Pedagogical Content Knowledge in Early Mathematics: What Teachers Know and How It Associates with Teaching and Learning

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—Lee Shulman
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LIST OF ABBREVIATIONS

DAP: Developmental appropriate practice

HIS-EM: High Impact Strategies in Early Mathematics

PCK: Pedagogical content knowledge

PCK-EM: Pedagogical content knowledge in early mathematics

TEAM: Tools for Early Assessment in Math

WJ III-AP: Woodcock-Johnson-III, applied problems

ZPD: Zone of proximal development
ABSTRACT

The study was designed to examine the profile of early childhood teachers’ content-specific knowledge, also referred to as pedagogical content knowledge, in early mathematics (PCK-EM) and investigate how PCK-EM relates to the quality of their mathematics teaching quality and students’ mathematical learning outcomes. A total of 182 teachers working with high need students from Pre-K through Grade 3 in an urban public school system participated in the study and analyzed a video of mathematics teaching through an online survey. The results painted a bleak picture regarding the profile of early childhood teachers’ PCK-EM: as a whole, teachers’ survey responses lacked an in-depth understanding of foundational mathematical knowledge, student mathematical learning, and effective mathematical teaching strategies. The results also suggested that teachers’ PCK-EM significantly predicted their quality of math teaching, beyond their years of teaching and hours in pre-service and in-service workshops related to math education. Finally, teachers’ conceptual understanding of foundational mathematics significantly predicted students’ learning at the classroom level. These findings highlight the significant role of early childhood teachers’ knowledge in promoting students’ mathematical learning and the need for helping these professionals improve their understanding of foundational mathematics and how it is taught.
CHAPTER I

INTRODUCTION

Effective Teaching and Pedagogical Content Knowledge

Learning is critical to individuals’ development throughout their lives and to the prosperity of society. How can knowledge and skills be obtained efficiently and effectively? Teachers play a crucial role in facilitating knowledge acquisition and socializing individuals. In fact, they are not only critical, but also irreplaceable to the educational enterprise. “We will sooner de-school society than de-teacher it” (Shulman, 1974, p. 319). “No microcomputer will replace them, no television system will clone and distribute them, no scripted lessons will direct and control them, no voucher system will bypass them” (Shulman & Sykes, 1983, p. 504).

The significant role of teachers has placed teaching competence in the spotlight. What makes teaching effective? Hill, Rowan & Ball (2005) made an extensive review of teaching effectiveness study and found that the relationship between teacher characteristics (which include but are not limited to teaching behavior and knowledge) and student achievement gains has been investigated for decades by two major approaches: process-product and educational production function. Process-product studies explore the correlations between teachers’ classroom teaching behaviors (process) and students’ learning outcomes (product). The teachers’ classroom behaviors that have been studied most include both affective factors (such as the warmth of the teacher) and
general principles of classroom management. Positive behavior-outcome relationships have been found when teachers efficiently use instructional time, establish smooth and efficient classroom routines, give sufficient feedback, encourage cooperative learning, provide scaffolding, hold appropriate expectations for learning outcomes and assign appropriate homework to students (Brophy, 2001; Evertson, Emmer, & Brophy, 1980; Evertson, 1981). Although there was some evidence implying the importance of generic teaching behaviors to move students’ thinking forward, the findings were usually of weak significance and mixed results.

The process-product approach of studying teaching competence has been criticized methodologically and conceptually. Methodologically, the approach relies exclusively on correlational analysis to detect the relationship between teaching processes and child learning outcomes (Ball & McDiarmid, 1990; Hill et al., 2005). Such analysis limits the possibility of investigating causal inferences between teaching behaviors and students’ achievement gains empirically. It also ignores the complexity and dynamic interactions between teaching and learning by merely studying the one-way impact from teacher to students. Conceptually, the focus of broad aspects of teaching behaviors overlooks the distinctiveness of subject specific teaching (Hill, et al., 2005), assuming that what works for history teaching and learning, for instance, would be effective in mathematics. The significance of teachers’ subject matter expertise is frequently ignored.

On the other hand, the educational production function approach explores the contribution of educational resources to students’ performances on standardized tests. Students’ family background and teachers’ and school resources are among the
educational resources students possessed. Teachers’ knowledge, not general classroom
teaching behaviors, is considered one of the educational resources possessed by students
and key for effective teaching and student achievement. With a belief that teachers’
understanding of a subject is powerful in impacting how they teach, proxy indicators
were identified to investigate the impact of teachers’ subject knowledge on students’
learning outcomes. Distal indicators in pre-service preparations such as degree earned,
certificates obtained, and courses taken in college were widely used. The investigation of
knowledge moves the assumption of exploring teaching effectiveness from teaching
behaviors to what teachers know. However, in contrast with the fact that teachers’
knowledge plays a significant role in teaching and learning, the attempts revealed
conflicting linkages between distal indicators of teacher knowledge and student
outcomes, suggesting that teacher preparation experience demonstrated by indirect
indicators is a poor proxy of teachers’ knowledge that matters for students’ learning.

To remedy this problem, further attempts were made to measure teachers’ subject
understanding more directly, including their performance on certification exams, tests on
advanced subjects (e.g. mathematics), as well as the number of advanced courses taken in
college. Teachers’ content knowledge measured in such a fashion was hypothesized to
positively contribute to students’ performance. Unfortunately, the hypothesis was overall
not tested favorably. While there is evidence that advanced mathematical courses taken
by teachers had positive impact on students learning (Brian Rowan, Chiang, & Miller,
1997), there was no extra effect once the number of advanced math courses taken was
more than five (Monk, 1994). In fact, elementary students taught by teachers with an
advanced degree didn’t seem to benefit from their teachers’ more rigorous subject
training: they performed worse than those students who were taught by teachers without a mathematics degree (Rowan, Correnti, & Miller, 2002). Teachers’ subject matter knowledge (measured by direct indicators such as test scores) was found to be significantly but weakly related to students’ achievement (Begle, 1979). In fact, a negative correlation was reported to a subgroup of students (Begle, 1979), suggesting that the better the teachers performed in advanced math courses at college, the less progress their students made in academic gains.

These findings, together, suggest a need to explore more effective approximations of subject matter expertise in producing students’ gains. Content understanding certainly lays the foundation for teaching; however, the knowledge examined by certification status, subject matter courses completed in college, or performance on college tests, may not be covered in curriculum and teaching to elementary or high school students. By exclusively focusing on academic knowledge from teacher preparation, the education production function approach fails to notice the importance of how knowledge is used in teaching a subject (Hill et al., 2005). The alignment of content understanding between teaching and learning, and the significance of unpacking content knowledge to students has yet to be addressed.

It is in this sense that the proposal of pedagogical content knowledge (PCK) has brought about a powerful tornado on the investigation of teaching competence. PCK is a theory of exploring and unpacking the unique professional knowledge in content teaching, which refers to knowledge of subject matter for teaching (Shulman, 1986, 1987). It started by asking questions about what is the unique professional knowledge required for teaching a subject well, how to distinguish knowledge held by common
adults and effective teachers, and how to distinguish between a content expert and a
teacher for the same subject.

According to PCK theory, the answer lies in the “blending of content and
pedagogy into an understanding of how particular topics, problems and issues are
organized, represented, and adapted to the diverse interests and abilities of learners, and
presented for instruction” (Shulman, 1987, p.4). PCK is central to effective teaching
because teachers need this understanding to structure the content of the lessons,
understand and anticipate students’ preconceptions or learning difficulties, and choose
specific representations or analogies to make the content comprehensible.

**Early Mathematics Teaching and Pedagogical Content Knowledge**

By highlighting the foundational role of content understanding and the simultaneous
integration of content understanding and pedagogy in successful teaching, the notion of
PCK provides a unique opportunity to investigate effective early mathematics teaching
from content understanding, knowledge of learners and pedagogical awareness.

Logically, the understanding of mathematics knowledge lays the foundation for teaching.
The question is what specific mathematical understanding is needed for teaching young
children. Is knowledge about advanced mathematics, such as calculus, a must for
teaching young children? PCK urges the necessity of considering the mathematical
knowledge required in teaching to a specific age group. Compared with teachers’ test
scores in certificate exams or advanced mathematics tests in college, it is more closely
related to the knowledge required in teaching young children.

Applying the notion of PCK to early mathematics teaching, it is necessary to align
the content of instruction with the capacity of young learners. The question then moves
to what mathematics young children are capable of learning and of vital importance for their future success. Should early mathematics education primarily emphasize learning counting and shape recognition? Is early math so simple that it requires little instruction in early childhood classrooms? Such questions have been sufficiently addressed in a number of recent publications by the National Council of Teachers of Mathematics & National Association for the Education of Young Children (NCTM, 2000; NCTM & NAEYC, 2002; 2010) and the Common Core State Standards for Mathematics (CCSS-M, 2010). Accordingly, early math is comprised of five knowledge domains, including number sense and operation, patterns and measurement, geometry and spatial awareness, algebra and data analysis. It also involves mathematics processes, such as problem solving, reasoning, communication, connections and representations, which are equally important for the development of young children’s mathematical competence (NCTM 2000).

The inclusion of diverse mathematics content is necessary but not sufficient for effective teaching. Early math is complex and abstract for young children; therefore, a profound understanding about foundational math topics is a must for sound teaching. For example, counting, the seemingly simplest math activity, involves intricate principles that young children have to learn one by one. To count a collection of objects, a child has to count each of the objects and use a number to label it, but only once (one-to-one correspondence). Memorizing the number words by order, however, does not lead to the understanding about the quantity associated with the numbers, the concept of more or less, or other functions of number such as referring to a position or an order. While adults can count flexibly and fluently without consciously thinking about the different meanings
such as the cardinal and ordinal nature of number words, young children learn the meanings separately and it takes time for them to make the connections (Cross, Woods, & Schweingruber, 2009). Without a sophisticated understanding of these and many other foundational mathematical concepts, it is less likely that teachers will be able to provide rich learning opportunities and foster young children’s mathematical competence in their classrooms.

Equally important to teachers’ mathematics knowledge for effective teaching is their understanding about the nature of learners and learning. Early childhood teachers are models of developmentally appropriate practice (DAP), given their knowledge about young children’s developmental stages and individual differences in social-cultural contexts (NAEYC, 1987; 2002; 2009). However, due to the nature of general training in early childhood teachers’ preparation, subject-specific understandings about learners and learning are often not sufficiently addressed. In fact, the knowledge gained from content-general assessments typically applied in the early childhood classroom may not provide adequate information about an individual child’s Zone of Proximal Development (ZPD, Vygotsky, 1978) in specific learning areas.

In an effort to understand early childhood teachers’ knowledge about learners and their early math learning abilities, Bowman, Katz, and McNamee (1982) studied the relationship between teachers’ judgment of preschoolers’ math abilities and students’ math performances. Preschool teachers in the study all revealed some understandings about what math knowledge they need to have in order to teach preschoolers, such as pattern, sorting and counting. When asked to characterize their children’s math abilities in terms of either high math learners (HMLs) or low math learners (LMLs), however,
these teachers made decisions largely based on students’ social emotional development, instead of their behaviors or performance related to math. When compared with the results of early math tests, not surprisingly, teachers’ judgments about students’ math abilities were rather inaccurate. The identification of HMLs and LMLs was inconsistent with the students’ performance on math tests; when there was consistency, it was mainly about HMLs.

The results of the study indicate a mismatch between teachers’ knowledge of the subject matter and their understanding of children as learners in early mathematics. Specifically, although teachers were found to possess some of the mathematical knowledge they are expected to teach young children, they were not necessarily able to interweave this understanding into assessing students’ understanding of learning math. In fact, the correct identification of HMLs in Bowman and her colleagues’ study was likely not based on teachers’ capability of assessing students’ math learning, but due to the coincidence of well-developed social-emotional competence and math skills in HMLs. Generic understanding about cognitive and socio-emotional development of young children was not enough for teaching early math; and lacking such knowledge would inevitably affect teachers’ curriculum choices and instructional decisions.

Is knowledge of content and learners’ cognition enough to facilitate students’ understanding then? An authentic teaching scenario (the author observed during a school visit) can provide some insights to answer this question.

A kindergarten classroom was making chains of paper loops with different colors. The teacher designed the activity to have students learn patterns and decorate the classroom. Children were free to pick up the loop colors and to generate their own patterns. Walking around the classroom, the teacher noticed a boy who seemed to have trouble making an AB pattern. He was making a Green/Yellow (GY)
pattern, but started adding irregular numbers of Green or Yellow loops after successfully following a couple of repeats. She drew the boy’s attention by pointing to the beginning of the chain, saying “Green-Yellow-Green-Yellow… What comes next?” “Yellow-Yellow-Green-Green-Green”, answered the boy excitedly while pointing at his paper loops. After a few similar repeated and unsuccessful efforts, the teacher asked the boy to have a “Clip-Clap-Clip-Clap” activity with her (putting hands on the lap for clip and putting hands together for clap). The boy’s started to show confusing on his face; the “clip-clap” analogy didn’t seem to help correct his mistake in making a pattern chain. It was time for outdoor activity; the teacher sighed and gave up.

In the above case, it is likely that the boy needs time and more exposure to comprehend the underlying principles for pattern. The episode also suggests that possessing knowledge of content (understanding the mathematical topic, i.e., patterns) and recognizing students’ cognition (assessing students’ understanding and identifying mistakes) is still not enough for successful teaching. In fact, these difficulties could be accrued in the case of novice teachers.

It is in this sense that knowledge of pedagogy, regarding how to present foundational math in appropriate ways to accommodate students’ needs, is also fundamental for teaching early math effectively. Along with DAP, generic pedagogies such as small group discussion, one-to-one scaffolding, and paired work are widely applied. In terms of mathematics, teachers are usually familiar with utilizing manipulatives and engaging multiple senses to promote children’s problem solving. However, simply providing a supply of materials and resources without intentional instruction or scaffolding may not lead to conceptual learning. It may, on the contrary, bring about low level of thinking and disruptive behaviors (Bowman, 2006). High quality math teaching is “more than a collection of activities” but “coherent, focused on important mathematics and well-articulated” (NCTM, 2000, p.14). To provide
differentiated instruction to meet the needs of children with diverse interests and capabilities, teachers must hold a variety of teaching strategies and a repertoire of mathematical representations, such as analogies, examples and interpretations. It is also necessary, for the teacher to know what strategies and representations may work best to promote higher-level thinking regarding specific students and subject topic.

**Overview of the Present Study**

In relation to the growing recognition of the importance of mathematics to the economic success of a society, the fields of early education and mathematics education have begun to pay more and more attention to the capacity of young children to learn math and the vitality of early math competence for individuals’ future achievement (Cross et al., 2009; Glenn Commission, 2000; NCTM & NAEYC, 2002; 2010). Young children do not become skilled at mathematics without instruction, however. Effective teaching is to successful early math learning as developmentally appropriate practice is to the wellbeing of a child. Unfortunately, while the latter is regarded as a gold standard for early education (NAEYC, 1987; 1997; 2009), the former or the effective early mathematics teaching has not been widely studied or clearly defined yet. The majority of studies on mathematics teaching focus on teachers in elementary, middle and high schools, and little is known about the characteristics of teachers’ mathematical knowledge in early childhood education.

While there is a dearth of studies to address content expertise in teaching early mathematics, early childhood teachers differ from their colleagues in the upper grades in many ways. One difference relates to teacher preparation. Unlike their peers at the upper elementary and high school levels, early childhood educators in the States are not trained...
to teach a specific subject such as mathematics. A majority of early childhood teachers have received little training in teaching mathematics, even those that have a bachelor’s degree in early education (Ginsburg & Golbeck, 2006). In fact, it has been reported that for example, many preschool teachers chose to teach this age because they thought it didn’t require teaching math (Ginsburg & Golbeck, 2006). It is true that the training for upper grade teachers aligns with the more specialized content and most adults have acquired and are using the math knowledge young children need to learn. However, because the understanding has become an integrated part of daily life, unpacking the abstract and complex underlying math concepts presents a quite different challenge. Teachers need to understand how to design and implement activities and take advantage of diverse teachable moments. The lack of subject preparation can pose serious obstacles for teaching foundational mathematics effectively.

As well, young children differ dramatically from their big brothers and sisters regarding the developmental stage and how fast they grow, which would in turn impact how they learn mathematics and the requirement for teaching. There is dramatic progress in physical development, language acquisition, social emotional development and cognitive development between 3 and 8 years. Compared with older school kids, however, these abilities and skills are not completely established with a wide range of individual differences, they still think more concretely and just begin to learn the meaning of various symbols. Curious in learning almost everything, children at this age are still immature and need to be guided and reassured. How can educators take advantage of the developing logical thinking, mathematical vocabulary, and other skills and experiences that children bring to early childhood classrooms while a large amount
of time and energy still needs to be spent for classroom management? The distinctive developmental stage for young children makes early mathematics teaching unique.

Early childhood teachers’ mathematics teaching pedagogy can be unique also because of the characteristics of young children and the teacher training in this grade level. Teacher preparation programs in early childhood usually involve less disciplinary-based instruction. They emphasize understanding about young children over disciplinary expertise. Because of the developing fine motor abilities and concrete learning style of young children, teaching strategies in the early childhood classroom have little similarity to those in the upper grades. For instance, hands-on and playful learning take priority in early childhood education. Small group and paired work, instead of large group activity, are also more common in early learning settings. However, the subject-general training and the general lack of preparation and knowledge in mathematics may cause many early childhood teachers to feel uncomfortable and inadequate to instruct mathematics, particularly in group situations.

Overall, early math teaching can be distinctively different from teaching other grade levels. Regardless of the subject-general training and the challenge of working with developing children, early mathematics competence builds foundations for future learning and is vital for school achievement (Duncan et al., 2007; Glenn Commission, 2000). Therefore, it is critically important to help young children develop such early competence through the provision of effective early mathematics teaching. For all these and many other reasons, early childhood teachers’ understanding about foundational math needs to be studied thoughtfully. Among diverse perspectives to conduct such investigation, PCK provides a unique opportunity to study early math teaching. It refines
the way we investigate mathematical expertise about content, students and pedagogy in an integrative way.

Utilizing data collected from a large project, the current study attempts to capture the profile of early childhood teachers’ PCK in early mathematics and its associations with the quality of mathematics teaching and students’ learning gains in mathematics. The inclusion of age span for math education varies across different statements and standards. For instance, NCTM & NAEYC (2002; 2010) advocated for “high quality, challenging and accessible” (p.1) math learning opportunities for 3- to 6- years old children; while Common Core State Standards for Mathematics (CCSS-M, 2010) addressed math learning standards from kindergarten to high school. With a belief that math learning starts early and needs to be extended coherently, the current study defines early childhood education as teachers who work with young children from Pre-Kindergarten (PreK) to 3rd Grade. Specific questions are:

1. What characterizes early childhood teachers’ mathematical content expertise from the lens of PCK?

2. What is the relationship between early childhood teachers’ PCK in mathematics and teaching effectiveness, including the quality of teaching math and students’ learning gains in mathematics over a year?
CHAPTER II
LITERATURE REVIEW

This chapter reviews the theoretical and empirical studies on PCK, including its nature, structure and validation. It starts by tracing the original conceptualization of PCK and moves to summarize the clarifications and extensions of PCK. Then it proposes a conceptual model of PCK in early mathematics (PCK-EM) before setting the research questions for the current study.

