpretation. This section presents briefly Itô's lemma of stochastic differentiation. By formally integrating Equation (27.1):

$$S(t, w) = S(0, w) + \int_0^t \mu(u, w) du + \int_0^t \sigma(u, w) dZ(u, w) , \quad (27.6)$$

this last equation describes Itô's process in terms of the original random variable

$S(t, w)$ rather than infinitesimal differences (differential form), as in Equation (27.1). Mathematically, Equations (27.1) and (27.6) are equivalent and (27.1) obtains clear meaning through integration, which is not developed here because of its complexity. Nevertheless, note that it is the second integral, $\int_0^t \mu(u, w) dZ(u, w)$, that presents the difficulties. The problem arises because uncertainty given by $dZ(u, w)$ in its limit does not have a precise meaning, and therefore the second integral cannot be treated like an ordinary Riemann integral. Itô's great accomplishment was to define the second integral as a random variable, that is, the limit in probability of a certain sequence of integrals of step functions multiplied by the uncertainty $dZ(u, w)$.

Suppose that a stochastic process is given by Equation (27.6) and that a new process $Y(t, w)$ is formed by letting $Y(t, w) = u[t, S(t, w)]$. Because stochastic calculus studies random changes in continuous time, the question arises: what is $dY(t, w)$? This question is important for both mathematical analysis and finance. The answer is given in Itô's lemma.

Itô's Lemma. *Consider the nonrandom continuous function $u(t, S): [0, \infty) \times R \rightarrow R$ and suppose that it has continuous partial derivatives u_t, u_S , and u_{SS} .*

Let

$$Y(t, w) = u[t, S(t, w)] \quad (27.7)$$

with

$$dS(t, w) = \mu[t, S(t, w)] dt + \sigma[t, S(t, w)] dZ(t, w). \quad (27.8)$$

Then the process $Y(t, w)$ has a differential given by

$$\begin{aligned}
dY(t, w) = & \{u_t[t, S(t, w)] + u_s[t, S(t, w)]\mu[t, S(t, w)] \\
& + \frac{1}{2}u_{ss}[t, S(t, w)]\sigma^2[t, S(t, w)]\} dt \\
& + u_s[t, S(t, w)]\sigma[t, S(t, w)] dZ(t, w)
\end{aligned} \tag{27.9}$$

The proof is presented in Gikhman and Skorokhod (1969, pp. 387–389), and extensions of this lemma may be found in Arnold (1974, pp. 90–99). Here the analysis is limited to three remarks.

- (1) Itô's lemma is a useful result because it allows the computation of stochastic differentials of arbitrary functions having as an argument a stochastic process that itself is assumed to possess a stochastic differential. In this respect, Itô's formula is as useful as the chain rule of ordinary calculus.
- (2) Given an Itô stochastic process $S(t, w)$ with respect to a given Wiener process $Z(t, w)$, and letting $Y(t, w) = u[t, S(t, w)]$ be a new process, Itô's formula gives the stochastic differential of $Y(t, w)$, where $dY(t, w)$ is given with respect to the same Wiener process — that is, both processes have the same source of uncertainty.
- (3) An inspection of the proof of Itô's lemma reveals that it consists of an application of Taylor's theorem of advanced calculus and several probabilistic arguments to establish the convergence of certain quantities to appropriate integrals. Therefore, Itô's formula may be obtained by applying Taylor's theorem instead of remembering the specific result in Equation (27.6). More specifically, the differential of $Y(t, w) = u[t, S(t, w)]$, where $S(t, w)$ is a stochastic process with differential given by Equation (27.1), may be computed by using Taylor's theorem and the following multiplication rules:

$$dt \times dt = 0 \quad dZ \times dZ = dt \quad dt \times dZ = 0 \tag{27.10}$$

as

$$dY = u_t dt + u_s dS + \frac{1}{2} (dS)^2 = u_t dt + u_s (\mu dt + \sigma dZ) + \frac{1}{2} u_{ss} (\mu dt + \sigma dZ)^2$$

By carrying out these multiplications and using the rules in Equation (27.10), Equation (27.9) is obtained.

27.3 STOCHASTIC DIFFERENTIAL-EQUATION APPROACH TO STOCK-PRICE BEHAVIOR<S1>

This section demonstrates how a stochastic equation can be used to describe the price behavior of an asset. We begin with Sample Problem 27.3

Sample Problem 27.3

As a special case of Equation (27.1) consider:

$$dS(t, w) = \mu S(t, w) dt + \sigma S(t, w) dZ(t, w)$$

in which μ and σ are constants as in Sample Problem 27.1. Assume that Equation (27.11) describes the price behavior of a certain stock with $S(0, w)$ given. The solution $S(t, w)$ of Equation (27.11) is given by

$$S(t, w) = S(0, w) \exp \left[\left(\mu - \frac{1}{2} \sigma^2 \right) t + \sigma Z(t, w) \right]. \quad (27.12)$$

To show that the behavior of Equation (27.12) is the solution of Equation (27.11), use Itô's lemma as follows. First, start with Equation (27.12), which corresponds to the function $Y(t, w)$. In this case, Equation (27.12) is a function of t and $Z(t, w)$, and instead of Equation (27.8):

$$dZ(t, w) = 0 \cdot dt + 1 \cdot dZ(t, w).$$

Next, compute the first- and second-order partials denoted by S_t , S_Z , and S_{ZZ} :

$$\begin{aligned}
S_t(t, w) &= \left(\mu - \frac{\sigma^2}{2} \right) S(t, w) \\
S_z(t, w) &= \sigma S(t, w) \\
S_{zz}(t, w) &= \sigma^2 S(t, w)
\end{aligned}$$

Collect these results and use Equation (27.9) to conclude that Equation (27.11) holds:

$$\begin{aligned}
dS(t, w) &= \left[\left(\mu - \frac{\sigma^2}{2} \right) S(t, w) + \sigma S(t, w) \cdot 0 + \frac{\sigma^2}{2} S(t, w) \cdot 1 \right] dt \\
&\quad + \sigma S(t, w) \cdot 1 \cdot dZ(t, w) \\
&= \mu S(t, w) dt + \sigma S(t, w) dZ(t, w)
\end{aligned}$$

This result is not only mathematically interesting; in finance it means that, assuming stock price are given by an Itô process as in Equation (27.11), then Equation (27.12) holds as well. To see if Equation (27.12) accurately describes stock prices in the real world, rewrite it as

$$\ln \frac{S(t, w)}{S(0, w)} = \left(\mu - \frac{\sigma^2}{2} \right) t + \sigma Z(t, w), \quad (27.13)$$

which is a random variable normally distributed with mean $(\mu - \sigma^2/2)t$ and variance $\sigma^2 t$. This means that if the stock price follows an Itô process, it has a log-normal probability distribution. Such a distribution for stock price is reasonable because it is consistent with reality, where negative stock prices are not possible; the worst that can happen is that the stock price reaches zero. For a detailed analysis of the properties of a log-normal random distribution, see Cox and Rubinstein (1985, pp. 201–204); for a list of bibliographical references on the empirical distribution of stock-price changes, see Cox and Rubinstein (1985, pp. 485–488). Sample Problem 27.4 provides further illustration.