**Original Proposal of PCK by Shulman**

It is widely agreed that teachers are professionals who must have specialized knowledge to teach well. It is less known, however, what this specialized knowledge is all about. Will pedagogy guarantee one to be the best possible teacher? Does a teacher need to be an expert in a specific subject? Will a content expert necessarily be a competent teacher for the same subject? These are among the many questions that have puzzled the field of teaching.

Pedagogical content knowledge (PCK) is one type of knowledge necessary for effective teaching. First proposed by Shulman (1986, 1987), the idea originated from his empirical research on medical diagnosis in the 1970s. Based on an interview (Berry, Loughran, & van Driel, 2008)¹, Shulman believed that one of the most conspicuous findings was that “there was no such thing as general diagnostic ability. […] Someone

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¹ This is an interview with Lee Shulman, conducted by the editors at the Annual Meeting of the American Educational Research Association, in Chicago, April 2007.
who was an absolute cracker jack diagnostican, when presented with cases of cardiovascular disease might look like an utter stumble-bum when presented with a case of rheumatology or neurology or of skeletal disease.” The capability of medical problem solving depends on one’s knowledge, experience, and supervision in specific areas of medicine.

Inspired by the striking findings in medical diagnosis, Shulman was unsatisfied with the fact that research in the teaching field had paid scant attention to teachers’ content knowledge. During the 1970s and 1980s, general pedagogical methods and instructional behaviors, regardless of specific subject matter, were a common focus of teaching effectiveness studies (i.e., process-product approach). The emphasis was on identifying teachers’ behaviors and strategies most likely to lead to gains in students’ achievement. For instance, Ball & McDiarmid(1990) found that the majority of the studies considered whether teachers follow an inquiry-based approach and to what extent concrete examples and manipulative were used. When content was included, it was not done subject by subject, but treated as a controlling variable. Researchers undoubtedly assumed that there was a general teaching ability analogous to general diagnostic ability in medical diagnosis.

Consequently, Shulman suggested research on effective teaching done “subject by subject” and paying attention to content-specific pedagogy. While generic teaching behaviors such as classroom management are valuable, they are not the sole sources of evidence to define knowledge bases of teaching (Shulman, 1987). High quality teaching goes beyond applying instructional principles, such as appropriate pace, to a sophisticated
professional knowledge. Planning, for instance, is important for teaching, but planning for teaching mathematics can be quite different from planning for teaching a history lesson. What is it that a mathematics teacher can do and understand but not a science teacher? To Shulman, the content aspect of teaching was a “missing paradigm of research on teaching” (Shulman, 1986, 1987).

In the meantime, another research camp (the educational production function approach) underlined educational resources to study teaching competence. Teachers’ knowledge was considered as one type of educational resource for students’ learning. Distal and direct indicators of teachers’ subject matter knowledge (SMK) were applied and linked to students’ performance in standardized tests. Distal indicators include degree earned and certificate status; direct indicators refer to performance on certificate exams and scores on advanced mathematics tests in college. Mixed findings were reported between teachers’ knowledge and students’ achievement. Neither was proved to be a reliable or strong predictor of students’ accomplishment, indicating that these measures are not effective indicators of knowledge required for subject teaching.

There was also a political reason for the avocation of PCK. The creation of the National Board System for teaching certification made it necessary to explore the differences between a content specialist (e.g. a mathematician) and a teacher in the same subject (e.g. a math teacher), between a mathematically competent adult and an expert math teacher. What is the exceptional knowledge that only a teacher knows and understands, not a specialist or a mathematically competent adult? The distinction is
salient for establishing teaching as a unique profession and guiding the process for licensing teaching professionals.

To address the significance of content understanding in teaching beyond general pedagogy, rectify the belief widely taken-for-granted that subject matter knowledge measured by traditional indicators is enough for productive teaching, and establish teaching as a unique profession, Shulman introduced PCK in evaluating teaching expertise (Shulman, 1987, 1986):

"Pedagogical content knowledge identifies the distinctive bodies of knowledge for teaching. It represents the blending of content and pedagogy into an understanding of how particular topics, problems or issues are organized, represented, and adapted to the diverse interests and abilities of learners, and presented for instruction. Pedagogical content knowledge is the category most likely to distinguish the understanding of the content specialist from that of the pedagogue" (Shulman, 1987, p. 4).

For Shulman, there is a type of professional knowledge that is only used in teaching, but not held by content experts or other mathematically competent adults. Teachers and adults do have some ordinary knowledge about mathematics; however, the commonly shared mathematical knowledge does not guarantee one will teach mathematics well. Not only does PCK move beyond broad content knowledge to content knowledge that is specific for teaching, it also distinguishes teachers from experts in the same subject, and expert teachers from novice teachers. It is specific to teaching, which consists of an “understanding of how to represent specific subject matter to the diverse abilities and interest of learners” (Shulman & Grossman, 1988, p.9). PCK is not knowledge of content itself but knowledge of how to teach a subject.
The statement also argued that both subject understanding and pedagogical skills are of crucial importance for effective teaching and learning. The simultaneous use of content and pedagogy to adapt to students’ needs differentiates expert teachers in a subject area from experts in the same subject area. For educators, subject matter understanding is part of the pedagogical reasoning process (Cochran, DeRuiter, & King, 1993).

“The key to distinguishing the knowledge base of teaching lies at the intersection of content and pedagogy, in the capacity of a teacher to transform the content knowledge he or she possesses into forms that are pedagogically powerful and yet adaptive to the variations in ability and background presented by the students” (Shulman, 1987, p.15)

More specifically, Shulman pointed out that PCK is a “craft knowledge”(van Driel, Verloop, & de Vos, 1998) that embodies the aspects of “content most germane to its teachability” (Shulman, 1986, p.9), including

“[T]he most regularly taught topics in one’s subject area, the most useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations – in a word, the ways of representing and formulating the subject that make it comprehensible to others… [it] also includes an understanding of what makes the learning of specific concepts easy or difficult: the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning” (Shulman, 1986, p.9).

How do we ensure this transformation? Shulman (1987) further suggested a model of pedagogical reasoning and action that involves a cycle through comprehension, transformation, instruction, evaluation, and reflection (see Table 1).
Table 1. A Model of Pedagogical Reasoning and Action (Adapted and redrawn with permission from Shulman, 1987)

**Comprehension**
Of Purposes, subject matter structures, ideas within and outside the discipline

**Transformation**

*Preparation:* Critical interpretation and analysis of texts, structuring and segmenting, development of a curricular repertoire, and clarification of purposes

*Representation:* use a representational repertoire, which includes analogies, metaphors, examples, demonstrations, explanations, and so forth

*Selection:* choice from among an instructional repertoire, which includes modes of teaching, organizing, managing, and arranging

*Adaptation and tailoring to student characteristics:* consideration of conceptions, preconceptions, misconceptions, and difficulties, language, culture, and motivations, social class, gender, age, ability, aptitude, interests, self-concepts, and attention

**Instruction**
Management, presentations, interactions, group work, discipline, humor, question, and other aspects of active teaching, discovery or inquiry instruction, and the observable forms of classroom teaching

**Evaluation**
Checking for student understanding during interactive teaching, testing student understanding at the end of lessons or units, evaluating one’s own performance, and adjusting for experiences

**Reflection**
Reviewing, reconstructing, reenacting and critically analyzing one’s own and the class’s performance, and grounding explanations in evidence

**New comprehension**
Of purposes, subject matter, students, teaching, and self
Consolidation of new understandings, and learning from experience
In particular, comprehending about purposes and structure of subject matter is the starting point for pedagogical reasoning. The “transformation” of subject matter requires teachers to critically interpret, explore, and select multiple ways to represent the information. These representations include analogies, metaphors, examples, problems, demonstrations, and classroom activities. They also have to choose instructional modes and adapt the material to accommodate students’ abilities, gender, prior knowledge, and preconceptions. Instruction then occurs based on preparation, representation, selection, and adaptation of transformation process. A new cycle of pedagogical reasoning and action will start, in which evaluation of students’ understanding is followed by teacher’s own reflection, and the students’ performance then leads to a new comprehension.

The pedagogical reasoning and action cycle is a process of continual reconstruction of subject matter knowledge for teaching based upon the needs and abilities of students (Buchmann, 1982; Gudmundsdottir, 1987, 1991), although “many of the processes can occur in different order” (Shulman, 1987, p.19). The articulation highlights the central role of content understanding, which starts from and ends at comprehension of the content to be taught. “For many educational scholars, Shulman’s most important contribution to the field has been his insistence that subject matter matters.” (Wilson, 2004 in Shulman, 2004, p.9). Shulman’s work has drawn attention to the discipline-specific nature of teaching, instead of the content-free tradition in educational research. To Shulman, content understanding is not only vital, but a leading factor and driving force of teaching.
For early childhood teachers, however, Shulman (1987) suggested that understanding students might serve a more fundamental role. Teaching young children mathematics, for example, may start from understanding about students, not the math content being taught. The cycle may begin with a teacher taking students’ age and individual differences into consideration, although always in the context of some concept to be taught:

Under some conditions, teaching may begin with a given group of students. It is likely that at the early elementary grades, or in special education classes, or other setting where children have been brought together for particular reasons, the starting point for reasoning about instruction may well be the characteristics of the group itself. […] Teachers may focus on comprehension of a particular set of values, or on the characteristics, needs, interests, or propensities of a particular individual or group of learners (Shulman, 1987, p.14).

**Clarification and Extensions of PCK after Shulman**

Researchers and practitioners have been enthusiastic about PCK since its inception. Recognizing its great potential to impact teaching and assessment of effective teaching, the researchers also became aware of the imprecision of Shulman’s proposition. Followers have further explored and interpreted PCK both theoretically and empirically by uncovering more details and expanding the initial proposal (e.g. Abell, 2008; Ball et al., 2001; Berry et al., 2008; Boyd et al., 2010; Cochran et al., 1993; Feiman-Nemser & Parker, 1990; Gess-Newsome, 1999a; Park & Oliver, 2008a; Smith & Neal, 1989). However, to date, there is no consensus in the definition of PCK, or its essential components and development.

It is worth noting also that while some of the researchers explicitly identify the construct as PCK (e.g. Abell, 2008; Appleton, 2002; Gess-Newsome, 1999a; Kersting,
2005; Kind, 2009; Loughran et al., 2004; Niess, 2011; Nilsson, 2008; Van Driel, Veal, & Janssen, 2001), others may use new terms without mentioning PCK, such as professional knowledge for effective teaching, content knowledge for teaching (e.g. mathematical knowledge for teaching, Hill, Schilling, & Ball, 2004), or craft knowledge (van Driel, Verloop, & de Vos, 1998). There is also a recent discussion about incorporating technology into subject teaching, which is referred to as technology pedagogical and content knowledge (TPACK) (Koehler & Mishra, 2009; Mishra & Koehler, 2006; Mueller, 2010; Rohaan, Taconis, & Jochems, 2009; Schmidt et al., 2009).

Regardless of the disagreement and varied interpretations, PCK has been widely accepted as a construct for studying teaching competence. It is agreed that PCK is “an understanding of how particular topics, problems or issues are organized, represented, and adapted to the diverse interests and abilities of learners, and presented for instruction” (Shulman, 1987, p.4). PCK is a specialized knowledge, which is needed, held and used by teachers. It is what teachers know about subject matter being taught and how to make it accessible to students (Carter, 1990); and teachers draw upon PCK for planning lessons, assessing pupil’s understanding and making instructional decisions.

The clarifications and extensions of studying PCK are summarized in three sections: the nature of PCK, the structure of PCK and the validation of PCK. Representative models and projects are presented to reveal the nature of PCK; the structure of PCK summarizes the conceptualization of sub-components of PCK; and the validation of PCK discusses empirical evidence and attempts to verify PCK as an indicator for successful teaching. It is noticed that PCK is discussed both broadly
crossing different disciplines and specifically tied to a particular subject such as mathematics. Pedagogical content knowledge is a subject-specific construct that requires disciplinary specific explorations. However, the understandings about the construct itself and the methodology for investigating teaching effectiveness are generic (Hill et al., 2005). Therefore, insights from examining PCK in higher-grade levels of mathematics and multiple disciplines would inspire the study of mathematics teachers’ PCK for young children.

**Nature of PCK: Representative Models and Projects**

This section summarizes the work from multiple perspectives by illustrating representative projects. Understanding about the nature of PCK, its conceptual model and research methods are closely tied and aligned. Without investigating them as a whole, it can be misleading by losing valuable insights from a larger scope. Therefore, a few well recognized PCK models and projects are selected to trace the evolution of theories, methodologies and empirical findings about PCK. The discussion will first cover broadly across different disciplines, and further specify into mathematics education.

As part of the research project entitled “Knowledge Growth in a Profession” led by Lee Shulman at Stanford University, Grossman (1990) pioneered the conceptual and empirical investigation of PCK. PCK is defined as “knowledge required to teach a specific school subject” (Shulman, 1986). The study focused on the nature of PCK and its sources in secondary beginning English teachers, as well as the potential influence of professional coursework on PCK. Grossman summarized possible sources (i.e. knowledge bases) of PCK: apprenticeship of observation, subject matter knowledge
(SMK), teacher education and classroom experience. While each of these sources provides a distinct opportunity to develop the knowledge about teaching English, PCK is evidenced by conceptions of purposes for teaching subject matter (i.e., English), knowledge of students understanding, curricular knowledge and knowledge of instructional strategies (i.e., components of PCK) (See Figure 1).

![Figure 1. Model of Teacher Knowledge. Adapted and redrawn with permission from Grossman, 1990 (redrawn).]

To investigate the role of teacher education in the growth and transformation of SMK into PCK, contrasting case study method was applied to recruit participants. The researcher interviewed three beginning teachers with good SMK preparations (began teaching without formal teacher education) and three teachers enrolled in a five-year teacher education program. Although the educational backgrounds and SMK preparations were similar, the two groups of teachers demonstrated distinctly different understandings and beliefs about the purposes, curriculum, and students’ learning in English. For instance, teachers without teaching education were more likely to interpret the goal of
teaching the subject and students’ learning relying on their own experiences of learning English; while those enrolled in a teacher education program reflected deeply about how subject-specific courses impacted their understanding about the discipline and students learning the subject. For teachers without subject-specific pedagogy training, the lack of instructional strategies and unfamiliarity with pedagogy prevented content understanding from unpacking into successful instructions. In terms of understanding students, beginning teachers without professional trainings ideally viewed students as bright and motivated, while the other group of beginning teachers (who enrolled in teacher education program) believed that students were of average capacity and motivation (an understanding closer to the reality). These beliefs were found to further impact the teachers’ understandings about the appropriate content for particular English courses and how the content should be organized.

Despite the small sample size and the atypical educational and teaching backgrounds of participants, the study provides deep and rich insights about the nature and development of PCK in secondary English teachers. The investigation was conducted during a time when there was a decreased emphasis on the importance of subject-specific pedagogy in teacher certification. In alternative route programs, talented college graduates were encouraged to enter teaching without subject-specific professional training. Unfortunately, based on Grossman’s study, teachers with good subject understanding in alternative route program were interpreting students’ difficulties inappropriately. In contrast, graduates of teacher education held an appropriate understanding about students learning the subject and internalized the idea of
instructional scaffolding. Teacher education, particularly subject-specific pedagogy courses, plays an important role in the growth and transformation of beginning teachers’ subject matter knowledge. The findings questioned the assumptions that subject matter knowledge is sufficient for initial teaching professionals and the taken-for-granted belief that classroom experience by itself can replace teacher education.

Grossman’s conceptual model of PCK, as well as the striking finding about how belief of teaching a subject impacts instruction, has received many echoes. A number of researchers adopted the idea of knowledge sources (i.e., knowledge bases) of PCK and the frame of studying PCK by more specific components. For instance, Magnusson, Krajcik, & Borko (1999) proposed a pentagon model of PCK for science teachers. Similarly, PCK is believed to be a transformation of other domains of knowledge for the purpose of teaching, including subject matter knowledge, pedagogy and context, and there is a reciprocal relationship between the base domains and PCK. Magnusson and her colleagues further expanded and specified five components of PCK: (1) orientation to science teaching, (2) knowledge and beliefs about science curriculum, (3) knowledge about students’ understanding of science, (4) knowledge of assessment of scientific literacy, and (5) knowledge of instructional strategies. More specifically, (1) orientation plays a central role in framing PCK; (2) beliefs about science curriculum are further divided into goals and objectives as well as specific curricular programs and materials; (3) knowledge of students’ understanding includes understanding about prerequisite ideas and skills to learn a topic, students’ diverse approaches of learning, and difficult areas for students to learn; (4) assessment knowledge refers to understanding about what
dimensions to assess and what methods can be used to make assessment; and (5) instructional knowledge can be further divided into three levels: subject specific strategies, topic specific strategies and topic specific activities (See Figure 2).

Figure 2. Components of Pedagogical Content Knowledge for Science Teaching (the original pentagon model). Adapted with permission from Magnusson et al., 1999.
The pentagon model has been widely used in studying PCK with various revisions, especially in the field of science education. For instance, Park and her colleagues revised the pentagon model of PCK by addressing the relationships among PCK components in studying secondary science teaching (See Figure 3). While adopting the framework of PCK by Magnusson et al. (1999), the orientation of teaching a subject was considered as a subcomponent of PCK similar to Grossman’s original framework. Moreover, the integration among PCK components has been highlighted and investigated empirically (Park & Chen, 2012; Park & Oliver, 2008).

By recoding the frequency of connections between PCK components and the total connections each component had, PCK map represented the interactions among PCK components. The analysis revealed that knowledge of students’ understanding and knowledge of instructional strategies and representations played a fundamental role in the integration, and is likely to play a leading role in teaching effectiveness. This was evidenced by more extensive connections between knowledge of students’ understanding, knowledge of instructional strategies and other components. Knowledge of assessment and curriculum had the most limited connections with other components. In addition, didactic orientation of science teaching was found to direct knowledge of instructional strategies and inhibit connections among other aspects of effective teaching (revealed by subcomponents of PCK) (Park & Chen, 2012). These findings, together, led researchers to notice the significance of coherence among PCK components and the strength of individual components both conceptually and empirically.
Figure 3. Pentagon Model of PCK for Teaching Science. Adapted with permission from Park & Chen, 2012.

The pentagon model and its revisions have several significant implications. Firstly, although PCK is subject-specific and needs to be studied subject by subject (Shulman, 1986), the theoretical models and research methods can be applied across disciplines. The model about sources and components of PCK proposed by Grossman in secondary English teachers has been transferred to studies of science teachers’ content expertise. Within the science education camp, it also made success across different fields such as biology (Park & Chen, 2012). Secondly, by specifying the components of PCK, it provides opportunities to quantify content expertise. More and more projects have started investigations beyond case studies based on the pentagon model, increasing the
possibility of making inferences about the professional knowledge of teaching a subject in a larger scope. Thirdly, it has moved the research of content expertise from studying a subject level (e.g. biology) to a topic level (e.g. photosynthesis). “[S]elect the essential topics regarding subject matter learnt for a particular grade” (Van Driel et al., 1998) and identify the “grain size” for depicting PCK at a topic level is more precise (Van Driel, & Verloop, 2008). The grain size investigation has advanced our understanding about the nature of PCK from different perspectives.

The dynamic nature of PCK has also been explored beyond investigations on specific components of PCK and their relationships. Originally, Shulman proposed the linear model of pedagogical reasoning and action\(^2\) to illustrate the development of PCK and acknowledged “many of the processes can occur in different orders” (1987, p.19) in a vague way. The dynamic nature of PCK has also been recognized but not fully addressed in the Pentagon model (Magnusson et al., 1999) and its derivatives (e.g. Park & Chen, 2012). As a breakthrough, Cochran and her colleague (1993), proposed a unique modification of PCK by highlighting its constructive nature and nonlinear path of development. Similar to Grossman’s work, the influence of other types of teacher knowledge on the development of PCK was addressed. PCK is defined as teachers’ integrated understanding of pedagogy, SMK, student characteristics and the environmental context of learning. Moreover, the development of PCK is an active process of teaching and learning; therefore all aspects of knowing must be simultaneously developed. Based on these understandings, Cohen and her colleagues proposed the name of Pedagogical Content Knowing (PCKg) and conceptualized its developmental model

\(^2\) See Table 1.
for teacher preparation. It is suggested that the components of PCKg can begin with limited focus and become more elaborate; and the simultaneous integration of PCKg components results in conceptual change, which eventually produces new knowledge (i.e., PCK) distinctively different from what was constructed (see Figure 4).

Figure 4. A Developmental Model of Pedagogical Content Knowing (PCKg) as a Framework for Teacher Preparation. Adapted with permission from Cochran, et al, 1993.

The dynamic and constructive nature of PCK has also been examined empirically. For instance, professional development programs that focused on teachers’ reflections of students’ misunderstandings and making associations between a) content knowledge and students’ understandings, and b) content understanding and instructional strategies have found to improve teachers’ PCK (Heller et al., 2012; Smith & Neale, 1989). Functioning relatively independent from each other, the three aspects of PCK (content understanding, knowledge of students’ vulnerabilities and pedagogical understanding to address students’ needs) were reported to become significantly related to each other among a
group of early childhood teachers who participated in a one-year professional training. The training was designed to increase their content understanding and how to apply that knowledge in understanding students’ learning and curricular design (Melendez, 2008). These findings, together, not only confirmed the dynamic nature of PCK in general, but also suggested that PCK can be improved by well-designed professional trainings beyond regular teaching and reflection.

Another major branch of the inquiries is to explore and portray PCK. Berry, Loughran and their colleagues are leading a unique and prestigious group that studies science teachers’ PCK. PCK is understood as the knowledge that teachers develop about how to teach particular topics leading to enhance students’ understanding of the subject matter. Based on a belief that PCK is yet to be defined and explored, the original purpose of this work was not to define PCK precisely or conceptualize its components. Instead, it aimed to capture and explore a holistic and multifaceted picture of teachers’ PCK for particular science topics. Resource Folios were developed to represent PCK, which comprises two components, Content Representation (CoRe) and Professional and Pedagogical experience Repertoires (PaP-eRs). CoRe is structured around three aspects: 1) main content ideas associated with a specific topic; 2) teaching procedures and purposes; and 3) knowledge about students’ thinking. PaP-eRs are short narratives about the thinking and actions of an expert teacher in teaching a specific aspect of content knowledge (Loughran et al., 2004; Loughran, Gunstone, Berry, Milroy, & Mulhall, 2001; Loughran, Mulhall, & Berry, 2008). PaP-eRs provides insights about teachers’
pedagogical reasoning in the context of teaching missing from the CoRes; and multiple PaP-eRs are needed to meet the complexity of teaching and learning.

Accordingly, through framed questions around main concepts in teaching particular content areas (e.g. biology), small group interviews were conducted to capture CoRe at a collective level. Prompts used in capturing CoRe are questions such as “what ideas one intended to teach students about particular concept” (e.g. photo synthesis) and “why it is important”, as well as “difficulties in teaching the idea” and “specific ways of ascertaining students’ confusion around the idea.” Based on several small group discussions, big ideas for a particular topic were then summarized by prompting questions, and eventually became components of CoRe (See Figure 5). Meanwhile, PaP-eRs were collected through lesson observations and teaching procedures as individualized combinations of big ideas in CoRe. It suggests knowledge and enactment as two dimensions of PCK (Heller, Daehler, Wong, Shinohara, & Miratrix, 2012; Park & Oliver, 2008a). The enactment involves a decision making process which draws upon situational understanding about content and learners.

The framework of CoRe and PaP-eRs has become influential and the collective work has been published to stimulate science teachers’ (in-service and pre-service) PCK development. Similar to the revised work of the pentagon model, it sought to study PCK at a more specific level (topic). By capturing, documenting and portraying the PCK of experienced science teachers from a teacher’s point of view, it has been confirmed for the first time that the unique professional knowledge can be organized at both individual and collective levels. More specifically, big ideas related to a particular topic are shared at the
level of professional communities, while the enactment of instruction is more an individualized combination based on personal experience, beliefs, and orientations (Friedrichsen et al., 2009; Loughran et al., 2001). Similar work has been done to capture PCK in tertiary educators (Padilla, Ponce-de-Leon, Rembado, & Garritz, 2008; Rollnick, Bennett, Rhemtula, Dharsey, & Ndlovu, 2008).

![Figure 5. CoRe and Associated PaP-eRs. Adapted with permission from Loughran et al., 2004.](image)

**Note:** CoRe (Content Representation)
PaP-eRs (Pedagogical and Professional experience Repertoires)
Lines from the PaP-eRs represent the links to particular aspects of the CoRe

The conceptualization of CoRes and PaP-eRs also has implications for teacher development and professional training. Methodologically, content knowledge for teaching can be embedded in classroom practice and not necessarily accessible for
teachers to articulate (Barnett & Hodson, 2001; Carter, 1990). Teaching can occur unconsciously without realizing the rationale of instructional decisions. At the same time, PCK can be explicit by reflection and accessible when teachers are making lesson plans (Danielson, 1996). Substantial differences in PCK have been found among experienced teachers around the same topic areas (Henze-Rietveld, 2006). Through promoted questions and reflection, the CoRes and PaP-eRs tools are helpful in supporting teachers to articulate and discuss their understanding of teaching and learning a particular science topic and enhancing their professional knowledge in practice. The systematic way of illustrating PCK from interviews and classroom observations provides a framework for professional trainings aiming to improve PCK (and eventually teaching effectiveness).

In the field of mathematical teaching and learning, Liping Ma and Deborah Ball pioneered the investigation of teaching competence in secondary education. Based on a comparative study between 23 “above average” teachers in the U.S. and 72 Chinese elementary math teachers with a wide range of teaching experience, about 10 percent of the teachers interviewed demonstrated accomplished conceptual understanding for the purpose of teaching, which Ma suggested calling Profound understanding of fundamental mathematics (PUFM) (Ma, 1999). PUFM underlines the significance of broad, deep, connected and coherent conceptual understanding required for productive math teaching. Although subject matter knowledge, rather than PCK, was used, PUFM is profoundly pedagogical (and can be considered as a form of PCK). Teachers’ who developed PUFM are more likely to have a proper expectation for student learning, and further promote students’ conceptual learning.
Following a similar line of thought, Deborah Ball and her colleagues have done extensive studies and revisions in understanding teaching effectiveness through the lens of Content Knowledge for Teaching (CKT). One of Ball’s contributions was to further articulate the importance of identifying the unique knowledge necessary for teaching a subject effectively. Building on a practice-based approach, the theory is generated upon job analysis, which places an emphasis on “the tasks in teaching and the mathematical demands of these tasks” (Ball & Bass, 2003; Ball, et al., 2008). By analyzing the skills and knowledge required in mathematical tasks, e.g., common teaching activities such as showing students how to solve problems and checking their work, it has been found that the unique knowledge that teachers need includes the capabilities of 1) anticipating students’ misunderstandings; 2) analyzing, reasoning and justifying students’ reasoning processes flexibly; and 3) selecting the most appropriate instructional materials to promote learning (Ball & Bass, 2000; Phelps & Schilling, 2004; Ball, et al., 2008; Bush, n.d; Kulikowich, 2007).

Aligned to these understandings, mathematical knowledge for teaching (MKT) has been conceptualized to unpack and explore teaching competence in elementary and middle schools; and corresponding measurement scales have been developed. MKT is the knowledge and skills needed for teachers to carry out the work of teaching a subject (i.e., mathematics) (Ball & Bass, 2000; Phelps & Schilling, 2004; Ball et al., 2008). It is composed of two relatively independent knowledge constructs: content knowledge (CK) and PCK. CK contains common content knowledge (CCK, subject understanding held by mathematically competent adults), specialized content knowledge (SCK, content
expertise held and used only by teaching professionals) and horizon knowledge (awareness of how mathematical topics are related over different grade levels included in the curriculum). PCK includes knowledge of content and students (KCS), knowledge of content and teaching (KCT) and curricular knowledge (see Figure 6). In particular, horizon knowledge is important because it would help teachers to know how a series of mathematical ideas are related as children move on into the next grade level. Specialized content knowledge is unique for teaching professionals and is pure content understanding without making inference from students. Overall, the subcomponents of MKT are considered as relatively independent knowledge categories.

Understanding about the constitution of PCK has been enriched by investigations of MKT. It has been confirmed that MKT is a multi-dimensional construct, and that general mathematical ability does not fully account for the knowledge and skills entailed in teaching mathematics. In the MKT framework, SCK was not defined as a component of PCK, however, the idea that content knowledge necessary for teaching contains not only common content knowledge shared by capable adults but also knowledge to flexibly reason, analyze and justify students’ thinking, is consistent with Shulman’s original conceptualization of PCK. Knowledge of content and students and knowledge of content and teaching are consistent with Pentagon models and further specified in math teaching, which is reflected in the capability of anticipating common student misconception, picking up the most appropriate tasks to assess students’ understanding, and using various and proper representations to promote students’ understanding (Ball, Thames, & Phelps, 2008; Ball & Bass, 2003; Bush, n.d.).
In the meantime, although MKT is a multi-dimensional construct, the relative independence of MKT components is not well supported, and the distinctions between/among its sub-components remain open. For instance, given the hypothesis that SMK and PCK are two discernable constructs; SCK (a sub-component of SMK) should be independent from sub-components of PCK such as KCS. However, both cognitive interview and factor analysis revealed that the boundaries between SCK and KCS are blurry; suggesting that the boundaries are ill defined conceptually (Kane, 2007). Although SCK is defined as content understanding used for teaching based on pure flexible mathematical reasoning without considering students or pedagogy, the results suggest that the content understanding of mathematics itself is likely to be an inseparable element of the mathematical knowledge needed in teaching (i.e., PCK).
In summary, the notion of PCK has sparked a plethora of studies on the nature and development of teachers' pedagogical content knowledge. The theoretical models and research methods are shared and applied across disciplines at different levels; and the findings not only questioned the assumptions that subject matter knowledge or general pedagogical knowledge itself is sufficient for sound teaching, but also provided a picture about what professional knowledge looks like. It is suggested that PCK is implicit but also measurable, dynamic and constructive, individual and canonical, which can be specific at both domain and topic level. The development of PCK is a process of teaching and learning that is constructive and mutually beneficial. Moreover, PCK can be improved by well-designed professional trainings beyond regular teaching and reflection.

**Structure of PCK: The Conceptualization of PCK Components**

A closer look at the conceptualization of the components/structure of PCK is necessary before generalizing the empirical evidence and comparing findings from different projects. The conceptualization of PCK components reflects understanding about key aspects of effective teaching and has greatly facilitated the empirical investigation of PCK; however, it is necessary to clarify the plausibility of this approach. In fact, current debates about PCK are partially related to whether it is reducible to other types of knowledge that have been well studied (i.e., knowledge bases/sources, e.g. in Grossman, 1990), or it can be considered as a distinct type of knowledge. The answer to this question significantly impacts and guides the conceptual and methodological investigations of PCK, because it suggests whether we can most effectively study teaching competence by making inferences from its knowledge bases or by investigating
PCK directly. Studying the knowledge bases and making inferences would be enough to inform effective teaching if the knowledge bases can be considered components of PCK. Otherwise, PCK must be studied directly.

Some researchers blurred the distinctions between PCK and its knowledge bases in studying teaching effectiveness. For instance, there are researchers who believe that PCK and subject matter knowledge (SMK) overlap. SMK refers to facts and concepts in a domain, why they are true, and how they are organized (Shulman, 1986). SMK undoubtedly plays a fundamental role in developing PCK, however, there is no conclusion yet whether it is a prerequisite of PCK, i.e., a different and separable type of knowledge from PCK (e.g. Fischer, 2005; Friedrichsen et al., 2009; Rollnick et al., 2008) or a co-developing element of PCK (e.g. Ball, Themes, & Phelps, 2008; Heller, Daehler, Shinohara, & Kaskowitz, 2004; Van Driel, et al., 1998). It is argued that PCK, “by its nature, contains both subject matter knowledge and general pedagogical knowledge” (Marks, 1990, p. 8), which could be considered as the pedagogical aspect of SMK (Veal & MaKinster, 1999). For these researchers, PCK is not distinguishable from SMK, therefore it is more reasonable to study SMK but refine its definition.

However, the majority of the researchers believe, along with Shulman, that PCK is a transformation of different types of knowledge bases (e.g. Wilson, Shulman, & Richert, 1988). Analogous to chemical reactions, although the generation of the product relies on its original materials, the complex is no longer any of the original substances (Gess-Newsome & Leaderman, 1999). Likewise, the development of PCK is an amalgam of many knowledge bases, however, once it is formed, it is no longer pure
subject understanding or pure pedagogy. It is in this sense that “[j]ust knowing the
content well was really important, just knowing general pedagogy was really important
and yet when you added the two together, you didn’t get the teacher” (Berry et al., 2008).
This understanding implies that PCK must be studied directly and cannot be inferred
from its knowledge bases.

The question then moves to what should be defined as the components of PCK.
The arguments underline the inclusion/exclusion of constituent components and their
specific definitions. Originally, Shulman (1986) mentioned two key aspects of PCK: 1)
understanding students’ prior knowledge, including common misunderstandings and
learning difficulties; and 2) capability of “transforming” the subject matter knowledge by
using multiple representations and instructional strategies to adapt to the diverse needs
and interests of students. Shulman didn’t claim explicitly that these are the two
components of PCK, while researchers in his team further specified the components of
PCK and investigated empirically (e.g. Grossman, 1990).

Most of the research groups agree with Shulman about the two core aspects and
their significance for productive teaching (see review Kind, 2009). Similar lines of
thought can be found in Dewey (1902) (who calls the transformation process
“psychologizing”), Ball (1990) (who used the term “representation”), and Veal &
MaKinster (1999) (who described it as “interpretation” and “specification”). Further
efforts have been made to refine the understanding of the two core elements in different
subject areas (e.g. Adler & Davis, 2006; Ball, Themes & Phelps, 2008; Carter, 1990;
Cochran et al., 1993; Doyle, 1990).
There are also additional types of knowledge being introduced and regarded as key components of PCK (Ball et al., 2008; Gess-Newsome, 1999a; Grossman, 1990; Park & Oliver, 2008a; Smith & Neal, 1989; van Driel, et al., 1998; Wilson & Shulman, 1987). For instance, content-specific orientation (or teaching orientation) has been explored by researchers studying teachers’ content expertise, but also questioned (Abell, 2008; Friedrichsen et al., 2009; Grossman, 1990; Magnusson et al., 1999; Park & Chen, 2012). To make it even more complex, researchers are using the same or similar terms to imply different constructs, or creating new terms by restructuring old terms.

Regardless of the differences in terminology and operational definitions, content understanding, content-specific cognition of learners, content-specific pedagogy, teaching orientation, and contextual knowledge are at the center of the debate. It is noted that each component of PCK will be discussed broadly and then applied to mathematic. To start, content understanding is knowledge about the subject one teaches. There is no doubt that one must be knowledgeable about the content he/she teaches. However, how does one interpret the findings about non-significant or even negative relationships between teachers’ content knowledge and student outcomes? Why was there a threshold of teachers’ subject matter understanding (i.e., the number of advanced math courses taken) that contributed to students’ learning (Monk, 1994)? Why were teachers with advanced subject knowledge in algebra more likely to believe that word problems are more difficult than symbolic problems for novices (Nathan & Petrosino, 2003, cf p.376, Hill, Ball and Schilling, 2008), the latter of which is in fact more abstract and harder to solve? Ball
&Bass (2003) suggested that knowledge gets compressed as one obtains more advanced understanding, which then prevents knowledge unpacking.

Another solution is to change the definition and evaluation methods of content understanding. It is in this sense that differentiating traditional understanding and evaluation of subject matter knowledge and the content knowledge required for teaching has been highlighted. Taking studies of mathematical teaching as an example, Skemp (1976) defined two types of mathematical understanding: relational understanding, knowing what to do and why, is markedly different from instrumental understanding, knowing what to do but not why (c.f. Hutchison, 1997). Similarly, Ball (1990) articulated two dimensions of teachers’ content knowledge for mathematics: capability of executing operations vs. representing operations precisely for students. That is to say, the capability of doing math (e.g. calculating fractions) is different from the understanding of the operational rules (i.e., representations) necessary for teaching.

Krauss and his colleagues (Krauss, Baumert, & Blum, 2008) proposed a four-level hierarchical classification of mathematical knowledge: 1) academic research knowledge generated in higher education; 2) a profound mathematical understanding of the mathematics taught at school; 3) a command of the school mathematics covered at the level taught, and 4) everyday mathematics knowledge adults retain after school. Despite the dissimilarities in categorization and description, the importance of content understanding beyond academic knowledge, that is, apart from being able to “do” math, has been recognized.
In a more recent effort, Ball and her colleagues (Ball et al., 2008) also highlighted and investigated different types of knowledge used in teaching elementary mathematics. Mathematical knowledge for teaching (MKT) addresses the differences between subject understanding held by mathematically competent adults and specialized content knowledge used only in teaching based on flexible mathematical reasoning. The definition shared insights with “profound understanding of fundamental mathematics (PUFM)” (Ma, 1999), which involves coherent understanding of content and how to communicate with students for the purpose of teaching and learning.

**Content specific knowledge about learners and learning** was considered as an indispensable foundation of PCK. As suggested by the pedagogical reasoning and action model, “comprehended ideas must be transformed in some manner if they are to be taught. […] [And the key is to] think one’s way from the subject matter as understood by the teacher into the minds and motivations of learners” (Shulman, 1987). Shulman (1986) viewed such knowledge composed of “an understanding of what makes the learning of specific topics easy or difficult: the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons” (p.9). Although the labels can be different, the idea of “students’ prior knowledge”, and gauging prior knowledge to infer learners’ “misunderstandings and learning difficulties” has been adopted by many researchers (e.g. Koehler & Mishra, 2009; Magnusson et al., 1999; Niess, 2008; Park & Oliver, 2008). In mathematical studies, Ball and her colleagues defined knowledge of content and students (KCS) as “content knowledge intertwined with knowledge of how students think about,
know, or learn this particular content” (Hill et al., 2008, p.375); and Clements & Sarama (2004; 2007) named content knowledge about learners and learning “common learning paths of knowledge development.”

**Content specific pedagogy** is frequently mentioned in the PCK models. Following Shulman’s original insights, a broad approach of instruction will not result in productive learning. Employing global principles may fail to capture the important features of subject teaching (Leach & Moon, 2000). Instead, “methods and activities within the context of subject matter” tied to “specific purposes and goals of instruction” are required (Brophy, 2001, p. 1). Magnusson et al. (1999) further categorized knowledge of instructional strategies at two levels: subject-specific strategies and topic-specific strategies (including both representations and activities). Subject-specific strategies are broadly defined as approaches applicable in teaching one subject (e.g. mathematics) as opposed to others (e.g. history). Topic specific strategies are pedagogies more narrowly applied to particular topics within the subject area. It refers to knowledge about representations and activities that can facilitate students’ learning around specific topics.