Sample Problem 27.4

The analysis of this problem applies the results of the last two sections to the example in Sample problem 27.1. Suppose that instead of Equation (27.11) Equation (27.14) is given. Then Equation (27.12) describes the solution of Equation (27.14):

$$S(t, w) = S(0, w) \exp \left[\left(0.001 - \frac{0.025^2}{2} \right) t + 0.025 Z(t, w) \right]. \quad (27.14)$$

From Equation (27.14), the exact evolution of the price of the stock as influenced by a mean, a variance, time, and uncertainty is obtained. Using Equation (27.14) for $t = 1$ to compute:

$$E \left[\ln \frac{S(t, w)}{S(0, w)} \right] = \left(\mu - \frac{\sigma^2}{2} \right) t = \left[0.001 - \frac{(0.025)^2}{2} \right] (0.1) = 0.0006875, \quad (27.15)$$

$$\text{Var} \left[\frac{S(t, w)}{S(0, w)} \right] = \sigma^2 t = (0.025)^2 (0.1) = 0.000625, \quad (27.16)$$

which means that for a given trading day, the price of the stock is expected to experience a continuous change of 0.068% with a standard derivation of 0.025%. Computing Equations (27.15) and (27.16) for any t can be done easily; the same computations cannot be performed readily in Equation (27.4).

Sample Problem 27.5

Here we give an intuitive description of Equation (27.1) with reference to Table 27-1 using Excel. In Table 27-1, we collect from Yahoo.com 40 recent daily closing Google Inc. prices. These prices are also illustrated graphically in Figure 27-1.

Table 27-1 Daily Price Data for Google Inc.

Day	Price	r_i	dZ
1	610.21		
2	630.08	0.03256	2.91992

3	625.26	-0.00765	-0.66558
4	624.22	-0.00166	-0.13180
5	624.15	-0.00011	0.00651
6	628.15	0.00641	0.58794
7	624.5	-0.00581	-0.50160
8	616.44	-0.01291	-1.13427
9	616.5	0.00010	0.02519
10	618.38	0.00305	0.28841
11	614.29	-0.00661	-0.57323
12	610.98	-0.00539	-0.46394
13	610.15	-0.00136	-0.10462
14	612	0.00303	0.28686
15	611.04	-0.00157	-0.12336
16	600.36	-0.01748	-1.54193
17	600.99	0.00105	0.11008
18	616.79	0.02629	2.36063
19	616.5	-0.00047	-0.02541
20	619.91	0.00553	0.50970
21	611.08	-0.01424	-1.25354
22	611.83	0.00123	0.12594
23	626.77	0.02442	2.19377
24	631.75	0.00795	0.72496
25	639.63	0.01247	1.12868
26	624.18	-0.02415	-2.13721
27	616.69	-0.01200	-1.05344
28	616.87	0.00029	0.04253
29	616.01	-0.00139	-0.10780
30	614.21	-0.00292	-0.24403
31	616.44	0.00363	0.34024
32	613.5	-0.00477	-0.40874
33	609.07	-0.00722	-0.62733
34	602.12	-0.01141	-1.00093
35	604.35	0.00370	0.34674
36	593.97	-0.01718	-1.51493
37	598.86	0.00823	0.75057
38	601	0.00357	0.33513
39	598.92	-0.00346	-0.29208

40	602.38	0.00578	0.53162
41	604.23	0.00307	0.29035

μ_{ri} -0.000185

σ_{ri} 0.011215

μ_{dz} 0

σ_{dz} 1

Figure 27.1 Daily Google Inc. Closing Prices

For modeling purposes, the Black–Scholes equation requires a mathematical expression for prices, such as shown in Table 27-1 and Figure 27-1. How can we check to see if the prices in these figures follow an Itô process? In column 3 of Table 27-1, we compute the daily Google returns and calculate the average historical return of the 40 daily returns. This daily average is 0.000185, which annualized becomes $0.000185 \times 250 = 0.04625$. The volatility of these 40 returns is computed as the annualized standard deviation of daily returns and is computed as $0.011215 \times \sqrt{250} = 0.177324$. Both these calculations assume 250 trading days per year. To complete checking the prices in Table 27-1 satisfy Equation (27.8), we solve for dZ . Since these prices are daily, we use the daily constant average return and constant daily volatility to obtain daily dZ s in column 4 of Table 27-1 using

$$dZ_i = \frac{r_i - \mu_{ri}}{\sigma_{ri}}.$$

We also compute the mean and variance of the dZ in the last column. Recall that earlier we postulated that $E(dZ) = 0$ and $\text{Var}(dZ) = 1$ is assumed. Indeed, the mean and variance of the dZ values in Table 27-1 are as postulated.

Note that if we were to graph the distribution of Google stock prices, it would follow a log-normal distribution while its returns would follow a normal distribution. Furthermore, the daily dZ s, as an approximation to the continuous random walk dZ of Equation (27.8), also are normally distributed with mean 0 and variance 1. Figures 27-2 and 27-3 are frequency approximations to a normal distribution and illustrate the distributions of returns and distributions of dZ s of Table 27-1.

Figure 27-2 Frequency Distribution of Returns

Figure 27-3 Frequency Distribution of Errors

27.4 *THE PRICING OF AN OPTION<SI>*

An **option** is a contract giving the right to buy or sell an asset within a specified period of time subject to certain conditions. The simplest kind of option is the **European call option**, which is a contract to buy a share of a certain stock at a given date for a specified price. The date the option expires is called the **expiration date (maturity date)** and the price that is paid for the stock when the option is exercised is called the **exercise price (striking price)**.

In terms of economic analysis, several propositions about call-option pricing seem clear. The value of an option increases as the price of the stock increases. If the stock price is much greater than the exercise price, it is almost certain that the option will be exercised; and, analogously, if the price of the stock is much less than the exercise price, the value of the option will be near zero and the option will expire without being exercised. If the expiration date is very far in the future, the value of the option will be approximately equal to the price of the stock. If the expiration date is very near, the value of the option will be approximately equal to the stock price minus the exercise price, or zero if the stock price is less than the exercise price. In general, the

value of the option is more volatile than the price of the stock, and the relative volatility of the option depends on both the stock price and maturity.