In particular, the significance of representations has been highlighted. Representations refer to “a wide variety of ways to capture an abstract concept or relationship” (NCTM, 2000, p.206), which can be 1) enactive, such as manipulative objects; 2) iconic, such as graph, table and picture; and 3) symbolic, such as written language (Bruner, 1966). Mathematical representations are media that display quantitative information (Hill, Charalambous, & Kraft, 2012), including symbols, equations, words, pictures, tables, graphs, manipulative objects, and actions as well as
spoken words and mental, internal ways of thinking about a mathematical idea (NCTM, 2000). As powerful thinking tools, representations help students to organize, record, and communicate math ideas, to select, apply and translate among representations to solve problems, and to model and interpret physical, social and mathematical phenomenon (NCTM, 2000).

**Teaching orientation** refers to educators’ overarching conceptions of teaching a particular subject (Grossman, 1990). It includes the purposes and goals for teaching a subject (e.g. mathematics) at a particular grade level (Magnusson et al., 1999). Compared with goals and objectives awareness in curricular understanding, it is much broader and grounded in a framework used for organizing teachers’ knowledge (Grossman, Wilson, & Shulman, 1989). In other words, it is a complex fusion of belief and values. Take high school math teaching as an example, teaching orientation means educators’ opinions about what is the discipline (i.e., mathematics), what are the most important things for high school students to learn about mathematics, how so and what is the best approach to achieve this purpose. Orientation of teaching exerts a powerful impact on how teachers perceive teaching and learning specific content and ultimately their teaching practice. It guides instructional decisions such as daily objectives, selection of curricular materials and methods of assessment (Borko & Putnam, 1996). Acceptance about what is the most valuable aspect of a subject for students to learn, as well as the best instructional approach to achieve the goal, substantially impacts teaching.

**Contextual knowledge (CxK)** is considered a crucial knowledge base for PCK. It may involve one or more type of knowledge about curriculum, school and community
Regardless of the inclusion and exclusion of certain types of knowledge, the importance of understanding curriculum and educational context has been highlighted for successful teaching. Curriculum knowledge, another type of contextual understanding, is knowledge about what should be taught to a particular group of students. Curricular understanding involves knowledge of learning goals and objectives and articulation of relevant topics within a specific subject; it also includes information about what students have learned in the previous levels and what they are expected to learn in higher grades (Grossman, 1990; Magnusson et al., 1999). Knowledge of curriculum appears in PCK because it reflects the organizing structure of subject knowledge being taught, which is salient for fruitful instruction. Awareness of how topics are arranged and possible utility of curriculum resources influences teachers’ evaluation of learners’ needs and scaffolding decisions, although understanding about learners’ developmental competencies and students’ process is not always included as part of curricular understanding.

The significance of acquaintance with school and community is also recognized as an inseparable component of educational contextual knowledge. Teaching and learning occurs within particular historical, social, and cultural contexts (Vygotsky, 1978; Rogoff, 1990). Teaching knowledge and content organization is contextualized, which is tremendously influenced by the school district. Understanding the context in which learning occurs (Ball et al., 2008; Ma 1999) would facilitate a group of pupils in learning specific content. This consideration is consistent with Bronfenbrenner’s argument that development must be considered within the whole ecological system (Bronfenbrenner,
1984). That system can be regarded as educational resources students are embedded in, and partially overlaps with Shulman’s proposal of “characteristic, knowledge of educational contexts, and knowledge of educational ends, purpose, and values, and their philosophical and historical grounds” (Shulman, 1987), a knowledge base needed for teaching professionals.

There are also some discussion about the relationships among PCK components, regardless of the disagreement in the identification and definition. Given the dynamic nature of PCK, how do these types of knowledge or components of PCK relate to one another? Take content understanding, knowledge about learners and pedagogy as an example, the components of PCK can be relatively independent and develop unevenly without coherence or integration. In Bowman and her colleague (1982)’s study about early childhood teachers’ knowledge of mathematics, teachers all demonstrated some mathematical knowledge expected to teach for young children. Unfortunately, they were not able to correctly identify students’ math capabilities. In Grossman (1990)’s study, the understanding of content itself won’t automatically transfer to an understanding of learners’ cognition. New teachers with adequate subject matter understanding but insufficient preparations to deal with student needs were found to have difficulty making decisions about optimal instructional solutions. Further, professional development training aiming to increase trainees’ knowledge about students’ understanding has been approved to drive teachers to pay more attention to thinking from students’ perspective. Unfortunately, with this increased understanding, the majority of them still struggled with how to respond to students’ misconceptions, or simply ignored them without stressing
those teachable moments (Smith & Neal, 1989). These findings, together, suggest that different aspects of PCK can function independently and a need for intentional integration in teacher training and preparation.

Although the different aspects of knowledge can function relatively independent from each other, the majority of researchers believe that teachers’ content understanding is the foundation for understanding students’ learning and application of pedagogy. Originally, Shulman (1987) highlighted content understanding as a driving force of instruction in the pedagogical reasoning and action model, although he admitted that for early childhood education, understanding of young children might serve as a starting point for lesson design. Wilson et al. (1987) observed that teachers with more content understanding were more likely to notice misconceptions, effectively deal with general class difficulties in the content area, and correctly interpret students’ insightful comments. Ball (2000) replicated and confirmed this observation. To accommodate diverse interests, understanding, abilities and experiences of students, it is essential to develop a special body of knowledge to interpret students’ learning.

Content understanding is also essential for pedagogical awareness and reasoning. With an increased understanding of the subject, teachers are able to think about principles underlining procedures more flexibly and concisely (Ball & McDiarmid, 1990; Capraro et al., 2005; Feiman-Nemser & Parker, 1990). On the other hand, weak subject matter preparation leads to rigid pedagogy. A shaky understanding of a mathematical concept (e.g. multiplication) would lead to a reliance on algorithmic approaches (i.e. “skill and drill”) or a limited instructional repertoire (Feiman-Nemser & Parker, 1990). Therefore,
effective teaching investigations should underline content understanding intermingling with pedagogy.

Overall, teaching is complex and demanding, requiring multiple aspects of knowledge. Educators implicitly and explicitly incorporate their understanding about the subject matter, pedagogy, students, educational contexts and teaching orientation into teaching. It is necessary to simultaneously integrate understanding about content, students, pedagogy, teaching orientation and contextual knowledge for effective teaching. However, different aspects of PCK can be relatively independent and may not develop in chorus or evenly. In the meantime, it is likely that content understanding plays a fundamental role in the formation of PCK. To achieve teaching effectiveness, sufficient content understanding needs to be integrated with multiple aspects of knowledge for teaching.

Validation of PCK: PCK and Teaching Effectiveness

Empirical evidence is needed to prove how the differences in PCK impact teaching and learning. The proposal of PCK aims to find an alternative to indicate quality subject teaching. Ironically, there is a lack of clarification regarding the relationships between PCK, teaching practice and student outcome. The disagreement in developing a shared operational definition and the corresponding measurement tool to assess PCK (Gess-Newsome, Cardenas, & Austin, 2011) made it difficult to compare results from different groups or make further inferences. To date, few attempts have been accomplished to verify this conceptualization but with mixed results.
Regardless of the inconsistent operational definitions for PCK and its components, the relationship between one aspect of PCK (i.e., its subcomponent) and teaching effectiveness has been investigated in multiple disciplines. For example, variations of content understanding within a subject have been found to influence teaching practice considerably (e.g. Ball, 2000; Carlsen, 1999; Grossman, 1990; Lederman & Gess-Newsome, 1992; Ma, 1999; Rollnick, Bennett, Rhemtula, Dharsey, & Ndlovu, 2008; Smith & Neal, 1989). As well, pedagogical skills are found to be essential for productive teaching. In studying a specific subject, students are more likely to learn when teachers give sufficient feedback, provide scaffolding, hold appropriate expectations for learning outcomes, and assign appropriate homework to students about the subject (Brophy, 2001; Evertson, Emmer, & Brophy, 1980; Evertson, 1981).

Understanding about students’ content-specific learning has also been approved to promote optimal instruction, refine professionals’ content understanding and boost pupils’ learning. Both teaching practice and students’ learning were improved by professional development that equipped teachers with understanding about how students learn specific content (Carpenter, Fennema, Peterson, & Carey, 1988; Saxe, Gearhart, & Nasir, 2001). In a longitudinal study on professional development upon cognitively guided instruction (CGI) model, Carpenter and colleagues (1989) reported facilitated changes in productive instruction by arming teachers with knowledge about children’s mathematical thinking. Teachers were taught to engage students in a variety of activities based on their prior knowledge and encourage students to comment and reflect on their mathematical thinking. The shift was directly related to students’ learning gains in both
conceptual understanding and problem solving (Carpenter, Fennema, Franke, Levi, & Empson, 1999; Elizabeth Fennema et al., 1996).

Empirical evidence also exists between teaching orientation and quality of teaching (e.g. Borko & Putnam, 1996; Grossman, 1990; Park & Chen, 2012). Teachers with similar subject matter preparation, but different in orientations were found to apply quite varied pedagogical practices (Grossman, 1990). The rigidity of teaching orientation was reported to impede teachers with substantive content knowledge from integrating students’ understanding in instructions (Park & Chen, 2012). Relatively resistant to change, teachers conceive subjects differently and bring this belief into teaching consistently.

There are some explorations about the impact of balance and integration among PCK components on teaching effectiveness. Coherence and integration correspondingly refer to the strength of PCK components and the capability of incorporating and connecting components of PCK. For example, Melendez (2008) explored the profile of content expertise in early childhood teachers, content understanding seemed to function independently from knowledge of students’ learning and pedagogical understanding to accommodate students’ needs within the content. A teacher with sufficient knowledge may not be able to incorporate the multiple aspects of understanding in teaching. With teaching experience and reflections, the subcomponents of PCK may become more integrated over time (Friedrichsen, Van Driel, & Lankford, in preparation; Friedrichsen et al., 2009). In contrast, experienced science teachers serving as mentors revealed highly integrated PCK components (Lankford, 2010; West, 2011). The imbalance and lack of
connection has also been found to prevent successful teaching, due to rigid instructions (Park & Chen, 2012). Therefore, the lack of coherence may impede the instruction from efficiency.

The balance and integration among PCK components would influence the quality of teaching, suggesting a necessity to consider all aspects of PCK simultaneously in the investigation of linking PCK to other indicators of teaching effectiveness. There are, however, very limited large-scaled investigations on the association between multiple aspects of PCK and teaching effectiveness. Among the scarce studies, even fewer reported promising evidence for PCK as an indicator for successful teaching. For instance, Baumert and his colleagues (Baumert, Kunter, Blum, Brunner, Voss, Jordan, Kusmann, Krass, Neubrand, & Tsai, 2010) conducted a study in Grade 10 mathematics teachers in Germany. PCK and SMK (i.e., subject matter knowledge) were considered distinguishable from each other and assessed to indicate teachers’ knowledge through open-ended questionnaires. PCK was defined as knowledge of mathematic tasks, students’ thinking, and multiple representations. There were positive relationships between both indicators of knowledge and students’ learning gains, mediated by teaching quality measured by cognitive activation (i.e., students’ thinking level and support for individual students). Furthermore, PCK was much more predictive for both classroom teaching and student outcomes, compared with SMK. More specifically, PCK scores predicted better quality of teaching, and explained 39% of variance in students’ learning gains over a year. The substantial associations between PCK, teaching quality and
students’ achievement over SMK suggested PCK could be an effective indicator for quality teaching and productive learning.

However, not all studies are in favor of PCK over SMK or GPK (i.e., General Pedagogical Knowledge) to predict quality teaching or students outcome. For instance, Gess-Newsome, Cardenas, & Austin (2011) evaluated a two-year professional training aiming to boost up teaching effectiveness and students’ learning for middle school biology teachers. The training highlighted SMK, GPK and PCK and defined PCK as a three-component construct including content understanding, pedagogical knowledge and contextual knowledge, measured through planning and reflection. The results, surprisingly, indicated that GPK was the only predictor for teacher practice, and SMK was the only indicator that explained variance in students’ learning gains. In contrast with the conceptualization, PCK was not proved as an effective indicator for productive teaching or learning based on this study.

There are some empirical studies exploring the association between PCK, teaching practice and students’ outcome in primary mathematics education. For instance, Ball and her colleagues found mathematical knowledge for teaching (MKT) significantly predicted student achievement after controlling for co-variants such as SES (Hill et al. 2005). The group also reported positive relationship between MKT and classroom instruction, which further predicted students’ achievement. However, there was no significant association between PCK and students outcome in the following study (Hill & Schilling, 2007). Kersting and her colleagues assessed elementary teachers’ PCK in mathematics by providing several short classroom teaching videos and asking teachers to
describe “how the teacher and students interacted around the mathematical content” (Kersting, 2008; Kersting, Givvin, Sotelo, & Stigler, 2010; Kersting, Givvin, Thompson, Santagata, & Stigler, 2012). Responses were then rated for understanding of content, students and pedagogy. Significantly positive relationships were found between teachers’ PCK and their MKT scores. However, MKT scores did not predict students’ learning gains in the study; positive associations were only found between PCK scores and students’ gains.

These mixed results suggest that the validity of PCK (demonstrated by MKT scores) has not been fully confirmed. By analyzing the associations among knowledge of a subject, teaching practice and students’ gains, the results provided some evidence for the predictive validity of the theory and its measurement. It is possible though, in the validating study conducted by Kersting and her colleagues, the non-significant relationship between MKT and students’ outcomes may be due to the use of different students’ assessment tools in the two research groups, as Kane (2007) commented on the validity study conducted by Ball’s group (Hill et al. 2005). Furthermore, by focusing on teachers’ flexibility in analyzing students’ unusual solutions or reasoning, MKT does not isolate PCK from subject matter knowledge. Therefore, it is impossible to make inferences about the unique contribution of PCK to teaching and learning from pure subject matter understanding.

In early math education, there is a scarcity of large-scaled studies to investigate the relationships between teachers’ PCK, teaching quality and students’ learning gains. Few projects have initiated studies to address the unique content knowledge required for
teaching and its association with teaching practice and students’ gains in early childhood years, let alone foundational math. Among the limited investigations, McCray & Chen (2012) applied scenario-based interview to explore preschool teachers’ PCK in mathematics and found substantial relationships between PCK and effective teaching. There were 22 preschool teachers and 113 Head Start preschoolers in the study. Mathematical language used during the instructions and students’ performance on standard tests was applied to indicate productive teaching. The sample size was small and the predictive coefficients were not strong, the indicator of teaching quality (i.e., math language) also limits its generalizability. It is possible that some unmeasured constructs such as teaching belief may be the key co-variable to account for the prediction. However, the significant association between PCK and both indicators of quality teaching is encouraging.

In summary, although PCK is conceptualized as a key indicator of professional knowledge for sound teaching, the hypothesis hasn’t been fully explored or validated, especially in early mathematics. Because there is no consensus on the definition of PCK, its essential components and development, it is difficult to compare results from different groups or make further inferences and generalizations. Applying the conceptualization of PCK summarized in the previous section revealed more empirical investigations on its sub-components. Given the significance of coherence and integration of PCK components, the results are not strong enough evidence to convince the conceptualization of PCK. Further, among the few existing large-scaled investigations attempted to link PCK with teaching effectiveness, little is known about how PCK would apply in early
math education. The indicators used for teaching quality also differ from math language used (McCray & Chen, 2012) to cognitive activation (Baumert et al., 2010). More large-scaled studies are needed to not only validate the assessment tool, but also to confirm the theoretical position that PCK is key to productive teaching.

**Conceptual Model of the Study: PCK in Early Mathematics (PCK-EM)**

To answer the question about what makes teaching foundational mathematics effective, the current study takes the approach of educational production function; which considers teachers’ knowledge a type of educational resource for students’ learning and addresses the subject specific nature of teaching. Further, the author agrees with Shulman that the content understanding needs to be redefined for the purpose of teaching and learning. Exam scores for certificate or advanced college classes do indicate some aspects of understanding; however, they are not closely aligned to the knowledge used in teaching a subject to a particular age group of students. As reflected in PCK, there is a special type of understanding that differentiates teachers from common adults, content experts, and veteran teachers from novice teachers.

More specifically, PCK features the simultaneous integration of subject understanding and pedagogical reasoning embedded in the instruction of a specific age group of students. It is an understanding that incorporates students and pedagogy into the context of subject teaching. PCK theory provides a unique lens to study early math teaching competence. First, PCK points out the need to differentiate how to do math procedurally from knowing the underlying rules and principles of math problems. Early childhood is a field that is reluctant to set up subject specific standards, and early math
was misunderstood as simple or insignificant in the past. Addressing the complex mathematical concepts, not the capacity of mechanically doing simple math would promote higher level thinking among young learners. Second, PCK highlights the importance of incorporating subject understanding into students’ learning. Early childhood teachers are experts regarding young children’s developmental characteristics and individual differences. However, with subject-general trainings, understanding about young children’s cognitive development may not apply to how young learners learn math. Third, previous studies on PCK have confirmed the significance of subject-specific pedagogy (Grossman, 1990). The lack of subject preparation and subject specific pedagogy training may overlook this important aspect in teaching foundational mathematics.

Building on the diverse empirical investigations of PCK, as well as the uniqueness of early math education, the current study proposes its own conceptual framework of PCK in early mathematics. PCK is a continual reconstructing of SMK for the purpose of teaching (Gudmundsdottir, 1987), demonstrated by a flexible content understanding to tailor students’ diverse backgrounds and needs (Buchmann, 1984). There are three dimensions of PCK: (1) **What**: content understanding aligned to the grade level(s); (2) **Who**: content-specific knowledge of learners’ cognitions and learning patterns; and (3) **How**: content-specific pedagogical knowledge. To teach early math effectively, teachers must coherently integrate the three aspects: understanding of foundational math, knowledge of young children’s mathematical learning patterns, and math-specific teaching strategies and representations (See Figure 7 for illustration of
pedagogical content knowledge in early mathematics, PCK-EM). The complexity of foundational math requires teachers to acquire a sophisticated understanding of underlying mathematical ideas and present them precisely to young children.

![Conceptual Model of Pedagogical Content Knowledge in Early Mathematics (PCK-EM)](image)

Figure 7. The Conceptual Model of Pedagogical Content Knowledge in Early Mathematics (PCK-EM).

**What: Content Understanding Aligned to Specific Age Groups**

*What* is an understanding of foundational mathematics for teaching young children. A teacher demonstrates understanding of foundational math through (1) *Breadth*: awareness of the relationships and connections among mathematical concepts; and (2) *Depth*: capability of “deconstructing” foundational math into its complex underlying ideas that young children need to learn. This definition is aligned with the standards recommended by National Council of Teachers of Mathematics (NCTM, 2000) and Common Core State Standards Initiative (CCSS-M, 2010), the horizontal knowledge addressed in MKT theory (Ball et al., 2008); the profound mathematical understanding of
the mathematics taught at school (Ma, 1999; Krauss et al., 2008), and the relational understanding highlighted by Skemp (1976) and Ball (1990).

At a broad level, teachers must be aware of mathematics concepts that should be covered in teaching young children. Based on understanding about young children’s learning capability and the importance of foundational math competence for their future success, NCTM (2000) recommended five content strands to be taught in early math: number and operations, measurement, algebraic thinking, geometry and spatial sense, and data analysis. CCSS-M (2009) uses a different framework but covers similar content domains. In contrast with taken-for-granted understanding, early math is much more than counting and shape recognition. Without awareness of the diverse mathematical concepts that should be taught, teachers will not give sufficient time and effort in addressing the range of ideas that are crucial for children’s future achievement.

In the early childhood teaching, there are usually multiple foundational math ideas involved in one activity. For example, young children love counting and making comparisons. Consider a group of children who are trying to compare whether more students would like to sing first or have snack first during the group time, a typical scenario in a preschool classroom. This seemingly simple activity actually involves at least three different math concepts: grouping, counting, and data analysis. Sort all students into two groups, count the total number of each group, and then make comparison about the quantity. Compared with higher-grade levels in which teaching may be specialized in a single domain such as probability, foundational math teaching is unique by involving several content domains simultaneously. Because mathematical
concepts and ideas are used synchronously, having a broad understanding about foundational math will allow teachers to cultivate math thinkers more effectively.

Understanding the relationships between and among math concepts is also of vital importance for sound teaching. There are coherent connections among mathematical ideas and concepts by a limited number of key principles. For instance, equal partitioning is one of the key principles for many math concepts and ideas. It lays the foundation for division (fraction), number sense and measurement. This is because 1) division is a type of equal partitioning based on attributes such as area and capacity; 2) the numeric property of number system assumes the equal differences between adjacent numbers and its extended patterns; and 3) measurement involves equal distance or other attributes between units. All are based on units upon equal partitioning. Making connections among these concepts would provide students a coherent picture about the knowledge system. Therefore, teachers with a comprehensive understanding and the capacity to see the connections among math concepts are more likely to design meaningful lessons, take advantage of incidental teachable moments, and scaffold students’ higher-level thinking.

Meanwhile, teaching professionals must possess a deconstructive understanding of complex math ideas that young children have to learn. Teaching requires making decisions based upon deep knowledge of content and children’s thinking (Ginsburg & Ertle, 2008). Mathematical activities young children engage in are embedded with complex ideas (Ginsburg & Ertle, 2008). The depth of conceptual understanding, i.e., the level of sophistication, needs to be unpacked in the context of early math instruction (Kilpatrick, Swafford, & Findell, 2001). In fact, foundational ideas have become such a
fundamental and integrated part of daily life that many adults take them for granted. They are also less likely to be examined in high school or college mathematics classes, when more focus is placed on advanced mathematics. Therefore, it is a must for teachers to help young children reason, reflect, apply and eventually grasp the concepts (Ball, 1990; Skemp, 1976). Without a sophisticated understanding of the underlying complex math ideas and the capability to unpack them, a teacher will not be prepared to help young children to learn these concepts.

Early Math Collaborative recommended 26 Big Ideas of mathematics to unpack the complex underlying mathematical ideas for early childhood educators (Erikson Institute’s Early Math Collaborative, 2013). The big ideas of mathematics are “clusters of concepts and skills that are mathematically central and coherent, consistent with children's thinking, and generative of future learning” (Clements & Sarama, 2009). In other words, there are key mathematical concepts that lay the foundation for lifelong mathematical learning and thinking. Consider a small group of kindergarteners using uni-cubes to learn how many ways they can make five. This seemingly simple activity actually contains complex ideas that young children need to learn and the teacher needs to unpack: a quantity (the whole) can be split into equal or unequal sets, and the parts can also be combined to form the whole (part-part-whole relationship) (Erikson Institute’s Early Math Collaborative, 2013). Understanding the Big Ideas of foundational math can help teachers move from seeing students’ learning procedurally (i.e., either mastering the skill or not) to what students really understand.
There are several concepts that are related to the big ideas of number composition and decomposition: a) there are various combinations that will produce five (e.g. 1+4=5, 2+3=5); b) the total stays the same regardless of the arrangement (i.e., the conservation concept, e.g. 2+3=4+1=5); and c) there are patterns of addends among different combinations of the same total quantity (i.e., the addends in different combinations increase and decrease by pattern, e.g.1 and 4, 2 and 3). Each of these ideas is complex, abstract and crucial to children’s development of a flexible and useful knowledge of addition. Teachers must be familiar with them in order to make each idea explicit. Together, knowing the big ideas of key math concepts and their relevant ideas would enable teachers to further students’ higher level thinking beyond mechanical memorization of facts.

**Who: Content-Specific Knowledge of Learners and Their Learning**

*Who* refers to knowledge about young children’s mathematical learning patterns. It is defined as understanding of young children’s learning progressions/trajectories of mathematical concepts (i.e., learning path) and likely misunderstanding and learning difficulties. More specifically, (1) understanding about learning path consists of two elements: prior knowledge (understanding of young children’s prior knowledge in learning specific mathematical concepts) and knowledge extension (awareness of relevant mathematical concepts that can be extended in learning specific mathematical concepts). (2) Misunderstandings refer to knowledge of students’ likely misunderstandings and learning difficulties around specific math content. Elaborated understandings of what students are like as learners, how they think about particular
topics, what comes before and next around specific mathematical concepts are essential for successful teaching. By emphasizing students’ learning characteristics embedded within specific content, this approach is in line with the consideration of developmental paths within mathematics (Sarama & Clements, 2004; 2007). The highlight of learning difficulties and misunderstanding aligns with one of the key aspects of PCK originally proposed by Shulman (1986), as well as the definition of knowledge of content and students (KCS) identified by several different projects (e.g. Ball et al., 2008; Magnusson et al., 1999; Park & Oliver, 2008).

Early childhood teachers’ understandings about young children have been vastly impacted by Piaget. Based on Piaget’s theory of cognitive development, children from three to nine are at the stages between preoperational and concrete operational. They begin to use symbols to represent objects, understand concepts like counting, and classify objects according to similarity in certain attributes. They also start to understand that one object may have different attributes, and demonstrate some organized and logical thoughts, although thinking tends to be tied to concrete reality (Piaget, 1969). Knowledge of these characteristics of children’s thinking allows a teacher to adjust expectations, plan lessons, and interpret students’ responses.

However, familiarity with cognitive developmental stages and individual differences can only provide a broad sense about how children develop. To teach foundational math effectively, it is necessary to interweave knowledge of students within the context of the subject (i.e., mathematics). This is because young children’s math thinking follows predictive developmental paths. Activities based on this understanding
would therefore be developmentally appropriate and effective (Sarama & Clements, 2008). A teacher must develop an understanding about not only how mathematical concepts are related to each other, but also in what order children are likely to grasp the ideas. For instance, to engage in the activity of number composition and decomposition, it is essential for teachers to know that they must first prepare students with not only counting skills, but also experience of taking away, adding together objects and how it relates to the quantity. Understanding about students’ prior knowledge serves as a starting point for lesson planning and scaffolding.

In a similar manner, to further students’ understanding, it is crucial for teachers to know how the concepts are extended as children proceed through their learning. Returning to the example of number composition and decomposition, developmentally, understanding of the part-part-whole relationship provides a conceptual foundation to solve missing-addend word problems in later years (Baroody, 2000), which is a precursor of sophisticated understanding of subtraction/addition. It is therefore necessary for teachers to know the conceptual alignment between the part-part-whole relationship and subtraction/addition understanding, as well as how missing-addend word problems would link the two. These understandings provide foundations to scaffold children moving forward in their conceptual learning.

Additionally, it is critical for teachers to be cognizant of students’ misunderstandings and learning difficulties within specific topics. Errors are an unavoidable and necessary part of learning; mistakes are also a window to students’ thinking that provides valuable teaching opportunities. For example, in learning number
composition and decomposition, young learners might not understand that the parts they created from one whole can be re-assembled and counted together to reach a single number. Instead, they may count each separately and not see the relationship between the two parts and the whole they begin with. Understanding about what problems students may encounter in learning specific content is necessary to engage a specific group of students in effective learning (Feiman-Nemser & Parker, 1990). Instruction focused on errors would therefore facilitate efficient strategies of problem solving (Palinscar & Brown, 1984). When confronted with unexpected student mistakes or questions, teachers are more likely to refine their content understanding, consider students’ perspectives and make instructional adjustments, which would greatly increase teaching effectiveness.

**How: Content-Specific Pedagogical Knowledge**

*How* is math-specific pedagogical knowledge that can facilitate young children’s mathematical understanding. Operationally, identifications of strategies and representations can be thought of as: (1) Pedagogy: knowledge of pedagogical strategies applies for young children and learning math in general; and (2) *Representations*: Knowledge of specific representations to present math ideas/concepts (e.g. illustrations, examples, models, demonstrations and analogies). It can be verbal or visual way of displaying quantitative information (Hill et al., 2012). Effective instruction should interweave content understanding and students’ thinking in conjunction with pedagogy. This definition of “how” concurs with several research projects about knowledge of content and teaching or pedagogical strategies unique for effective subject teaching.
Knowledge about pedagogical strategies as they apply to young children and learning math is a must for effective teaching. Students’ cognition and characteristics such as age-related developmental stages, individual differences, motivation and interest are the common aspects that should be considered in teaching practice (Developmentally Appropriate Practice, NAEYC, 1987; 1999; 2009). Particularly for early childhood education, educators’ understanding of young children plays a significant role in creating the best possible teaching. For example, due to the un-maturated and developing brain and executive function, young children cannot concentrate for a long time. Therefore, small group and one-on-one guidance are widely used to draw their attention. Likewise, it takes a long time for young children to grasp a full understanding about number system and apply the knowledge flexibly; therefore similar math learning activities must be arranged with different numbers and materials to reinforce the comprehension.

It is necessary to apply pedagogical strategies to accommodate students’ developmental stage and individual differences; however, merely considering general age-related characteristics of students is insufficient to inform subject (i.e., mathematics) instruction. For example, manipulatives and multiple sensational inputs are commonly applied in early childhood settings to accommodate the concrete learning styles of young children. However, productive math teaching also requires pedagogical awareness that incorporates students’ learning into content at a more specific level. Educators must consider specific mathematical materials and how to organize and place them (Copley,
2010) together with more general aspects of teaching such as instructional grouping, and represent the content in a comprehensible way. It requires an understanding more sophisticated than giving young children multiple sensory inputs in general. Learning is enhanced only when suitable representations of the content and students’ cognition are taken into consideration.

More specifically, effective math teachers should own a repertoire of representations and know the relative strength and weakness of different representations. Representational fluency involves understanding and relating different representations, which can be enactive, iconic or symbolic. Different representations illustrate different aspects of a complex concept or relationship, and examples of math-specific representations are equations, variables, words, pictures, tables, graphs, geometric shapes, manipulative objects, and actions. Students at various developmental levels may have their own learning styles and vast individual differences; and good problem solvers are those who can skillfully translate among vocal, iconic and symbolic representations. Therefore, educators should be aware of the developmental path of the representation using for young children, and own a repertoire of representations to accommodate learners’ needs.

Let’s return to the example of teaching number composition and decomposition for preschoolers. Exemplar instructions must take the complexity of the mathematical ideas and knowledge of students’ cognitions, including misunderstandings, individual differences and learning path, into account. Teachers may consider: 1) providing enough cubes to allow children to keep previous combinations available while working on other
arrangements. In this way, children would “see” that there are different combinations and the total amount stays the same; 2) writing down the written symbols alongside the combinations as a reminder and prompt to make connections between the quantity objects represent and the quantity written numbers stand for; and 3) providing two colors of cubes and arrange the combinations in a way to help young children themselves discover the pattern, etc. In other words, the selection of materials and organizing of learning activities should provide opportunities to address novices’ learning about the key aspects of the subject content.

Overall, The PCK-EM model intentionally focuses the content knowledge aspect for teaching. We take the stance that content understanding (i.e., “what” component in PCK-EM model) is an essential element of PCK, but distinguish its definition from SMK by linking it to knowledge required in teaching the specific grade level and highlighting the importance of deep conceptual understanding beyond “knowing how to do math.” We also agree with Shulman (1986) about the two key aspects of PCK, understanding about students’ cognition and pedagogical strategies and representations, and incorporate other groups’ insights to further specify these aspects of knowledge in the context early math teaching.

The key aspects of effective instruction demonstrated by the dimensions of PCK (what, who and how) impact each other simultaneously. Specifically, content understanding serves as a foundation for successful teaching, a broad awareness about content covered in early math and their relationships and a sophisticated understanding of the mathematical ideas young children need to learn is essential. Without a solid
understanding of the mathematics content young children need to learn, it is impossible for teachers to know what learners bring into the classroom, what misunderstanding and difficulties they are likely to have, or how to make the math content comprehensible to students. In other words, a solid understanding of the math content is fundamental and prerequisite for developing other aspects of knowledge required in teaching (i.e., other components of PCK-EM).

At the same time, understanding about young children’s misunderstanding and the capability of applying appropriate representations to tailor students’ needs plays a significant role in teaching and learning. Although content understanding is essential, it is insufficient for effective teaching unless understanding about foundational math is incorporated with young children’s learning and applied in instruction. More specifically, comprehension of how students learn math, including their learning patterns around particular mathematical content and likely misunderstandings and learning difficulties is of vital importance for productive teaching. These understandings, together, integrate with an awareness of varied usefulness of representations and strategies (i.e., pedagogies) about how to organize the lesson and class to achieve effective teaching. The assumption of PCK-EM is consistent with Park and Chen (2012)’s study on high school biology teachers’ PCK.

Orientation of teaching early math is not explicitly included in the PCK-EM model because incorporating teaching orientation as a knowledge base for PCK has been questioned. The consideration relates to the question about whether PCK is a knowledge base, a skill set, or a specific disposition. There is no doubt that teaching is greatly
impacted by orientation and belief of instructors during the process of knowledge transformation and researchers have explored it from different disciplines (Grossman, 1990; Magnusson et al., 1999; Park & Chen, 2012). However, the current study takes the stance that PCK is knowledge required for productive teaching, and regards orientation as a complex belief system beyond knowledge, not an inseparable component of knowledge (Abell, 2008; Fennema & Franke, 1992; Friedrichsen et al., 2009). In other words, teaching orientation is viewed as an overarching factor impacting teaching rather than a knowledge aspect of teaching, therefore is not included in PCK-EM model.

Curriculum knowledge is not considered in the PCK-EM model as a distinguished aspect. This is because there are limited curricula available in early childhood settings, especially for mathematics (Ginsburg & Golbeck, 2006). Within these limited choices, the teaching of math is still highly varied in terms of whether one uses a curriculum, which curriculum is used, and the fidelity of implementation. Meanwhile, teachers’ knowledge of curriculum and curriculum enactment can be partially revealed by examining content understanding and pedagogical knowledge, i.e., what content strands and processes should be covered in early math teaching and how to plan instructional tasks (Anhalt, Ward, & Vinson, 2006; Doyle, 1990). Therefore, it makes more sense not to include curricular understanding as a separate component of PCK.

**Setting the Stage: Why Early Math & Research Questions**

The growing importance of mathematics to society and to children’s development calls for delving into early mathematics education. Math is the language of science, engineering, and technology. Without math competence, one cannot learn other
disciplines such as physics and chemistry well. Moreover, computers have become an inseparable part of life and work; the ever-changing information-technology society requires a workforce that is competent in mathematics (Cross et al., 2009; Glenn Commission, 2000; NCTM & NAEYC, 2010). With limited mathematical reasoning and problem solving skills, one would not function successfully in contemporary society.

Despite the significance of math competence on the function of individuals and society, many Americans are struggling with mathematics. As reported by United States Center for Educational Statistics (2007), about 22% adults in the States were unable to solve 8th grade math problems, which are skills necessary to function in daily life and work settings. As well, children in the U.S. are struggling to achieve mathematics either compared to other countries or to their more successful peers in the States. According to PISA (Programme for International Student Assessment of the Organisation, 2012), an international study conducted by the Organization for Economic Co-operation and Development (OECD) to evaluate education systems in 64 regions, high school students in the United States performed significantly lower in math, compared with high school students from other regions. But the failure in math is not limited in high school; there is an apparent achievement gap as early as kindergarten. Children from poor families showed lower levels of mathematics achievement at the entry of kindergarten (Huttenlocher, Jordan, & Levine, 1994; Starkey, Klein, & Wakeley, 2004) and fell further behind (Starkey & Klein, 2000).

These troubling data raise several questions: Why are so many adults not functionally math-literate? Why are high school students falling behind in math learning?
Why does the achievement gap in mathematics start before formal schooling? There is no simple answer, however, just as Rome was not built in a day, children’s math understanding is not developed over night. A group of researchers from Missouri made some effort to connect these struggles. The researchers tested 180 seventh graders about core math skills needed to function as adults (i.e., functional numeracy measures); the results indicated that mathematical understanding at the beginning of formal schooling was associated with functional numeracy knowledge in adolescence (Geary, Hoard, Nugent, & Bailey, 2013), the latter of which further predicted employability and income in adulthood (i.e., Hanushek & Woessmann, 2008; Rivera-Batiz, 1992). Similar longitudinal studies have shown that understanding and growth in early mathematics was a chief predictor for later school success and real world outcomes (Duncan et al., 2007; Geary et al., 2013; Watts et al., 2014). Together, they suggest that early childhood is a critical stage for learning mathematics that builds the foundation for later success.

Contrary to the pervasive assumption that math is too abstract for concrete thinkers; young children are capable of learning math. Children of all ages have some knowledge of mathematics (Clements & Sarama, 2008; Ginsburg & Ertle, 2008) and most them enter school with a wealth of knowledge and cognitive skills. By the age of three or four, preschoolers have encountered many mathematical experiences and demonstrated an impressive body of mathematical understanding (Baroody, Cibulskis, Lai, & Li, 2004; Ginsburg, Klein, & Starkey, 1998; Ginsburg, Cannon, Eisenband, & Pappas, 2008). The evidence indicates that young children’s thinking can be complex and abstract.
Although young children are capable of learning complex and abstract mathematical ideas, the understanding is heavily influenced by experience and instruction. For instance, although toddlers and preschoolers can possess some understanding of number sense, it is usually implicit (Balfanz, 1999; Zvonkin, 1992). This understanding needs to be transferred to formal number knowledge, a process that may not necessarily be smooth. To take advantage of children’s potential for learning, the experience of informal math must be connected and reinforced formally in early childhood education (Baroody, et al., 2004; Tudge, Li & Stanley, 2008). Compared with other subjects such as literacy, math learning requires more intentional guidance to make the knowledge explicit. It is more likely for young children to be exposed to and enjoy a story than a math problem. Therefore, appropriate learning opportunities provided by adults (and older children) are necessary and important for young children to learn mathematics.

Unfortunately, little attention has been paid to teaching mathematics to young children before they enter formal schools. A study of pre-kindergarten across 11 states showed that only 8% of learning activity was math-relevant, which includes any activity involving counting, time, shapes, and/or sorting (Early et al., 2006) (Early, Barbarin, Bryant, Burchinal, Chang, Clifford, et al., 2005). Similarly, a recent study indicates that among 100 classrooms researchers visited on multiple sites in a mid-west city, 90% of teachers in early childhood classrooms conducted literacy-related activities, but only 21% carried out mathematics activities (Chicago Program Evaluation Project, 2007). Preschool teachers were frequently found to provide little support for children’s
mathematical development and seldom use mathematics terminology (Balfanz, 1999; Clements & Sarama, 2007; Frede et al., 2009; Lee & Ginsburg, 2007). When mathematics activities occurred, they were often presented as secondary goals of teaching (Cross et al., 2009). The critical math competence is not well supported during its crucial developing period.