The first rigorous formulation and solution of the problem of option pricing was achieved by Black and Scholes (1973) and Merton (1973). Consider a stock option denoted by C whose price at time t can be written:

$$C(t, w) = C[t, S(t, w)] \quad (27.17)$$

in which C is a twice continuously differentiable function. Here $S(t, w)$ is the price of some stock upon which the option is written. The price of this stock is assumed to follow Itô's stochastic differential equation:

$$dS(t, w) = \mu[t, S(t, w)]dt + \sigma[t, S(t, w)]dZ(t, w). \quad (27.18)$$

Assume, as a simplifying case, that $\mu[t, S(t, w)] = \mu S(t, w)$ and $\sigma[t, S(t, w)] = \sigma S(t, w)$. For notational convenience, w is suppressed from the various expressions, and sometimes t as well. Therefore, Equation (27.18) becomes

$$dS(t) = \mu S(t)dt + \sigma S(t)dZ(t). \quad (27.19)$$

Consider an investor who builds up a portfolio of stocks, options on the stocks, and a riskless asset (for example, government bonds) yielding a riskless rate r . The nominal of the portfolio, denoted by $P(t)$, is

$$P(t) = N_1(t)S(t) + N_2C(t) + Q(t) \quad ,$$

(27.20)

where

N_1 = the number of shares of the stock;

N_2 = the number of call options; and

Q = the value of dollars invested in riskless bonds.

Assume that the stock pays no dividends or other distributions. By Itô's lemma, the differential of the call price using Equations (27.17) and (27.19) is

$$\begin{aligned} dC &= C_t dt + C_S dS + \frac{1}{2} C_{SS} dS^2 \\ &= \left(C_t + C_S \mu S + \frac{1}{2} C_{SS} \sigma^2 S^2 \right) dt + C_S \sigma S dZ \\ &= \mu_C C dt + \sigma_C C dZ \end{aligned} \quad (27.21)$$

Observe that in Equation (27.21):

$$\mu_C C = C_t + C_S \mu S + \frac{1}{2} C_{SS} \sigma^2 S^2, \quad (27.22)$$

$$\sigma_C C = C_S \sigma S. \quad (27.23)$$

In other words, $\mu_C C$ is the expected change in the call-option price and $\sigma_C^2 C^2$ is the variance of such a change per unit of time. Itô's lemma simply indicates that if the call-option price is a function of a spot stock price that follows an Itô process, then the call-option price also follows an Itô process with mean and standard derivation parameters that are more complex than those of the stock price. The Itô process for a call is given by Equations (27.22) and (27.23).

The change in the normal value of the portfolio dP results from the change in the prices of the assets because at a point in time the equations of option and stock are given — that is, $dN_1 = dN_2 = 0$. More precisely:

$$\begin{aligned} dP &= N_1 (dS) + N_2 (dC) + dQ \\ &= (\mu dt + \sigma dZ) N_1 S + (\mu_C dt + \sigma_C dZ) N_2 C + rQ dt \end{aligned} \quad (27.24)$$

Let w_1 be the fraction of the invested in stock, w_2 be the fraction invested in options, and w_3 be the fraction of the invested in the riskless asset. As before, $w_1 + w_2 + w_3 = 1$, that is, all of the funds available are invested in some type of asset. Set $w_1 = N_1 S / P$, $w_2 = N_2 C / P$, $w_3 = Q / P = 1 - w_1 - w_2$. Then Equation (27.24) becomes

$$\frac{dP}{P} = (\mu dt + \sigma dZ)w_1 + (\mu_C dt + \sigma_C dZ)w_2 + (r dt)w_3 \quad .$$

(27.25)

At this point, the notion of **economic equilibrium** (also called **risk-neutral** or **preference-free pricing**) is introduced in the analysis. This notion plays an important role in modeling financing behavior, and its appropriate formulation is considered to be a major breakthrough in financial analysis.

More specifically, design the proportions w_1, w_2 so that the position is *riskless* for all $t \geq 0$ — that is, let w_1 and w_2 be such that

$$\text{Var}\left(\frac{dP}{P}\right) = \text{Var}(w_1 \sigma dZ + w_2 \sigma_C dZ) = 0. \quad (27.26)$$

In the last equation, Var_t denotes variance conditioned on $S(t)$, $C(t)$, and $Q(t)$.

In other words, choose $(w_1, w_2) = (\bar{w}_1, \bar{w}_2)$ so that

$$\bar{w}_1 \sigma + \bar{w}_2 \sigma_C = 0. \quad (27.27)$$

Then from Equation (27.25), because the portfolio is riskless it follows that the portfolio must be expected to earn the riskless rate of return, or:

$$E_t\left(\frac{dP}{P}\right) = [\mu \bar{w}_1 + \mu_C \bar{w}_2 + r(1 - \bar{w}_1 - \bar{w}_2)] dt = r(t) dt. \quad (27.28)$$

Equations (27.27) and (27.28) yield the Black–Scholes–Merton equations:

$$\frac{\bar{w}_1}{\bar{w}_2} = -\frac{\sigma_C}{\sigma}, \quad (27.29)$$

and

$$r = \mu \bar{w}_1 + \mu_C \bar{w}_2 - r \bar{w}_1 - r \bar{w}_2 + r, \quad (27.30)$$

which simplify to

$$\frac{\mu - r}{\sigma} = \frac{\mu_C - r}{\sigma_C} \quad .$$

(27.31)

Because of the law of one price, Equation (27.31) says that the net rate of return

per unit of risk must be the two assets and describes an appropriate concept of economic equilibrium in this problem. If this were not the case, there would exist an arbitrage opportunity until this equality held. Using Equation (27.31) and making the necessary substitutions from Equations (27.22) and (27.23), the partial differential equation of the pricing of an option is obtained:

$$\frac{1}{2}\sigma^2 S^2 C_{SS}(t, S) + rSC_S(t, S) - rC(t, S) + C_t(t, S) = 0. \quad (27.32)$$

The equation along with the boundary conditions for call options fully characterize the call price: $C(S, T) = \text{Max}(0, S - X)$, $S \geq 0, 0 \leq E \leq T$. The solution to the differential equation (27.32) given these boundary conditions is the Black–Scholes formula.

27.5 A REEXAMINATION OF OPTION PRICING<SI>

To illustrate the notion of economic equilibrium once again, consider the nominal value of a portfolio consisting of a stock and a call option on this stock and write:

$$P(t) = N_1(t)S(t) + N_2(t)C(t) \quad (27.33)$$

using the same notation as in the previous section. Equation (27.33) differs from Equation (27.20) because the term $Q(t)$ has been deleted. Now concentrating on the two assets of the portfolio — that is, the stock and the call option — and using Equations (27.33) and (27.21), the change in the value of the portfolio is given by

$$dP = N_1 dS + N_2 dC = N_1 dS + N_2 [(C_t + \frac{1}{2}C_{SS}\sigma^2 S^2)dt + C_S dS]. \quad (27.34)$$

Note that $dN_1 = dN_2 = 0$, since at any given point in time the equations of stock and option are given. For arbitrary quantities of stock and option, Equation (27.34) shows that the change in the nominal value of the portfolio dP is stochastic because

dS is a random variable. Suppose the quantities of stock and call option are chosen as that

$$\frac{N_1}{N_2} = -C_s. \quad (27.35)$$

Note that C_s in Equation (27.35) denotes a hedge ration and is called **delta**.