Is school time too limited in early childhood education to include math teaching? In contrast to the popular belief, school time is not too limited to include math instruction. Take preschoolers and kindergarteners as an example, about 40 percent of their time in school was participating in activities not associated with any instructional purposes (Cross et al., 2009). The majority of this non-instructional time was routine activities such as transitioning, waiting in line, or eating meals and snacks (Early et al., 2005). These non-instructional periods could become invaluable moments for math learning. Unfortunately, few preschool or kindergarten teachers appeared to make the most use of the learning opportunities arising during transitional periods or employ strategies for integrating math activities.

Although there are many factors impacting the set-up of non-instructional time, the fact that little math is incorporated raises a question about whether teachers are well equipped with the math knowledge for teaching for understanding. It is frequently stated that early childhood teachers do not have good mathematical content knowledge (e.g., Ginsburg & Golbeck, 2006); however, few quantitative studies of teachers’ math knowledge at this grade level exist. What do early childhood teachers know about foundational math that young children can learn? Because early mathematics is crucial
and teachers play a central role in math learning, it is extremely important to find answers to this question.

In the meantime, although there have been studies on mathematical knowledge for teaching in upper elementary and other grades, to what extent and how appropriate these findings can apply to teaching foundational math is being questioned. Early math teaching differs from upper grade levels in many ways. There are usually multiple mathematical ideas involved in one of the play activities young children are involved in and intertwined with other learning areas, therefore, teachers must have a broad understanding about foundational math to cultivate math thinkers more effectively. It also requires teachers to obtain a comprehensive understanding about the connections among math concepts and big ideas in mathematics young children have to learn. In the meantime, although early math content deals primarily with small numbers and rudimentary concepts, the underlying structure is abstract and complex. The seemingly simple concepts adhere to similar principles as advanced mathematical topics such as algebra and statistics. Preparing teaching professionals adequately in unpacking the complexity of foundational mathematics can therefore pose distinctive challenges for students’ understanding and sense making.

Unfortunately, the training of early childhood teachers is basically subject-general. Professionals working with young children have little math-specific training during the pre-service and in-service trainings. While Grossman’s pioneer work on the development of PCK warned the field of teacher education that subject matter knowledge or classroom teaching experience themselves is not sufficient for teaching professionals;
it also highlighted the positive impact of subject-specific courses in supporting teachers’ understanding about the subject and students’ learning of the subject (Grossman, 1990). The lack of subject preparation and subject-specific pedagogy learning can pose serious obstacles for teaching foundational mathematics effectively.

As well, young children differ from their big brothers and sisters physically, cognitively, and emotionally. On the one hand, their fine motor skills are still developing; their attention span tends to be short, they think more concretely and are just beginning to learn the meaning of various symbols and self-regulation. Therefore, teaching young children requires effort to meet their developmental needs. On the other hand, however, young children start to understand that one object may have different attributes, and demonstrate some organized, logical thoughts, although the thinking tends to be tied to concrete reality. A growing body of literature has indicated that many mathematical competencies, such as sensitivity to set size, pattern, and quantity, are present very early in life (Cross et al., 2009; Ginsburg, Lin, Ness, & Seo, 2003). Young children have more mathematical knowledge than we previously believed. Responding to young children’s developmental needs while promoting their mathematical thinking makes math teaching in early childhood unique.

To help young children construct understanding, it is also necessary to implement teaching strategies about the math content in an age-appropriate and flexible way, which makes early math teaching unique. For example, paper-pencil and deskwork are not developmentally appropriate for young learners. Hands-on and playful learning take priority in early mathematic learning and teaching. In recent years, math-related
verbalizations by teachers/parents have also drawn the field’s attention as a critical component of early math teaching strategies (Levine, Suriyakham, Rowe, Huttenlocher & Gunderson, 2011). The studies imply the significance of applying effective representations to make the math content comprehensible for young learners. Young children’s learning styles differ profoundly from older school children; therefore, teaching strategies in the early childhood classroom resemble little in the upper grades. For all these and many other reasons, early childhood teachers’ mathematical expertise needs to be studied with considerable care and thoughtfulness.

While there are many ways to conduct such investigation, the notion of PCK provides a promising framework to study early math teaching. The field of early childhood education is reluctant to specify standards for particular subject domains in order to prevent the fact-and-skill driven approach (Bowman et al., 2001). Teacher preparation is also subject-general. Therefore, the content aspect of teaching was a “missing paradigm” (Shulman, 1987, 1986). PCK theory, however, urges the necessity of studying content expertise for teaching, and highlights the specialized type of understanding that is only required and used in teaching. It implies that common mathematical understanding (knowledge shared among common adults) may not guarantee successful teaching for foundational math; it also suggests the significance of math-specific understanding regardless of the subject-general teacher preparation. Building on the literature of math teaching in the upper grades, our understanding of early childhood teachers’ math knowledge warrants an independent investigation in its own right.
Besides the uniqueness of early math education and limited investigation about how PCK applies to early childhood education, many questions about teaching effectiveness remain unanswered. On the one hand, we have gained a considerable amount of knowledge about PCK and teaching effectiveness, and educators have continued to write about how teachers’ PCK may guide instructional practice and improve learning outcomes. On the other hand, a great deal remains unclear regarding how to define and measure the PCK construct. These issues include but are not limited to: 1) the conceptualization of PCK and how it applies to specific subject area and grade levels; 2) the most feasible way of assessing PCK for particular subject and specific grade level; 3) the dynamic nature of PCK components and whether there is a central fundamental element, such as content understanding; and 4) the relationship between PCK, teaching quality, and students’ learning outcome.

How can the construct be studied when it is still being defined and explored conceptually, methodologically and empirically? The challenge poses a “chicken and egg problem” for researchers who are “attempting to understand the knowledge for teaching” (Alonzo, 2007, p. 132). On the one hand, there are arguments about: 1) the nature and structure of content knowledge for teaching; 2) how to measure the expertise upon diverse understandings and assumptions of the construct; and 3) evidence about the predictability of PCK on teaching quality and students’ learning (Abell, 2008; Alonzo, 2007; Hill, Rowan, & Ball, 2005; Rohaan et al., 2009). The soundness of the description about what teachers know and do not know from the lens of PCK relies heavily on the
validity of the conceptualization and measurement of PCK. Therefore, the lack of consensus makes it difficult to make an assertion about teachers’ content understanding.

On the other hand, the investigation on the relationship between PCK, teaching quality and student outcome would advance our understanding about the construct, the measurement and promote potential revisions for the theory and assessment (Alonzo, 2007). Researchers have acknowledged that this is a dilemma when studying PCK because one is a foundation for the other, but neither is fully developed. With this in mind, the current study would take this opportunity to explore the construct, the tool, and the empirical investigation simultaneously. Guided by the framework of PCK-EM, the present study attempts to investigate the following questions:

**Question 1: What is the profile of early childhood teachers’ PCK-EM?**

**Sub question 1.1 What is the distribution of each dimension of PCK-EM?**

Little is known about what early childhood teachers know and do not know about the content knowledge necessary to teach early mathematics. In the PCK-EM model, this issue is directly examined through 3 dimensions involving: (1) “what”: what teachers understand about mathematics content in terms of the depth and breadth around particular big ideas of foundational mathematics; (2) “who”: what teachers know about the students’ cognition in mathematics regarding learning paths and misunderstanding around particular mathematics topics; and (3) “how”: what teachers know about how to transmit mathematical ideas by appropriate pedagogical strategies and mathematical representations. What level of understanding do teachers have about teaching mathematics to young children (based on 3 dimensions of PCK-EM)? Are teachers’
scores on each of the 3 dimensions normally distributed? Are they universally high across all 3 dimensions? Examining the knowledge required for teaching foundational mathematics through the lens of PCK-EM may shed new light on the characteristics of early childhood teachers’ mathematical knowledge that are essential for effective teaching.

Sub question 1.2 Are some dimension(s) of PCK-EM better developed than others? Teaching is complex and involves not only content understanding, but also familiarity with learners and corresponding pedagogies. While the simultaneous use of these knowledge pieces is expected to be a prerequisite for sound teaching, it is unclear yet whether the different aspects of knowledge develop simultaneously. Are teachers’ scores evenly distributed across the 3 dimensions of PCK-EM? If not, then what skills are better developed than others? Answering this question is a necessary starting point for designing appropriate teacher education and training programs.

Sub question 1.3 What are the relationships among the 3 dimensions of PCK-EM? Do different dimensions of PCK-EM develop independently from each other? More specifically, for instance, does a good understanding of the math content necessarily relate to a grasp of how young children learn the same concept and how to effectively represent the idea? Although early childhood teachers may be knowledgeable about the development of young children and DAP, it is unclear whether this knowledge translates directly to the teaching of mathematics content, to an understanding of how young children learn math, and to the discovery of how to effectively present mathematical ideas to young learners. Melendez (2008) made a thorough investigation of early
childhood teachers’ PCK; however, PCK was studied only very broadly across different subject areas with a limited sample size of 52 pre-kindergarten and kindergarten teachers.

**Sub question 1.4 Are there different groups of teachers regarding their PCK-EM profiles?** At individual level, do all teachers develop their content expertise reflected by the three dimensions of PCK-EM in a similar way? What are the differences between expert teachers and teachers’ with less professional knowledge of teaching early mathematics?

**Question 2: What are the relationships between early childhood teachers’ knowledge and practice, as well as their students’ learning gains?**

It is necessary to obtain empirical evidence to support the conceptualization of PCK as an indicator (more effective than traditionally defined subject matter understanding or general pedagogy) of quality teaching and students’ learning, especially in foundational math. While it is proposed that PCK underlies teaching effectiveness, empirical evidence is needed to verify this theoretical hypothesis. More specifically, analyses will examine the degree to which knowledge indicator(s) reliably and strongly predict teaching practice and students’ accomplishment. By adopting the conceptualization of PCK, the content knowledge required in teaching is defined in a way that is consistent with the content students need to learn, among other requirements for effective teaching. Evidence linking PCK to both teaching practice as well as student learning outcomes would also provide support for the validity of the assessment tool.

The association between teachers’ knowledge and practice, as well as students’ learning has not yet been fully explored. While many existing studies have increased our
understanding of PCK and teaching effectiveness, the majority of the investigations were small-scaled without linking to teaching practice or students’ learning gains (e.g. Grossman, 1990). To date, the few attempts that have been made to verify this conceptualization have produced mixed results (e.g. Baumert et al., 2010; Gess-Newsome et al., 2011; Hill et al., 2005; Kersting, et al. 2010); and these studies are difficult to compare, partly due to differences in the conceptual definition and measurement of PCK (Gess-Newsome et al., 2011), and variation in the specific indicators used to assess teaching quality and students’ learning gains. In Baumert and his colleagues’ study (2010), for instance, teaching quality was revealed by cognitive activation (i.e., students’ thinking level and support for individual students). In McCray & Chen’s paper, it was indicated by math-related language in classroom teaching. Ball’s group rated classroom-teaching videos to determine teaching quality. Using more effective indicators of math teaching quality such as on-site classroom observation would further this investigation.

Moreover, little is known about how PCK applies to early math education as an indicator of effective teaching. In McCray and Chen (2012)’s study of early childhood teachers’ PCK, the sample was limited to 22 pre-kindergarten teachers, and the indicator of teaching quality was math language used in teaching, which severely narrows the generalizability of the results. More investigations of larger scale with other indicators of teaching quality are needed to enhance our understanding of the role of PCK in teaching quality and student outcomes.

Sub question 2.1What is the relationship between early childhood teachers’ knowledge and the quality of their classroom teaching in mathematics? It is essential
to investigate how teachers’ content understanding of math contributes to actual classroom teaching. Do teachers with higher levels of understanding deliver higher quality math lessons? Although there have been some prior explorations of the relationship between knowing and doing, the results of these studies were mixed, and the indicators of teaching quality that were used are difficult to compare (e.g., Baumert et al., 2010; Hill et al., 2005; McCray & Chen, 2012). Exploring the relationships between knowledge and teaching in early math teaching would partially validate the PCK-EM survey tool.

**Sub question 2.2 What is the relationship between early childhood teachers’ knowledge and their students’ learning gains in mathematics?** To answer the question about “What subject matter understanding does a teacher need in order to be effective,” it is necessary to link the indicator of teachers’ professional knowledge to students’ learning within the same subject (i.e., content area). There are few empirical studies on teachers’ mathematical knowledge and its impact on child learning outcomes in early childhood education (McCray & Chen, 2012). Do students whose teachers have higher levels of PCK make greater progress in learning mathematics? In addressing this question, previous research teams that have studied elementary math teaching have reported mixed results (Hill et al., 2005; Kersting, et al. 2010). Understanding about mathematical concepts, knowledge about students’ misunderstandings, and awareness of mathematical representation were hypothesized as key for successful teaching. The proposed research is designed to obtain empirical evidence concerning this hypothesis, which has strong implications for practice.
CHAPTER III

METHODOLOGY

This study attempted to characterize early childhood teachers’ PCK-EM and explore its relationships with other indicators of effective teaching: mathematical teaching quality and students’ learning gains. The sample consisted of 182 teachers working with young children from pre-kindergarten to 3rd grade in a large, urban area in the Midwest, U.S. A. The profile of early childhood teachers PCK-EM was explored by analyzing narrative responses to an online, video-elicited, open-ended survey collected at the beginning of a school year. The quality of teaching was evidenced by on-site classroom observation during a similar time period. Students’ learning outcomes were indicated by two assessments at the beginning and the end of the same school year.

The study was part of a four-year, school-based professional training aiming to build early childhood teachers’ teaching competence in foundational mathematics. The profile of PCK was explored by using data collected from all teachers recruited. To investigate a) the relationship between PCK and teaching quality; and b) the association between PCK and students’ learning outcomes over a year, data from a subgroup of the total sample were analyzed (i.e., teachers and students from the comparison group without professional training). As an initial stage of investigation, the research design of the current study did not involve the program evaluation regarding the impact of professional training. However, the recruitment of the participants and the assignment to
intervention and comparison groups occurred at the school level. Therefore, the introduction of the sample starts with a debriefing about the professional training project, how the participating schools were recruited, what the student body characteristics looked like at the school level, and further describes the resulting sample. The section then explains the measurements, research design, data collection procedure, and data analyses plan.

**Methods for Sample Selection**

Participants were recruited from 16 public schools for the Early Math Innovations Project: *Achieving High Standards for Pre-K – Grade 3 Mathematics: A Whole Teacher Approach to Professional Development* by the Early Math Collaborative at Erikson Institute. The overarching goal of the initiative is to increase teachers’ mathematics teaching competence and help low income minority students in Pre-K to 3rd grade reach or exceed state learning standards in mathematics through a four-year innovative professional development (PD) program. Among the participating schools, 8 were intervention schools (received training) and 8 were comparison schools (assessed for program evaluation purposes).

Assignment to treatment versus comparison conditions occurred at the school level and all teachers from Pre-K through 3rd grade in each school were encouraged to participate in the program evaluation. The selection of the intervention schools was a year-long joint effort of the implementation team and the school district administrators. Based on the lists of schools with high needs recommended by the network school district leaders, the implementation project approached school administrators in a group meeting
to explain the intervention. For schools that expressed willingness to participate at administration level, an on-site visit was scheduled to collect more specific information about students’ mobility rate and the school’s collaborative atmosphere. Later, the implementation team made a second visit to those schools with acceptable students’ mobility rate and collaborative atmosphere. Without the presence of administrator, teachers were informed about the program and the voluntary participation in the program evaluation. Faculty members then indicated their willingness to participate on a sliding scale (the two end points indicate strong unwillingness and willingness to participate respectively). The final decisions about the partnership were made based upon collective information and opinions from principals, faculties and on-site visits (see Figure 8).

Schools that could implement the PD and had both faculty and administrative support to continue the project over the course of the study (4 years) were identified (between January and June 2011).

Comparison schools were later selected in the same network by matching with intervention schools following a similar process of listing, approaching, informing and decision-making. To ensure that treatment schools and comparison schools had similar demographic composition and achievement indicators, propensity score matching techniques (Staurt, 2010) were applied to find comparable schools. The estimated propensity model was developed upon percentages of 1) 3<sup>rd</sup> grade students who met math standards in 2009, 2) 3<sup>rd</sup> grade students who exceeded math standards in 2009, 3) students who were English Language Learners (ELLs), 4) students who were identified as minority, 5) students receiving free or reduced-price lunch, and 6) students’ mobility.
Among 65 non-treatment schools, 8 were selected based on closeness to each treatment school on the propensity score using nearest neighbor matching between June 2011 and August 2011 (and willingness to participate at both administrators and faculty levels). Note that Two of the identified comparison schools dropped out after selection (July 2011-August 2011), but before pre-test data collection occurred. At that time, the organization of schools in the school district had changed such that the 3 district areas from which the treatment schools were selected were represented by 6 networks. Two replacement comparison schools were then selected based on their characteristics from schools in the 6 networks.

Regarding the research incentives, all teachers who responded to the on-line survey received a $50 gift card after submitting their response. Intervention teachers received a stipend for working outside schools, in addition to the intellectual benefits and support. They were given children’s story books related to math teaching during the training. Comparison school teachers, on the other hand, received two science-themed books for the year they participated. They were also rewarded with membership to either “RAZ Kids” or “Reading A-Z”, an online independent reading site with printable, leveled books (some are available in more than one language). At a school level, comparison schools received a set of 36 children’s books suitable for supporting the teaching of foundational mathematics by the end of the program. Professional development vouchers from the institute (the implementation team is affiliated to) were given to comparison schools (4 for each of the first 3 years and 8 for the final year). By the end of the
Network chief recommended schools to work with

Implementation Team met with principals as a group to introduce the project

Implementation team met all faculties from Pre-K to 3rd grade without administrator to inform the project and gather information about willingness of participation team visited schools on site to obtain more information such as students’ mobility, school’s overall atmosphere for long term partnership

Implementation team met all faculties from Pre-K to 3rd grade without administrator to inform the project and gather information about willingness of participation schools of site to obtain more information such as students’ mobility, school’s overall atmosphere for long term partnership

Implementation team integrated information and made a final decision about partnership

Figure 8. The Process of School Recruitment.
intervention, the implementation team provided two half-day workshops for all Pre-K-3rd grade teachers at comparison schools. Teachers from comparison schools are prioritized for future professional training opportunities.

Table 2. Descriptive Statistics of School-level Matching Characteristics

<table>
<thead>
<tr>
<th>Baseline School-Level variables</th>
<th>Overall Mean (SD)</th>
<th>Intervention Mean (SD)</th>
<th>Comparison Mean (SD)</th>
<th>ES</th>
<th>Significance</th>
</tr>
</thead>
<tbody>
<tr>
<td>% free-reduced price lunch</td>
<td>.92 (.06)</td>
<td>.92 (.04)</td>
<td>.91 (.07)</td>
<td>.26</td>
<td>.62</td>
</tr>
<tr>
<td>% meet state math standards</td>
<td>.49 (.09)</td>
<td>.50 (.08)</td>
<td>.48 (.11)</td>
<td>.17</td>
<td>.74</td>
</tr>
<tr>
<td>% exceed state math standards</td>
<td>.23 (.15)</td>
<td>.21 (.12)</td>
<td>.24 (.17)</td>
<td>-.21</td>
<td>.68</td>
</tr>
<tr>
<td>% English language learners</td>
<td>.25 (.15)</td>
<td>.19 (.10)</td>
<td>.33 (.11)</td>
<td>-1.37</td>
<td>.02</td>
</tr>
<tr>
<td>Mobility rate</td>
<td>.17 (.05)</td>
<td>.17 (.07)</td>
<td>.17 (.02)</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>% Hispanic</td>
<td>.59 (.30)</td>
<td>.62 (.33)</td>
<td>.55 (.28)</td>
<td>.22</td>
<td>.67</td>
</tr>
<tr>
<td>% Black</td>
<td>.26 (.34)</td>
<td>.29 (.37)</td>
<td>.23 (.32)</td>
<td>-.17</td>
<td>.74</td>
</tr>
</tbody>
</table>

Overall, 16 schools from 6 networks were selected, 8 were intervention (i.e., treatment) schools and 8 were comparison schools with comparable student body characteristics. As shown in Table 2, on average, over 90% of the students in the participating schools were enrolled in free/reduced lunch. The students’ academic performance (mathematics) on the statewide test ISAT\(^1\) suggests that about half of the students were not meeting the state math standards, and only one out of five students in these schools exceeded state math standards. About one out of four students was ELL (the percentage was slightly lower in intervention schools). The students’ mobility rate, a factor important for four-year PD and program evaluation, was around 20%. Hispanic and Black students on average represented about 60% and 26% of the whole population.

\(^1\) ISAT is a third grade norm and criterion-referenced test. The ISAT math tests cover the five content areas in the state standards, and the ISAT reading tests assess vocabulary development, reading strategies, and reading comprehension.
respectively. The sampling is likely to address the research questions of the study because it targeted early childhood teachers working with high needs children from low-income families and with low academic performances.

Sample Description

Teacher Participants

There were 182 teachers in the study. The number of teachers from each school ranged from 3 to 18 (\(M = 11, SD = 4\), depending on the size of the school). The distributions were roughly comparable across grade levels and between comparison and intervention schools. More specifically, there were about 30 teachers from each early elementary grade and about 40 teachers from Pre-K and Kindergarten respectively, and 8 teachers (about 5% of the sample) were working in mixed age classrooms (e.g. 1\textsuperscript{st} and 2\textsuperscript{nd} split) (see Table 3).

Table 3. The Distributions of Teachers by Grade Level

<table>
<thead>
<tr>
<th>Grade</th>
<th>Pre-K</th>
<th>K</th>
<th>1st Grade</th>
<th>2nd Grade</th>
<th>3rd Grade</th>
<th>K-1st Grade</th>
<th>1-2 Split</th>
<th>2-3 Split</th>
</tr>
</thead>
<tbody>
<tr>
<td>Overall</td>
<td>39</td>
<td>46</td>
<td>29</td>
<td>30</td>
<td>30</td>
<td>3</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>%</td>
<td>21.3%</td>
<td>25.1%</td>
<td>15.8%</td>
<td>16.4%</td>
<td>16.4%</td>
<td>1.6%</td>
<td>1.6%</td>
<td>1.1%</td>
</tr>
</tbody>
</table>

Note: \(N = 182\).
This is corresponding to the teachers who took PCK-EM Survey.
Pre-K: Pre-Kindergarten; K: Kindergarten; K-1 Split: Kindergarteners and 1\textsuperscript{st} graders mixed class;
1-2 split: 1\textsuperscript{st} and 2\textsuperscript{nd} graders mixed class; 2-3 Split: 2\textsuperscript{nd} and 3\textsuperscript{rd} graders mixed class.

As shown in Table 4, 95.5% of the participants were females, and the majority (84.8%) was between 25 to 54 years old. One out of four teachers in the sample was new to the grade they were teaching (less than two years), and the rest of them had taught in the current grade for more than 2 years. Half of the participants were White, one third
were Latino and one tenth were Black. Participants’ educational backgrounds and mathematics teaching and professional training experience were also collected.

Table 4. The Background Information of Participating Teachers

<table>
<thead>
<tr>
<th>Age Span</th>
<th>24 and under</th>
<th>25-34</th>
<th>35-44</th>
<th>45-54</th>
<th>55-64</th>
<th>65 and over</th>
</tr>
</thead>
<tbody>
<tr>
<td>%</td>
<td>2.7%</td>
<td>30.4%</td>
<td>29.5%</td>
<td>25.0%</td>
<td>9.8%</td>
<td>2.7%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Ethnicity</th>
<th>African-American or Black</th>
<th>American Indian or Alaska Native</th>
<th>Asian</th>
<th>Caucasian or White</th>
<th>Hispanic or Latino</th>
<th>Native Hawaiian or other Pacific Islander</th>
<th>Other</th>
</tr>
</thead>
<tbody>
<tr>
<td>%</td>
<td>10.6%</td>
<td>0.9%</td>
<td>7.1%</td>
<td>46.0%</td>
<td>31.9%</td>
<td>0%</td>
<td>1.8%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Years of teaching</th>
<th>Mean (SD)</th>
<th>Range (Min, Max)</th>
<th>0-5 years</th>
<th>6-10 years</th>
<th>11-15 years</th>
<th>&gt;15 years</th>
</tr>
</thead>
<tbody>
<tr>
<td>%</td>
<td>12.85 (9.27)</td>
<td>(1, 41)</td>
<td>27.5%</td>
<td>20.9%</td>
<td>19.8%</td>
<td>31.9%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Certificate</th>
<th>Early Childhood Teacher Certificate</th>
<th>Elementary education Certificate</th>
<th>Bilingual Endorsement</th>
<th>Special Education Certificate</th>
<th>Others</th>
</tr>
</thead>
<tbody>
<tr>
<td>%</td>
<td>32.8%</td>
<td>69.9%</td>
<td>39.9%</td>
<td>16.9%</td>
<td>71.0%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Pre-service math education/methods classes</th>
<th>Mean (SD)</th>
<th>Range (Min, Max)</th>
<th>0 class</th>
<th>1-2 classes</th>
<th>3-5 classes</th>
<th>&gt;5 classes</th>
</tr>
</thead>
<tbody>
<tr>
<td>%</td>
<td>3.15 (3.79)</td>
<td>(0, 30)</td>
<td>15.4%</td>
<td>41.2%</td>
<td>30.8%</td>
<td>12.6%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>In-service math education/methods workshop</th>
<th>Mean (SD)</th>
<th>Range (Min, Max)</th>
<th>0 hour</th>
<th>1-5 hours</th>
<th>6-15 hours</th>
<th>&gt;15 hours</th>
</tr>
</thead>
<tbody>
<tr>
<td>%</td>
<td>8.87 (12.95)</td>
<td>(0, 80)</td>
<td>29.7%</td>
<td>25.3%</td>
<td>28.6%</td>
<td>16.5%</td>
</tr>
</tbody>
</table>

*Note: N = 115–182*

2 This is corresponding to 7.1% early childhood special education certificate and 9.8% K-2 special education certificate.

3 The sample size varies because the information was collected through the overall sample and a subgroup of teachers at different time points.
While 96% of the participants had Bachelor’s degree in a variety of majors, about 72% of them also earned a master’s degree relevant to education. On average, participants had about 13 years teaching experience, ranging from 1 year to 41 years. Among them, one third of the teachers had been teaching professionals for more than 15 years, while a similar portion of teachers have been in this field for less than 5 years; the rest of the sample (about 40%) had been teaching for 6 to 15 years.

Regarding pre-service and in-service math training experience, the average number of the math classes the teachers had taken during pre-service training was around 3, ranging from 0 to 30. While the majority of the teachers took 1-5 classes, about 15% of them had not taken any pre-service math class, and about 10% of them took more than 5 classes related to math education or methods during pre-service training. The teachers in the sample took about 9 hours of in-service math education/methods workshops ($SD = 13$), spanned from 0 to 80 hours. For every six teachers in the sample, two teachers didn’t take any in-service math workshop, three teachers went to math education or method workshop between 1 to 15 hours, and one participated in more than 15 hours of math workshops.

Overall, the sample consisted of early childhood teachers (from Pre-K to 3rd grade) working with high needs children. They represented diverse ethnicity groups, varied ages, and teaching experiences. The teachers were well educated, with diverse learning experiences and in-service training experiences for teaching foundational math. Their overall group profiles were similar to teaching professionals in the urban school district.
Student Participants

Below is a summary about the number of students assessed (Table 5). There were 756 students assessed at pre-test from comparison schools. On average, about 8 students from each classroom were assessed ($SD = 3$), and ranged from 2 to 17. The number of students was comparable across grade levels, with fewer 2nd graders and 3rd graders. About half of the students assessed were females. Only children’s data from comparison schools is used and presented for the purpose of the current study.

Table 5. The Distribution of Students by Grade Level

<table>
<thead>
<tr>
<th></th>
<th>Pre-K</th>
<th>Kindergarten</th>
<th>1st Grade</th>
<th>2nd Grade</th>
<th>3rd Grade</th>
<th>Overall</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pre-test</td>
<td>152</td>
<td>203</td>
<td>171</td>
<td>105</td>
<td>125</td>
<td>756</td>
</tr>
<tr>
<td>%</td>
<td>20.1%</td>
<td>26.9%</td>
<td>22.6%</td>
<td>13.9%</td>
<td>16.5%</td>
<td>100.0%</td>
</tr>
<tr>
<td>Pre-Post</td>
<td>140</td>
<td>181</td>
<td>156</td>
<td>95</td>
<td>112</td>
<td>684</td>
</tr>
<tr>
<td>%</td>
<td>20.5%</td>
<td>26.5%</td>
<td>22.8%</td>
<td>13.9%</td>
<td>16.4%</td>
<td>100.0%</td>
</tr>
</tbody>
</table>

Note:
The table only includes information about students from comparison schools. Pre-test are those students who had pre-test score for at least one of the two child math assessments (TEAM and WJ-AP). Pre-Post are those students who had both pre- and post- test records for at least one of the child math assessments (TEAM and/or WJ-AP).

TEAM: Tools for Early Assessment in Math, indicating students’ mathematics performance.

Research Design

The current study involved data collected within one school year (i.e., pre-test in fall 2011 and post-test in spring 2012), corresponding to the first year of data collection of the larger project. To answer the first question about what characterizes early childhood teachers’ PCK, coded responses from PCK-EM survey were used for descriptive analysis, correlational analysis, and latent profile analysis. Data from intervention and

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4 Due to the state level test in the 3rd grade, more 2nd graders and 3rd graders were involved in preparing the test and were less likely to consent and to be assessed for extra tests.
comparison teachers at pre-test were applied to better represent what early childhood teachers know and do not know about teaching foundational mathematics (see Table 6).

Table 6. Research Design of Studying PCK-EM and its Relationship to Teaching and Learning in Mathematics

<table>
<thead>
<tr>
<th>Research Question</th>
<th>Teacher Sample</th>
<th>Teacher Indicator</th>
<th>Students Sample</th>
<th>Students Indicator</th>
<th>Time point</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 the profile of PCK</td>
<td>I &amp; C</td>
<td>PCK-EM</td>
<td>NA</td>
<td>NA</td>
<td>Pre-test</td>
</tr>
<tr>
<td>2.1 the relationship between PCK and teaching quality</td>
<td>C&lt;sup&gt;5&lt;/sup&gt;</td>
<td>PCK-EM, HIS-EM</td>
<td>NA</td>
<td>NA</td>
<td>Pre-test</td>
</tr>
<tr>
<td>2.2 the relationships between PCK and students' learning gains</td>
<td>C&lt;sup&gt;6&lt;/sup&gt;</td>
<td>PCK-EM, HIS-EM</td>
<td>C</td>
<td>WJ-AP, TEAM</td>
<td>Pre-test for teachers, post for students (controlling for students’ pretest scores)</td>
</tr>
</tbody>
</table>

Note:
I: intervention school subjects, C: comparison school subjects; NA: not applicable.
PCK-EM: Pedagogical Content Knowledge in Early Mathematics survey, indicating the content knowledge for teaching foundational mathematics.
HIS-EM: High Impact Strategies in Early Mathematics observation, indicating the quality of teaching mathematics.
TEAM: Tools for Early Assessment in Math, indicating students’ mathematics performance.

Regarding the relationship between teachers’ PCK and teaching quality, coded responses from the PCK-EM survey and teaching quality observation scores (HIS-EM) during the same time period (i.e., pre-test) were analyzed. This analysis involved data from comparison teachers only. In this way, it would answer questions such as whether

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<sup>5</sup> Independent t-test suggested that overall, teachers in the comparison school showed lower PCK (M = 2.27) than intervention school (M =2.45), t (180) =2.17, p< .05; there was no significant differences in HIS-EM scores between comparison (M =4.11) and intervention schools (M =4.02), t (170) = .42, p = .68.

<sup>6</sup> Independent t-test suggested that on average, there were no significant differences in students’ math performance at pre-test, indicated by TEAM T scores (t (1501) = .23, p = .82) and WJ-AP standardized scores (t (1542) = .12, p = .91).
(and to what extent) PCK is predictive of teaching quality; what specific aspects of “knowing” and “doing” are related.

To answer the question about the predictive validity of PCK and students’ learning gains, data from comparison school subjects (i.e., teachers and students) were utilized. More specifically, it explored the relationship between teachers’ PCK scores at the beginning of the school year (pre-test) and students’ scores in TEAM and WJ-AP at the end of the school year (post-test), after controlling for students’ performance at pre-test. Data from intervention schools was not included because it was hypothesized that teachers’ PCK-EM would change as they participate in the intervention (i.e. professional trainings) and further impact students’ learning. Applying the PCK-EM scores at pre-test for intervention schools would not reflect the impact of changed PCK on students’ learning.

**Measures**

Because there are few assessment tools tied to the unique nature of early childhood education, the project developed its own measures to capture and track teacher development: a survey of pedagogical content knowledge in early mathematics (PCK-EM) and the high impact strategies in early mathematics (HIS-EM). PCK-EM is an online video-cued survey. HIS-EM is an on-site, live observation to capture math teaching quality. Two assessments, WJ-AP subtest (Woodcock, McGrew, & Mather, 2001) and TEAM (Clements et al., 2011) were applied to assess students’ mathematical performance. Another survey called “about my teaching” was administrated together with PCK-EM survey to collect participants’ demographic information and teaching and
learning experience related to mathematics. The data collected from PCK-EM survey is
the primary focus of the current study. However, understanding and explanation about its
psychometric properties is still limited (Zhang et al., 2014); therefore, the PCK-EM
survey will be explained in detail and with examples.

The Primary Measure

The pedagogical content knowledge in early mathematics (PCK-EM) survey
is a video-elicited, open-ended survey to capture educators’ content knowledge for
teaching mathematics from Pre-K through 3rd grade, based on Ball’s work (Ball, 1988),
PCK studies on early childhood teachers across content areas (Melendez, 2008), and
foundational mathematics (McCray, 2008). Two videos of authentic teacher-led math
lessons are provided. After watching each video, teachers are asked to answer 9 open-ended
questions aligned to the multiple facets of PCK regarding: the central and relevant
math concepts in the video, students’ likely misunderstanding and prior knowledge of the
particular topic, and instructional strategies to make the content accessible to students
who are more advanced and those who are struggling (see “prompted questions” in the
Appendix A). The measure takes about 45 minutes to complete and is coded on a 5-level
rubric. The pilot study suggested that inter-coder reliability was over 90%, using the
percentage of agreement among four coders, with one point or 0 point discrepancies
among four coders considered consistent.

7Although there are two videos used, the current study will only analyze responses from one particular
video “Number 7” about number composition and decomposition in a kindergarten classroom. This is
because videos are of different content focus, and from different grade levels. Previous studies from
different cohorts of teachers have revealed positive and significant correlations between teachers’ responses
to two different videos. To simplify the investigation without introducing confounding factors such as
grade level, content and difficulty of analyzing different videos, only one video was applied for current
study.
The video stimulus. In the teacher-led mathematic lesson, content domains relate directly to the common mathematical concepts taught in preschool, kindergarten and early elementary levels. The content is intentionally selected and edited to focus on a math topic tied to a specific big idea of foundational math. The big ideas of mathematics are “clusters of concepts and skills that are mathematically central and coherent, consistent with children's thinking, and generative of future learning” (Clements & Sarama, 2009). Understanding the Big Ideas of foundational math can help teachers move from seeing students either master procedural fluency and skills or not to what students really understand, and enable teachers to further students’ high level thinking beyond mechanical memorization of facts.

Take the video “Number 7” as an example; the lesson is about number composition and decomposition related to number sense and operations. Below is a brief description of the scenario:

A small group of four kindergarten children are learning different ways to make seven by using uni-fix cubes. The teacher asks children to use manipulatives and give out different answers. She corrects students’ mistakes (such as miscounting) and scaffolds when necessary (e.g. suggested using cubes to check correctness). She then uses a chart to reflect the combinations, writes down the written symbols and extends its application to coins (pennies and nickels).

For young children to learn the mathematical concept about number composition and decomposition, the following complex underlying big ideas need to be deconstructed, including: (1) a quantity (whole) can be decomposed into equal or unequal parts; (2) the parts can be composed to form the whole (Erikson Institute’s Early Math Collaborative, 2013); (3) there are different combinations; and (4) the sum stays the same regardless of the combination.
**The prompted questions.** Nine open-ended questions are provided to elicit teachers’ PCK around the mathematical topic(s) appeared in the video.

(1) What is the *central mathematical* concept of this activity? Please justify your answer.

(2) What are *other* important *mathematical* concepts you think are related to the central mathematical concept of this activity? Please justify your answer.

(3) What *prior mathematics* knowledge do children need to have in order to understand the central mathematical concept of this activity?

(4) Do the *children* appear to *understand* the central mathematical concept of this activity? Please provide evidence that supports your assessment.

(5) Based on your assessment, *what would you do next* to reinforce or extend children’s understanding? Please justify your answer.

(6) What are some *common mathematical misunderstandings* children might have when learning this central mathematical concept?

(7) What has the teacher *done or said* to help the children understand the central mathematical concept? (Consider things like: materials, setting, lesson design, teacher language, pacing, interactions, etc.) Were those instructional choices *effective*? Please justify your answer.

(8) How could the teacher *change this activity* to meet the needs of a child who is *struggling*? (Consider things like: materials, setting, lesson design, teacher language, pacing, interactions, etc.) Please justify your answer.
(9) How could the teacher change this activity to meet the needs of a child who is advanced? (Consider things like: materials, setting, lesson design, teacher language, pacing, interactions, etc.) Please justify your answer.

The scoring. According to the PCK-EM conceptual model (detailed in chapter 2), the overall responses from each respondent are coded based on 6 subcomponents (not the answers question by question). A 1-5 scale is applied to differentiate understanding from low (more obvious, behavioral, or procedural) to high (sophisticated/conceptual) levels within each dimension. Definitions are consistent across dimensions so that the scores are comparable among the subcomponents (and therefore what/who/how dimensions) (see Appendix D for complete rubrics). The coding was informed by PCK studies from different research groups (e.g., Gardner & Gess-Newsome2011; Kersting, 2008; Loughran et al., 2004).

(1) WHAT_Depth: Understanding of a specific big idea or big ideas, demonstrated by the capability of “deconstructing” a foundational math concept into its complex underlying ideas that young children need to learn.