Then,

$N_1 dS + N_2 C_s dS = 0$, and inserting Equation (27.35) into Equation (27.34) yields:

$$\begin{aligned} dP &= -N_2 C_s dS + N_2 [(C_t + \frac{1}{2} C_{ss} \sigma^2 S^2) dt + C_s dS] \\ &= N_2 (C_t + \frac{1}{2} C_{ss} \sigma^2 S^2) dt \end{aligned} \quad (27.36)$$

Let $N_2 = 1$ in Equation (27.36) and observe that in equilibrium the rate of return of the riskless portfolio must be the same as the riskless rate $r(t)$. Therefore:

$$\frac{dP}{P} = r dt. \quad (27.37)$$

Equation (27.37) can be used to derive the partial differential equation for the value of the option. Making the necessary substitutions in Equation (27.36):

$$\frac{(C_t + \frac{1}{2} C_{ss} \sigma^2 S^2) dt}{-C_s S + C} = r dt,$$

which upon rearrangement gives Equation (27.32). Note that the option-pricing equation is a second-order linear partial differential equation of the parabolic type. The boundary conditions of Equation (27.32) are determined by the specification of the asset. For the case of an option that can be exercised only at the expiration date t^* with an exercise price X , the boundary conditions are

$$C(t, S = 0) = 0, \quad (27.38)$$

$$C(t = t^*, S) = \text{Max}(0, S - X). \quad (27.39)$$

Observe that Equation (27.38) says that the call-option price is zero if the stock price is zero at any date t ; Equation (27.39) says that the call-option price at the expiration

date $t = t^*$ must equal the maximum of either zero or the difference between the stock price and the exercise price.

The solution of the option-pricing equation for a call and a put option subject to the boundary conditions are given in Equations (27.40a) and (27.40b) for $T = t^* - t$, as

$$C(T, S, \sigma^2, X, r) = SN(d_1) - Xe^{-rT}N(d_2), \quad (27.40a)$$

$$P(T, S, \sigma^2, X, r) = Xe^{-rT}(-N(d_2)) - S(-N(d_1)), \quad (27.40b)$$

where N denotes the cumulative normal distribution, namely:

$$N(y) = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^y e^{-x^2/2} dx.$$

In Equation (27.40a), T is time to expiration (measured in years) and d_1 and d_2 are given by

$$d_1 = \frac{\ln(S/X) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}, \quad (27.41)$$

$$\begin{aligned} d_2 &= \frac{\ln(S/X) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \\ &= d_1 - \sigma\sqrt{T} \end{aligned} \quad (27.42)$$

It can be shown that

$$\frac{\partial C}{\partial T} > 0, \quad \frac{\partial C}{\partial S} > 0, \quad \frac{\partial C}{\partial \sigma^2} > 0, \quad \frac{\partial C}{\partial X} < 0, \quad \frac{\partial C}{\partial r} > 0. \quad (27.43)$$

These partial derivatives justify the intuitive behavior of the price of an option, as was indicated in the beginning of the previous section. More specifically, these partials show the following:

- (1) As the stock price rises, so does the option price.
- (2) As the variance rate of the underlying stock rises, so does the option price.
- (3) With a higher exercise price, the expected payoff decreases.
- (4) The value of the option increases as the interest rate rises.
- (5) With a longer time to maturity, the price of the option is greater.

Before giving an example, it is appropriate to sketch the solution of Equation (27.32) subject to the boundary conditions of Equations (27.38) and (27.39). Let t denote the current trading period that is prior to the expiration date t^* . At time t , two outcomes can be expected to occur at t^* . (1) $S(t^*) > X$ — that is the price of the stock at the time of the expiration of the call option is greater than exercise price, or (2) $S(t^*) \leq X$. Note that the first outcome occurs with probability $P_0 \equiv P[S(t^*) > X]$ and the second occurs with probability $1 - p$. Obviously, the only interesting possibility is the first case when $S(t^*) > X$, because this is when the price of the call has positive value. If $S(t^*) \leq X$, then $C(t^*) = 0$ from Equation (27.39). Again from Equation (27.39), if $S(t^*) > X$, then the price of the call option at expiration $C(t^*)$ can be computed from the expiration of Equation (27.39):

$$C(t^*) = E\left\{\text{Max}\left[0, S(t^*) - X\right]\right\} = E\left[S(t^*) - X\right]. \quad (27.44)$$

What is the price of a call option if the first outcome materializes at t instead of t^* ? This can be answered immediately by appropriate continuous discounting. Using Equation (27.44):

$$C(t) = e^{-r(t^*-t)} E\left[S(t^*) - X\right]. \quad (27.45)$$

Recall, however, that $C(t)$ in Equation (27.45) holds only with probability p .

Combine both possibilities to write:

$$\begin{aligned} C(t) &= p \cdot e^{-r(t^*-t)} E\left[S(t^*) - X\right] + (1 - p) \cdot 0 \\ &= p \cdot e^{-r(t^*-t)} E\left[S(t^*) \mid S(t^*) > X\right] - p \cdot e^{-r(t^*-t)} X \end{aligned} \quad (27.46)$$

Detailed calculation in Jarrow and Rudd (1983, pp. 92–94) shows that because the price of the underlying stock is distributed log normally, it follows that

$$p = P[S(t^*) > X] = N \left[\frac{\ln \frac{S(t)}{X} + (r + \frac{1}{2} \sigma^2)(t^* - t)}{\sigma \sqrt{t^* - t}} \right] ,$$

(27.47)

$$p \cdot E[S(t^*) | S^* > X] = S(t) e^{r(t^* - t)} N \left[\frac{\ln \frac{S(t)}{X} + (r + \frac{1}{2} \sigma^2)(t^* - t)}{\sigma \sqrt{t^* - t}} \right] .$$

(27.48)

Combining Equations (27.47) and (27.48) with $T = t^* - t$ into Equation (27.46) yields Equation (27.40a).

It is worth observing that two terms of Equations (27.40a) and (27.40b) have economic meaning. The first term, $SN(d_1)$, denotes the present value of receiving the stock provided that $S(t^*) > X$. The second term gives the present value of paying the striking price provided that $S(t^*) > X$. In the special case when there is no uncertainty and $\sigma = 0$, observe that

$$\begin{aligned} N(d_1) &= N(d_2) = N(\infty) = 1 ; \text{ and} \\ C &= S(t) - e^{-r(t^* - t)} X \end{aligned} \quad (27.49)$$

that is, a call is worth the difference between the current value of the stock and the discounted value of the striking price provided $S(t^*) > X$; otherwise the call price would be zero. When $\sigma \neq 0$ — that is, when uncertainty exists and the stock price is volatile — the two terms in Equation (27.49) are multiplied by $N(d_1)$ and $N(d_2)$, respectively, to adjust the call price for the prevailing uncertainties. These two probabilities can also be given an economic interpretation. As mentioned earlier, $N(d_1)$ is called delta; it is the partial derivative of the call price with respect to the

stock price; $N(d_2)$ gives the probabilities that the call option will be in the money, as Equation (27.47) shows.

Assuming that investors in the economy have risk-neutral preferences, it will be possible to derive the Black–Scholes formula without using stochastic differential equations. Garven (1986) has shown that to derive Equation (27.40a) knowledge is required of normal and log-normal distributions and basic calculus, as presented in Appendix 27A.

Sample Problem 27.6

Equations (27.40a) and (27.40b) indicate that the Black–Scholes option-pricing model is a function of only five variables: T , the time to expiration, S , the stock price, σ^2 , the instantaneous variance rate on the stock price, X , the exercise price, and r , the riskless interest rate. Of these five variables, only the variance rate must be estimated; the other four variables are directly observable. A simple example is presented to illustrate the use of Equation (27.40). The values of the observable variables are taken from *Yahoo! Finance*.

On Wednesday, March 16, 2011, at 3:58 pm EDT, IBM Corp. had a stock price of \$152.93. The July 11 call-option with a strike price of 150.00 was priced at \$10.50. We estimate the riskless rate at 0.25% from the US Treasury bill rate. The only missing piece of information is the instantaneous variance of the stock price.

Several different techniques have been suggested for estimating the instantaneous variance. In this regard, the work of Latané and Rendleman (1976) must be mentioned; they derived standard derivations of continuous price-relative returns that are implied in actual call-option prices on the assumption that investors behave as if they price options according to the Black–Scholes model. In the example the implicit variance is

calculated by using a numerical search to approximate the standard derivation implied by the Black–Scholes formula with these parameters: stock price $S = 152.93$, exercise price $X = 150$, time to expiration $T = 121/365 = 0.3315$, riskless rate $r = 0.0025$, and call-option price $C = 10.50$. The approximated implied volatility is found to be $\sigma = 0.264$.

Sample Problem 27.7

Using the information about the implied volatility presented above and a stock price of $S = 155$, we present the following example. Given $S = 155$, $X = 150$, $T = 0.3315$, $r = 0.0025$, and $\sigma = 0.264$, use Equation (27.40) to compute C . Using Equation (27.41) and (27.42) calculate:

$$d_1 = \frac{\ln\left(\frac{155}{150}\right) + \left[0.0025 + \left(\frac{0.264^2}{2}\right)\right]0.3315}{0.264\sqrt{0.3315}} = 0.29717,$$

$$d_2 = \frac{\ln\left(\frac{155}{150}\right) + \left[0.0025 - \left(\frac{0.264^2}{2}\right)\right]0.3315}{0.264\sqrt{0.3315}} = 0.14517.$$

From a standard normal distribution table, giving the area of a standard normal distribution, $N(0.29717) = 0.616836$ and $N(0.14517) = 0.557923$. Finally,

$$C = 152.93 \times 0.616836 - 150e^{-0.0025 \times 0.3315} \times 0.557923 = \$10.71.$$

These calculations show that as the price of the underlying stock increases from 152.93 to 155, the call price increases as indicated in (27.43) from \$10.50 to \$10.71, while all other variables are the same.

Sample Problem 27.8

Using the information from the above example, we will calculate a call with a strike price of \$160 using Equation (27.40a)

$$d_1 = \frac{\ln\left(\frac{155}{160}\right) + \left[0.0025 + \left(\frac{0.264^2}{2}\right)\right]0.3315}{0.264\sqrt{0.3315}} = -0.127419,$$

$$d_2 = \frac{\ln\left(\frac{155}{160}\right) + \left[0.0025 - \left(\frac{0.264^2}{2}\right)\right]0.3315}{0.264\sqrt{0.3315}} = -0.268425.$$

From a standard normal distribution table, giving the area of a standard normal distribution, $N(-0.127419) = 0.449314$ and $N(-0.268425) = 0.394208$. Finally,

$$C = 152.93 \times 0.449314 - 160e^{-0.0025 \times 0.3315} \times 0.394208 = \$9.63.$$

As expected, the price of this call option, $C = \$9.63$, with $X = 160$ has a lower calculated price than the call option with $X = 150$, as indicated in (27.43).

This simple example shows how to use the Black–Scholes model to price a call option under the assumptions of the model. The example is presented for illustrative purposes only, and it relies heavily on the implicit estimate of the variance, its constancy over time, and all the remaining assumptions of the model. The appropriateness of estimating the implicit instantaneous variance is ultimately an empirical question, as is the entire Black–Scholes pricing formula. Boyle and Ananthanarayanan (1977) studied the implications of using an estimate of the variance in option-valuation models and showed that this procedure produces biased option values. However, the magnitude of this bias is not large.

One additional remark must be made. The closeness of a calculated call option price to the actual call price is not necessary evidence of the validity of the Black–Scholes model. Extensive empirical work has taken place to investigate how market prices of call options compare with price predicted by Black–Scholes; see MacBeth and Merville (1979) and Bhattacharya (1980).

27.6 REMARKS ON OPTION PRICING<SI>

For a review on the literature on option pricing, see the two papers by Smith (1976, 1979). It is appropriate here to make a few remarks on the Black–Scholes option-pricing model to clarify its significance and its limitation.

First, the Black–Scholes model for a European call as original derived, and as reported here, is based on several simplifying assumptions.

- (1) The stock price follows an Itô equation.
- (2) The market operates continuously.
- (3) There are no transaction costs in buying or selling the option or the underlying stock.
- (4) There are no taxes.
- (5) The riskless rate is known and constant.
- (6) There are no restrictions on short sales.

Several researchers have extended the original Black–Scholes model by modifying these assumptions. Merton (1973) generalized the model to include dividend payments, exercise-price changes, and the case of a stochastic interest rate. Roll (1977) had solved the problem of valuing a call option that can be exercised prior to its expiration date when the underlying stock is assumed to make known dividend payments before the option matures. Ingersoll (1976) studied the effect of differential taxes on capital gains and income, while Scholes (1976) determined the effects of the tax treatment of options on the pricing model. Furthermore, Merton (1976) and Cox and Ross (1976) showed that if the stock-price movements are discontinuous, under certain assumptions the valuation model still holds. These and other modifications of the original Black–Scholes analysis indicate that the model is quite robust about the relaxation of its fundamental assumptions.

Second, it is worth repeating that the use of Itô's calculus and the important insight concerning the appropriate concept of an equilibrium by creating a riskless hedge portfolio have let Black and Scholes obtain a closed-form solution for option pricing. In this closed-form solution, several variables do not appear, such as (1) the expected rate of return of the stock, (2) the expected rate of return of the option, (3) a measure of investor's risk preference, (4) investor expectations, and (5) equilibrium conditions for the entire capital market.

Third, the Black–Scholes pricing model has found numerous applications. Among these are: (1) pricing the debt and equity of a firm; (2) the effects of corporate policy and, specially, the effects of mergers, acquisitions, and scale expansions on the relative values of the debt and equity of the firm; (3) the pricing of convertible bonds; (4) the pricing of underwriting contracts; (5) the pricing of leases; and (6) the pricing of insurance. Smith (1976, 1979) summarized most applications and indicates the original reference. See also Brealey and Myers (1988).

Fourth, Black (1976) showed that the original call-option formula for stocks can be easily modified to be used in pricing call options on futures. The formula is

$$C(T, F, \sigma^2, X, r) = e^{-rT} [FN(d_1) - XN(d_2)] \quad ,$$

(27.50)

$$d_1 = \frac{\ln(F/X) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \quad , \quad (27.51)$$

$$d_2 = \frac{\ln(F/X) - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \quad . \quad (27.52)$$

In Equation (27.50), F now denotes the current futures price. The other four variables are as before — time to maturity, volatility of the underlying futures price, exercise price, and risk-free rate. Note that Equation (27.50) differs from Equation (27.40) only in one respect: by substituting $e^{-rT}F$ for S in the original Equation (27.40),

Equation (27.50) is obtained. This holds because the investment in a futures contract is zero, which causes the interest rate in Equations (27.51) and (27.52) to drop out.

Fifth, three important papers by Harrison and Kreps (1979) and Kreps (1981, 1982) consider some foundational issues that arise in conjunction with the arbitrage theory of option pricing. The important point to consider is this: the ability to trade securities frequently can enable a few multiperiod securities to span many states of nature. In the Black–Scholes theory, there are two securities and uncountable many state of nature, but because there are infinitely many trading opportunities and because uncertainty resolves nicely, markets are effectively complete. Thus, even though there are far fewer securities than states of nature, markets are complete and risk is allocated efficiently. An interesting result of Harrison and Kreps (1979) is that certain self-trading strategies can create something out of nothing when there are infinitely many trading opportunities. The doubling strategies are the well-known illustrations of this phenomenon. Harrison and Kreps introduce the concept of a simple strategy to eliminate free lunches and conjecture that a non-negative wealth constraint could rule out the doubling strategies. Duffie and Huang (1985) gave an interpretation of admissible strategy as a limit of a sequence of simple strategies and use an integrability condition on the trading strategies. Dybvig and Huang (1986) showed that under certain condition and the non-negative wealth constraint are functionally equivalent.

Finally, for an intensive survey of numerous empirical tests, see Galai (1983).

27.7 SUMMARY<SI>

This chapter has discussed the basic concepts and equations of stochastic calculus (Itô's calculus), which has become a very useful tool in understanding finance theory

and practice. By using these concepts and equations, the manner by which Black and Scholes derived their famous option-pricing model was also illustrated. Although this chapter is not required to understand the basic ingredients of security analysis and portfolio management discussed in Chapters 1–26, it is useful for those with trading in advanced mathematics to realize how advanced mathematics can be used in finance.

QUESTIONS AND PROBLEMS<SI>

1. Briefly describe Itô's lemma and Itô's process.
2. Describe the meaning of the process of a stock price $S(t, w)$ that follows the stochastic process

$$dS(t, w) = \mu S(t, w)dt + \sigma S(t, w)dZ(t, w).$$

3. Derive the Black–Scholes call option model by using Itô's lemma and the stochastic process of a stock price $S(t, w)$ in question 2.
4. Suppose the stock price $S(t, w)$ in question 2, what is the process of the variable $S^n(t, w)$? (Hint: show that $S^n(t, w)$ also follows the same stochastic process.)
5. Assume the stock price $S(t, w)$ in question 2. Let $\mu = 0.15$ and $\sigma = 0.2$ in the first two years and $\mu = 0.28$ and $\sigma = 0.35$ in the last year. Today stock price is \$50.
 - (a) What is the probability distribution of the stock price at the end of two years?
 - (b) What is the probability distribution of the stock price at the end of three years?
6. Assume the stock price $S(t, w)$ follows the stochastic process

$$dS(t, w) = \mu dt + \sigma dZ(t, w).$$

Is this process more appropriate than the process in question 2? Why or Why not?

7. Assume the stock price $S(t, w)$ in question 6. Let $\mu = 1.5$, $\sigma = 2$ and now stock price is \$110.
- What is the probability distribution of the stock price in the next year?
 - What are 95% confidence limits for the stock price in the next year?
8. Assume the stock price $S(t, w)$ in question 2 again. Suppose the expected return of stock price equal 7% and volatility is 25% per year, now the stock price is \$100.
- What is the expected stock price at the end of six months?
 - What is the standard deviation of the stock price at the end of six months?
 - What is the expected stock price next year?
9. Assume the stock price $S(t, w)$ in question 2 again. Derive the process of Y under the condition as follows:
- $Y = 3S(t, w)$
 - $Y = S^n(t, w)$
 - $Y = e^{r(T-t)}S(t, w)$.
10. Assume the stock price $S(t, w)$ in question 5, what is the probability that the stock price will be greater than \$130 in two years?

APPENDIX 27A: AN ALTERNATIVE METHOD TO DERIVE THE BLACK-SCHOLES OPTION PRICING MODEL<SI>

Perhaps it is unclear why it is assumed that investors have risk-neutral preferences when the usual assumption in finance courses is that investors are risk averse. It is feasible to make this simplistic assumption because investors are able to create riskless portfolios by combining call options with their underlying securities. Since

the creation of a riskless hedge places no restrictions on investor preferences other than nonsatiation, the valuation of the option and its underlying asset will be independent of investor risk preferences. Therefore, a call option will trade at the same price in risk-neutral economy as it will in a risk-averse or risk-preferent economy.

27A.1 Assumptions and the Present Value of the Expected Terminal Option Price<S2>

To derive the Black–Scholes formula, it is assumed that there are no transaction costs, no margin requirements, and no taxes; that all shares are infinitely divisible, and that continuous trading can be accomplished. It is also assumed that the economy is risk neutral.

In the risk-neutral assumptions of Cox and Ross (1976) and Rubinstein (1976), today's option price can be determined by discounting the expected value of the terminal option price by the riskless rate of interest. As was seen earlier, the terminal call-option price can take on only two values: $S_t - X$, if the call option expires in the money, or 0 if the call expires out of the money. So today's call option price is

$$C = \exp(-rt) \text{Max}(S_t - X, 0), \quad (27A.1)$$

where

C = the market value of the call option;

r = riskless rate of interest;

t = time to expiration;

S_t = the market value of the underlying stock at time t ; and

X = exercise or striking price.

Equation (27A.1) says that the value of the call option today will be either $S_t - X$ or 0, whichever is greater. If the price of stock at time t is greater than the exercise price, the call option will expire in the money. This simply means that an

investor who owns the call option will exercise it. The option will be exercised regardless of whether the option holder would like to take physical possession of the stock. If the investor would like to own the stock, the cheapest way to obtain the stock is by exercising the option. If the investor would not like to own the stock, he or she will still exercise the option and immediately sell the stock in the market. Since the price the investor paid (X) is lower than the price he or she can sell the stock for (S_t), the investor realizes an immediate profit of $S_t - X$. If the price of the stock (S_t) is less than the exercise price (X), the option expires out of the money. This occurs because in purchasing shares of the stock the investor will find it cheaper to purchase the stock in the market than to exercise the option.

Assuming that the call option expires in the money, then the present value of the expected terminal option is equal to the present value of the difference between the expected terminal stock price and the exercise price, as indicated in Equation (27A.2):

$$\begin{aligned} C &= \exp(-rt) E [Max(S_t - X, 0)] \\ &= \exp(-rt) \int_x^{\infty} (S_t - X) h(S_t) dS_t \end{aligned} \quad (27A.2)$$

where $h(S_t)$ is the log-normal density function of S_t . To evaluate the integral in (27A.2), rewrite it as the difference between two integrals:

$$\begin{aligned} C &= \exp(-rt) \left[\int_x^{\infty} S_t h(S_t) dS_t - X \int_x^{\infty} h(S_t) dS_t \right] \\ &= E_x(S_t) \cdot \exp(-rt) - X \cdot \exp(-rt) \cdot [1 - H(X)] \end{aligned} \quad (27A.3)$$

where

$E_x(S_t)$ = the partial expectation of S_t , truncated from below at x ; and

$H(X)$ = the probability that $S_t \leq X$.

Equation (27A.3) says that the value of the call option is present value of the partial expected stock price (assuming the call expires in the money) minus the present value of the exercise price (adjusted by the probability that the stock's price will be less than the exercise price at the expiration of the option). The terminal stock price S_t , can be rewritten as the product of the current price (S) and the t -period log-normally distributed price ratio S_t/S , so $S_t = S(S_t/S)$. Equation (27A.3) can also be rewritten as

$$\begin{aligned} C &= \exp(-rt) \left[S \int_{x/S}^{\infty} \frac{S_t}{S} g\left(\frac{S_t}{S}\right) \left(\frac{dS_t}{S}\right) - X \int_{x/S}^{\infty} g\left(\frac{S_t}{S}\right) \left(\frac{dS_t}{S}\right) \right] \\ &= S \exp(-rt) E_{x/S}\left(\frac{S_t}{S}\right) - X \exp(-rt) \left[1 - G\left(\frac{X}{S}\right) \right] \end{aligned} \quad (27A.4)$$

where

$$\begin{aligned} g\left(\frac{S_t}{S}\right) &= \text{log normal density function of } S_t/S; \\ E_{x/S}\left(\frac{S_t}{S}\right) &= \text{the partial expectation of } S_t/S, \text{ truncated from below at } x/S; \\ G\left(\frac{X}{S}\right) &= \text{the probability that } S_t/S \leq X/S. \end{aligned}$$

27A.2 Present Value of the Partial Expectation of the Terminal Stock Price<S2>

The right-hand side of Equation (27A.4) is evaluated by considering the two integrals separately. The first integral, $S \exp(-rt) E_{x/S}(S_t/S)$, can be solved by assuming the return on the underlying stock follows a stationary random walk. That is

$$\frac{S_t}{S} = \exp(Kt), \quad (27A.5)$$

where K is the rate of return on the underlying stock per unit of time. Taking the natural logarithm of both sides of Equation (27A.5) yields:

$$\ln\left(\frac{S_t}{S}\right) = (Kt).$$

Since the ratio S_t/S is log normally distributed, it follows that Kt is log normally distributed with density $f(Kt)$, mean $\mu_K t$, and variance $\sigma_K^2 t$. Because $S_t/S = \exp(Kt)$, the differential can be rewritten as $dS_t/S = \exp(Kt) t dK$; $g(S_t/S)$ is a density function of a log-normally distributed variable S_t/S ; so following Garven (1986), it can be transformed into a density function of a normally distributed variable Kt according to the relationship $S_t/S = \exp(Kt)$ as

$$g\left(\frac{S_t}{S}\right) = f(Kt) \left(\frac{S_t}{S}\right). \quad (27A.6)$$

These transformations allow the first integral in Equation (27A.4) to be rewritten as

$$S \exp(-rt) E_{x/S} \left(\frac{S_t}{S} \right) = S \exp(-rt) \int_{\ln(x/S)}^{\infty} f(Kt) \exp(Kt) t dK.$$

Because Kt is normally distributed, the density $f(Kt)$ with mean $\mu_K t$ and variance $\sigma_K^2 t$ is

$$f(Kt) = (2\pi\sigma_K^2 t)^{-1/2} \exp\left[-\frac{1}{2}(Kt - \mu_K t)^2 / \sigma_K^2 t\right].$$

Substitution yields:

$$\begin{aligned} S \exp(-rt) E_{x/S} \left(\frac{S_t}{S} \right) &= S \exp(-rt) (2\pi\sigma_K^2 t)^{-1/2} \\ &\times \int_{\ln(x/S)}^{\infty} \exp[Kt] \exp\left[-\frac{1}{2}(Kt - \mu_K t)^2 / \sigma_K^2 t\right] t dK \end{aligned} \quad (27A.7)$$

Equation (27A.7)'s integrand can be simplified by adding the terms in the two exponents, multiplying and dividing the result by $\exp\left(-\frac{1}{2}\sigma_K^2 t\right)$. First, expand the

term $(Kt - \mu_K t)^2$ and factor out t so that

$$\exp[Kt] \exp\left[-\frac{1}{2}(Kt - \mu_K t)^2 / \sigma_K^2 t\right].$$

Next, factor out t so

$$\exp(Kt) \exp\left\{-\frac{1}{2}t \left[(K^2 - 2\mu_K K + \mu_K^2) / \sigma_K^2\right]\right\}.$$

Now combine the two exponents

$$\exp\left\{-\frac{1}{2}t \left[(K^2 - 2\mu_K K + \mu_K^2 - 2\sigma_K^2 K) / \sigma_K^2\right]\right\}.$$

Now, multiply and divide this result by $\exp(-\frac{1}{2}\sigma_K^2 t)$ to get:

$$\exp\left\{-\frac{1}{2}t \left[(K^2 - 2\mu_K K + \mu_K^2 - 2\sigma_K^2 K + \sigma_K^4 - \sigma_K^4) / \sigma_K^2\right]\right\}.$$

Next, rearrange and combine terms to get:

$$\begin{aligned} \exp\left\{\left(-\frac{1}{2}t\right) \left[(K - \mu_K - \sigma_K^2)^2 - \sigma_K^4 - 2\mu_K \sigma_K^2\right] / \sigma_K^2\right\} = \\ \exp\left[(\mu_K + \frac{1}{2}\sigma_K^2)t\right] \exp\left\{-\frac{1}{2} \left[Kt - (\mu_K + \sigma_K^2)t\right]^2 / \sigma_K^2 t\right\} \end{aligned} \quad (27A.8)$$

In Equation (27A.8), $\exp[(\mu_K + \frac{1}{2}\sigma_K^2)t] = E(S_t/S)$, the mean of the t -period log-normally distributed price ratio S_t/S . So, Equation (27A.7) becomes:

$$\begin{aligned} S \exp(-rt) E_{x/S} \left(\frac{S_t}{S} \right) = S E \left(\frac{S_t}{S} \right) \exp(-rt) (2\pi\sigma_K^2 t)^{-1/2} \\ \times \int_{\ln(x/S)}^{\infty} \exp(Kt) \exp\left\{-\frac{1}{2} \left[Kt - (\mu_K + \sigma_K^2)t\right]^2 / \sigma_K^2 t\right\} \end{aligned} \quad (27A.9)$$

Since the equilibrium rate of return in a risk-neutral economy is the riskless rate,

$E(S_t/S)$ may be rewritten as $\exp(rt)$:

$$\begin{aligned} S E \left(\frac{S_t}{S} \right) \exp(-rt) = S \exp(rt) \exp(-rt) \\ = S \end{aligned}$$

So Equation (27A.9) becomes

$$\begin{aligned} S \exp(-rt) E_{x/S} \left(\frac{S_t}{S} \right) = S (2\pi\sigma_K^2 t)^{-1/2} \\ \times \int_{\ln(x/S)}^{\infty} \exp\left\{-\frac{1}{2} \left[Kt - (\mu_K + \sigma_K^2)t\right]^2 / \sigma_K^2 t\right\} t dK \end{aligned} \quad (27A.10)$$

To complete the simplification of this part of the Black–Scholes formula, define a standard normal random variable y :

$$y = \left[Kt - \left(\mu_K + \sigma_K^2 \right) t \right] / \sigma_K t^{1/2}.$$

Solving for Kt yields:

$$Kt = \left(\mu_K + \sigma_K^2 \right) t + \sigma_K t^{1/2} y,$$

and therefore:

$$t dK = \sigma_K t^{1/2} dy.$$

By making the transformation from Kt to y , the lower limit of integration becomes

$$\left[\ln(x/S) - \left(\mu_K + \sigma_K^2 \right) t \right] / \sigma_K t^{1/2}.$$

Further simplify the integrand by noting that the assumption of a risk-neutral economy implies:

$$\exp \left[\left(\mu_K + \frac{1}{2} \sigma_K^2 \right) t \right] = \exp(rt).$$

Taking the natural logarithm of both sides yields:

$$\left(\mu_K + \frac{1}{2} \sigma_K^2 \right) t = (rt).$$

Hence, $\left(\mu_K + \frac{1}{2} \sigma_K^2 \right) t = \left(r + \frac{1}{2} \sigma_K^2 \right) t$.

The lower limit of integration is now:

$$-\left[\ln(S/x) + \left(r + \frac{1}{2} \sigma_K^2 \right) t \right] / \sigma_K t^{1/2} = -d_1.$$

Substituting this into Equation (27A.10) and making the transformation to y yields:

$$S \exp(-rt) E_{x/S} \left(\frac{S_t}{S} \right) = S \int_{-d_1}^{\infty} \exp\left(-\frac{1}{2} y^2\right) / (2\pi)^{1/2} dy.$$

Since y is a standard normal random variable (distribution is symmetric around zero) the limits of integration can be exchanged:

$$\begin{aligned} S \exp(-rt) E_{x/S} \left(\frac{S_t}{S} \right) &= S \int_{\infty}^{-d_1} \exp\left(-\frac{1}{2} y^2\right) / (2\pi)^{1/2} dy \\ &= S N(d_1) \end{aligned} \quad (27A.11)$$

where $N(d_1)$ is the standard normal cumulative distribution function evaluated at

$y = d_1$.

27A.3 Present Value of the Exercise Price under Uncertainty<S2>

To complete the derivation, the integrals that correspond to the term $X \exp(-rt)$

$[1 - G(X/S)]$ must be evaluated. Start by making the logarithmic transformation:

$$\ln\left(\frac{S_t}{S}\right) = Kt.$$

This transformation allows the rewriting of $g(S_t/S)$ to $(S/S_t)f(Kt)$ as

mentioned previously. The differential can be written as

$$d\frac{S_t}{S} = \exp(Kt) t dK.$$

Therefore,

$$\begin{aligned} X \exp(-rt)[1 - G(X/S)] &= X \exp(-rt) \int_{\ln(X/S)}^{\infty} f(Kt) t dK = \\ &X \exp(-rt) (2\pi\sigma_K^2 t)^{-1/2} \int_{\ln(X/S)}^{\infty} \exp\left[-\frac{1}{2}(Kt - \mu_K t)^2 / \sigma_K^2 t\right] t dK \end{aligned} \quad (27A.12)$$

The integrand is now simplified by following the same procedure used in simplifying the previous integral. Define a standard normal random variable Z :

$$Z = \left[\frac{Kt - \mu_K t}{\sigma_K t^{1/2}} \right].$$

Solving for Kt yields:

$$Kt = \mu_K t + \sigma_K t^{1/2} Z,$$

and $t dK = \sigma_K t^{1/2} dZ$. Making the transformation from Kt to Z means the lower limit of integration becomes

$$\frac{\ln(X/S) - \mu_K t}{\sigma_K t^{1/2}}.$$

Again, note that the assumption of a risk-neutral economy implies:

$$\exp\left(\mu_K + \frac{1}{2}\sigma_K^2\right) = \exp(rt).$$

Taking the natural logarithm of both sides yields:

$$\left(\mu_K + \frac{1}{2}\sigma_K^2\right)t = rt,$$

or:

$$\mu_K t = \left(r - \frac{1}{2} \sigma_K^2\right) t.$$

Therefore, the lower limit of integration becomes:

$$\begin{aligned} \frac{-\left[\ln(S/x) + \left(r - \frac{1}{2} \sigma_K^2\right) t\right]}{\sigma_K t^{1/2}} &= -\left(d_1 - \sigma_K t^{1/2}\right) \\ &= -d_2 \end{aligned}$$

Substitution yields:

$$\begin{aligned} x \exp(-rt) [1 - G(X/S)] &= x \exp(-rt) \int_{-d_2}^{\infty} \exp\left[-\frac{1}{2} Z^2 / (2\pi)^{1/2}\right] dZ \\ &= x \exp(-rt) \int_{-\infty}^{d_2} \exp\left[-\frac{1}{2} Z^2 / (2\pi)^{1/2}\right] dZ \quad (27A.13) \\ &= x \exp(-rt) N(d_2) \end{aligned}$$

where $N(d_2)$ is the standard normal cumulative distribution function evaluated at $Z = d_2$.

Substituting, Equations (27A.11) and (27A.13) into Equation (27A.4) completes the derivation of the Black–Scholes formula:

$$C = S N(d_1) - X \exp(-rt) N(d_2). \quad (27A.14)$$

This appendix provides a simple derivation of the Black–Scholes call-option pricing formula. Under an assumption of risk neutrality the Black–Scholes formula was derived using only differential and integral calculus and a basic knowledge of normal and log-normal distributions.

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