(2) WHAT_Breadth: Awareness of mathematical concepts related to a specific big idea or big ideas.

(3) WHO_Prior Knowledge: Understanding of young children’s prior knowledge in learning a specific big idea or big ideas

(4) WHO_Misunderstandings: Knowledge of students’ likely misunderstandings and learning difficulties around a specific big idea or big ideas.
(5) HOW_Strategy: Knowledge of pedagogical strategies (either from the video or for own teaching) that can facilitate, reinforce and/or extend students’ understanding of a specific big idea or big ideas.

(6) HOW_Representation: Knowledge of specific representations (illustrations, examples, models, demonstrations and analogies) that can make clear a specific big idea or big ideas to facilitate, reinforce and/or extend students’ understanding.

More specifically, the coding of “depth” is looking for evidence of understanding about the specific big idea(s) in the video, demonstrated by the respondent’s capability of “deconstructing” a foundational math concept into its complex underlying ideas that young children need to learn. The lowest level response (level 1) is limited to behavioral or procedural description of the video and merely describes some obvious ideas without critical inference at a conceptual level. The medium level response (level 3) identifies a specific big idea or big ideas by making some inferences beyond procedures and explicit behaviors. However, the mathematical understanding is still limited without deconstructing the concept into underlying ideas that young children need to learn. The highest-level response (level 5) demonstrates a repertoire of the multiple aspects of the specific big idea(s) that are at conceptual level and generalizable. Anchor descriptions and examples are provided for level 1, 3, and 5. The understanding between 1 and 3, 3 and 5 is assigned as level 2 and 4 respectively (see Appendix D. for the complete rubrics and examples).
Step 1: Survey Takers watch video and answer questions

Step 2: Researchers retrieve narrative responses

Step 3: Coding: Coders rate responses according to 6 subcomponents
"depth" "breadth" "prior knowledge" "misunderstanding" "pedagogy" and "representation") at a 1-5 scale

Step 4: Scoring: Researchers generate scores for PCK-EM

PCK_WHAT: Average of “depth” and “breadth”
PCK_WHO: Average of “prior knowledge” and “misunderstanding”
PCK_HOW: Average of “pedagogy” and “representation”
PCK-EM: Average of “(PCK_WHAT”, “PCK_WHO” and “PCK_HOW”

Figure 9. Flow chart of PCK-EM survey and data generation process.

For instance\(^8\), referring back to the “Number 7” video about number composition and decomposition, examples of a low-level answer would vaguely mention “simple

\(^8\) For More anchor examples, see Appendix D.
addition” or “represent 7 with manipulatives” at a procedural level, and believe that this is the central mathematical idea in the video. Medium level responses would correctly identify the concept of “number combination” without articulating the big ideas. High-level responses are those demonstrating understanding about big ideas of the lesson such as “there are different/various ways/combinations of decomposing the same number.” Here is a flowchart to summarize how PCK-EM survey works.

The rationale of designing the survey and scoring. The design of the assessment has taken the target population of teachers (pre-kindergarten to early elementary), the goals of early math teaching, and current advances in studying PCK (theoretically and methodologically) into consideration. The PCK-EM survey intends to make inference about teachers’ professional knowledge based on their lesson analysis ability. The open-ended survey format is user-friendly for early childhood teachers. The nature of early math teaching (informal and foundational) has determined that a traditional paper-pencil subject matter math test would be inappropriate for teachers working with young children and has little direct connection with their instruction.

Meanwhile, effective stimulus is necessary to elicit knowledge because PCK can be implicit and contextualized, being associated with particular students, events and classroom (Barnett & Hodson, 2001; Carter, 1990). Videos that provide authentic classroom-teaching scenarios similar to what teachers need to deal with in their day-to-day teaching can “reflect the complexities of teaching more faithfully than a test item does” (Shulman, 1988; Kersting, 2008). The prompted questions ask survey-takers about the mathematical ideas presented in the video, students’ math understanding, as well as
how teaching could occur effectively to facilitate learners’ comprehension. By confronting teachers with problems that require judgments, decisions, choices and actions regarding understanding about the content, the students and pedagogy, similar challenges as professionals teach young children, the prompted questions and the video stimulus are likely to elicit unique professional knowledge required in teaching foundational math.

In fact, expert teachers demonstrated far different responses from novice teachers during and after watching the same classroom teaching videos (Bransford, 2000). There is also evidence of the positive relationship between teachers’ lesson analysis capacities and students’ learning (Kersting et al., 2010; Roth et al., 2011; Seago, 2003). Many research projects have started investigating PCK at a grain size through specific topics (e.g. Van Driel et al., 1998; Loughran et al., 2008). Together, it is suggested that video-based lesson analysis abilities around specific math topics can be a proxy to infer teachers’ inert and unconscious knowledge (i.e., PCK).

As well, the content of the video is carefully selected. The video “number 7” applied in PCK-EM survey involves number composition and decomposition, a type of number system knowledge. It features mathematical topics aligned with the grade level, i.e. Pre-kindergarten to 3rd grade. Compared with content covered in certificate exam or advanced tests in college, it is more likely to match the understanding required for teaching young children. The specific mathematic topics selected for the survey are also important concepts for future success. Early math competence is associated with later school success (Duncan et al, 2007; Geary et al., 2013; Jordan, Kaplan, Ramineni, & Locuniak, 2009), which further predicts adult employability and income (Cohen,
Dowker, Heine, Kaufmann, & Kucian, 2013; Hanushek & Woessmann, 2008; Rivera-Batiz, 1992). However, not all math understanding in the early years predicts later success. Early number system knowledge\(^9\), predicted functional numeracy more than six years later; but skills using counting procedures to solve arithmetic problems did not (Geary et al., 2013). Therefore, PCK-EM survey intentionally selected a teaching scenario about number system knowledge.

There is a consistent line of thought from the PCK-EM conceptual model to the design of the prompted questions and the coding framework (i.e., measurement model). The conceptual model of PCK-EM guides generation of the questions in PCK-EM survey and its coding framework to capture different facets of content knowledge used for teaching (i.e., PCK). In line with the conceptual model of PCK-EM, the questions are intended to reveal respondents’ integrated understanding of the subject content (mathematics), students’ learning of specific content, and instructional strategies to make the content accessible to learners. The coding framework is aligned with the PCK-EM conceptual model.

There are, however, differences between the conceptual model of PCK-EM and its practical application (i.e., coding framework), because of the shift from conceptualizing PCK at a subject level to studying PCK at a topic level. In the conceptual model of PCK-EM, multiple domains of mathematics, their relationships and how they apply to young children’s learning are included at a broad level. The PCK-EM survey (and its coding rubric), on the other hand, is making inference about teachers’ PCK in

\(^9\) Number system knowledge refers to understanding about the systematic relations among Arabic numerals and skill at using this knowledge to solve arithmetic problems.
early mathematics based upon their lesson analysis ability on one (or several) video(s).

Therefore, it is necessary to make adjustments by operationally applying the general
theoretical conceptualization (i.e., PCK-EM conceptual model) to a particular
mathematical topic (i.e., coding for PCK-EM survey responses). Following this thought,
the coding uses the specific big ideas that appear in the video to determine the relevance
and sophistication of understanding. In other words, the rating considers to what extent
the respondents are able to identify concepts, strategies etc. related to the specific big
idea(s) that are the goal for children’s understanding, and how appropriate it is to
promote children’s understanding about the big idea(s). See Appendix C for “general
principles of coding.”

Another major change involves replacing the subcomponent of “learning path”
with “prior knowledge.” The revision is also related to the shift from mathematics in
general as a subject to a specific mathematical topic. Theoretically, there is a dimension
of students’ learning path around mathematics concepts and ideas, including prior
knowledge and extensions. However, when it applies to a specific topic in foundational
math, learning path is more likely to overlap with relevant mathematical ideas. In fact,
the pilot study on responses to “number 7” video showed very limited evidence existing
for “knowledge extension”, one of the two aspects of “learning path.” When there was
some evidence, it was more likely to overlap with “breadth.” Therefore, revision was
made to the coding framework, in which prior knowledge replaces the overall learning
path, and information about knowledge extension is combined with “breadth.”
Because of the content-specific nature of PCK, it is necessary to apply the coding rubric to a specific video. Similar to Co-Res developed by Loughran and his colleagues (2004), topic-specific PCK is detailed in “anchor responses” for a particular video stimulus (See Appendix D for anchor answers of video “Number 7”). It was generated based on the lesson analysis of specific videos used in the survey, experts’ responses to the same survey, big ideas in foundational math (Early Math Collaborative, 2014), learning paths of particular mathematical topics(Clements & Sarama, 2004; Platas, 2008), and math-specific pedagogies for young children (Battista, 2004). The anchor answers serve to effectively facilitate the process of reaching inter-coder reliability and as reference to keep on-going consistency among coders.

**Supplementary Measures**

**About my teaching** is an online survey collecting teachers’ demographic information administered together with PCK-EM survey. There are two versions of the survey corresponding to two time points (pre-test in fall 2011 and follow up in spring 2013). Questions include teachers’ age span, ethnicity, years of teaching, educational backgrounds (such as degree and certificate earned), and experiences of taking mathematics courses and workshops. See Appendix F and G for more details.

**High impact strategies in early mathematics (HIS-EM)** is an observation tool to identify and measure the quality of mathematics teaching practice in preschool through third grade. It attempts to identify high-impact teaching strategies that reflect pedagogical content knowledge (PCK) aligned with research in mathematics classroom teaching(Borko, Stecher, Alonzo, Moncure, & McClam, 2005; Stecher et al., 2006) and
principles and standards of early mathematics teaching and learning recommended by NCTM (2000) and CCSS-M (2010). The observation is from the start to the finish of a single teacher-directed mathematics lesson scored on a 7-point Likert scale. The observer captures quality of the lesson through 9 dimensions regarding teachers’ understanding of the mathematical content, knowledge of students’ individual characteristics and the application of appropriate teaching strategies. High quality of mathematical teaching was demonstrated by clear learning goals that highlight concept development, the use of developmental appropriate learning formats and math-specific representations, the sensitivity to individual differences and the capability of creating math learning community to engage young learners.

Confirmatory Factor Analyses (CFAs) revealed that the 9 aspects of HIS-EM were highly correlated with each other, suggesting a single underlying mathematics teaching proficiency. The alpha for internal consistency among the nine dimensions was .97. According to a pilot study in schools serving primarily low-income, ethnically diverse students, the tool has demonstrated acceptable concurrent validity. Teachers’ HIS-EM scores were associated with students’ learning gains over a year $(p < .01, N = 181)$ (Cerezci & Brownell, 2015).

**Tools for Early Assessment in Math (TEAM)** is a research-based instrument assessing young children’s mathematical thinking on multiple topics (Clements et al., 2011). By providing materials and illustrations beyond oral direction, the assessment allows for a variety of responses (e.g. manipulative and oral). The assessment results
provide information about each student’s skill level based on their solution strategies and error types.

The assessment is available in two grade spans, Pre-K-2 and 3-5. There are two parts, A and B. Part A measures young children’s knowledge of number such as recognition of number and subitizing, verbal counting and object counting; number composition and decomposition, adding, place value and multiplication and division. Part B measures understanding about geometry such as shape recognition, shape composition and decomposition, and spatial imagery. It also includes items on geometric measurement and patterning using geometric shapes.

The assessment is administered individually without time limits on responses. Pre-K-2 assessment includes suggested start points by grade level, for instance, students from kindergarten is suggested to be assessed from item 13 in part A and item 94 in part B. The 3-5 (i.e., the grade levels) assessment begins with a pre-screen that students complete on their own\(^\text{10}\). Each student is then placed in the formal assessment at a point customized to his or her skill level based on results from the pre-screen. It stops after four consecutive incorrect responses for each part. The developer also provides a lookup table to convert the raw score into T score (IRT score) for data analysis purpose.

**Woodcock-Johnson-III, applied problems (WJ-AP)** is the 10\(^{th}\) subtest of Woodcock-Johnson-III (Woodcock et al., 2001); it is a standardized, norm-referenced measure of math achievement. The subtest generally takes 10 minutes to administer. The

\(^{10}\) In this particular project, only Pre-K-2 version of assessment was applied. At the time when the project started, the “3-5” assessment was available and the developer of TEAM suggested applying the “Pre-K-2” assessment to pre-K through 3\(^{rd}\) graders without ceiling effect.
test was done one-on-one, and all child responses were verbal and gesture-based (i.e., pointing to the selected answer). Testing begins with an item corresponding to the child’s age and ends when the child makes six consecutive mistakes. Raw scores on this assessment can be converted to age estimate and standardized scores for meaningful interpretation in relation to national norms, the standardized scores can also be used for statistical modeling purposes.

**Procedure of Data Collection**

The majority of data collection occurred at two time points, fall 2011 (pre-test) and spring 2012 (post-test) for all teacher level and child assessments\(^\text{11}\). Complementary demographic information was collected by the end of the 2\(^{\text{nd}}\) year data collection (spring 2013).

The **PCK-EM survey** was administered consecutively with “*about my teaching*” *survey* (to collect demographic information of participating teachers, see Appendix F and Appendix G). At pre-test, teachers who enrolled in the program (both intervention and comparison) received an email notice with an online linkage to the survey and a confidential ID number. In the follow-up data collection (spring 2013), teachers who stayed in the program received another email notice about the complementary survey of demographic information. The entire survey could be done from any computer with Internet access, and in general took about an hour to finish. Technical support and

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\(^{11}\) In the fall 2011 and spring 2012, data were collected from teachers and students in both intervention and comparison schools through surveys, classroom observations and standardized tests. Because the current analysis won’t evaluate the program impact, only the relevant information will be presented.
assistance were available from the implementation project staff via phone or email (and the contact information was included in the same email).

A pilot study was conducted to develop the coding rubrics and reach inter-rater reliability. Coders received intensive training that involved reading about early mathematics (e.g. CCSS, Big Ideas, PCK-EM Manual, etc.), watching the stimulus videos and coding responses. After two-week intensive discussions about the conceptualization of PCK, the big ideas covered in specific video stimuli (such as the video of “Number 7”) and getting familiar with the coding framework and rubrics, coders independently coded 10 responses retrieved from PCK-EM survey. After reaching inter-rater reliability (2 adjacent scores from 1 to 5 were considered acceptable among 4 coders, and the overall agreement must be over 80%), coders started coding about 20 responses per week.

Three coders were involved in coding responses to PCK-EM survey. The coding was completed within 6 weeks. Inter-coder reliability was examined by assigning the same set of responses (20%) to two coders; repeated within-coder reliability was checked through giving each coder a portion of their previously coded responses (4 weeks apart). The whole process was blind to the coders. Coders were given feedback regarding inconsistencies; discrepancies were then discussed and resolved before moving to the next set of coding.

Inter-rater reliability was calculated using percent-within-one (PWO) analysis, percentage of exact agreement and intra class correlations (ICCs). The PWO ranged between 90% to 97% between coders, and 40% to 65% if using exact match criteria.
Individual ICCs were between .48 and .84; and average ICCs were between .65 and .91. Intra-rater repeated reliability was calculated using PWO analysis as well as percentage of exact agreement. PWO ranged from 87.5% to 100%, with exact match percentage ranged from 50% to 62.5%.

**HIS-EM** The program data coordinator scheduled observations in coordination with the participating teachers. Trained observers then did one-time, on-site observations using the HIS-EM tool. The observation started and ended at the beginning and end of a teacher-directed math lesson, as defined by the teacher and lasted about 30 minutes on average. To keep reliability, observers met twice to code a video-recorded mathematical lesson during the field observation period, and the scores were compared to master scores developed by the research team.

**TEAM & WJ-AP** Young children’s mathematical achievement was assessed via two math assessments: WJ-AP subtest (Woodcock, McGrew & Mather, 2011) and TEAM (Clements &Sarama, 2011). Once the teachers agreed to participate in the research, all parents in the teacher’s class were asked to provide consent for their children to participate in the study (teachers distributed consent forms to parents). Parents’ consent forms were available in English, Spanish, Polish, Urdu, and Arabic. When necessary, a few parent-teacher gatherings were held to provide more information about the child tests. Among the consented students, 10 students from each classroom were randomly selected and administered the two math assessments.

The assessment occurred during the same time period as teacher observation. Child assessors trained through a third party went to the classroom and conducted the
child assessments. At the first time point of data collection (i.e., fall 2011), child assessors randomly selected 10 children (ideally) from within each participating classroom. During spring 2012, post assessment was conducted for any child assessed at pre-test. When the parent’s consent form indicated that the language spoken at home was neither English nor Spanish, the assessor conferred with the teacher about students’ ability to be accurately assessed in English and made decision about administrating the assessment upon a few minutes of casual communication with the child. Each assessment usually took about 20 to 35 minutes and was held in a quiet location in or near the classroom. All assessed children received a book (random topics without explicit math content).

**Attrition**

At the teacher level, attrition was due to voluntary participation. The design of the PCK-EM survey determined that the data would only represent those who were willing to and eventually participated (i.e., 182 out of 220 teachers, 82.7%, responded the survey). For teaching observations, all teachers in the comparison schools who consented were observed; however, there were 3 teachers (3.5% of the sample) who took the online survey, but were resigned or removed from the positions at school. For those teachers, there were no observation data (nor child data) available. Therefore, the number of teachers who had both PCK and teaching quality data was 82.

There were several reasons for students not being assessed. While there were 85 teachers from the comparison schools who took PCK-EM survey at pre-test, 74 teachers had children from their classes being assessed at pre-test. This was because 1) some
teachers resigned or were removed from their positions at school before the child assessment began ($N = 3$); 2) for teachers working with special needs or gifted children, no child assessments were administered ($N = 8$).

At the child level, children were randomly selected from those who were 4 years old by September 1, 2011, were enrolled in the study schools and participating teachers’ classrooms, had parents’ consent to be assessed, and were able to complete the students’ assessments in either English or Spanish at pre-test. At post-test, only those who had a pre-test and did not have an IEP and/or a 504 plan (revealed at post-test) were tested. Children who had a pre-test but were no longer in the same school were excluded from the post data collection. About 9.5% ($N = 72$, from 756 at pre-test to 684 at post-test) of the students were either no longer at the school at post-test or were identified for special education service between pre- and post-test (IDP/504 plan). There were also a few missing child assessments that could be attributed to assessor errors; therefore some children only had one of the assessments scores available. This applied to 0.9% ($N = 7$) of the students’ sample in pre-test and 2.8% ($N = 21$) of the students’ sample in post-test.

**Data Analysis**

The purpose of this study was to investigate the profile of early childhood teachers’ knowledge of teaching mathematics through the lens of PCK-EM, and examine its relationships with math teaching quality and students’ learning gains in mathematics over a school year. Descriptive statistics analysis was conducted on measures of math teaching quality and child outcome. Analyses were conducted using SPSS Statistics 22.0 (IBM Corp, 2013) and Mplus7.0 (Muthén & Muthén, 2012) for the first question and

**Question 1. The Profile of PCK-EM**

What is the profile of early childhood teachers’ PCK-EM? The question can be further divided into four:

**Sub question 1.1 What is the distribution of each dimension of PCK-EM?**

Descriptive analysis was calculated to describe the distribution of the teachers’ professional knowledge, including mean, standard deviation (SD), range, skewness and kurtosis of each PCK-EM dimension (i.e., “what” “who” and “how”). The mean of each dimension indicates the average level of understanding for the corresponding aspect of knowledge, and the standard deviation and range suggests the variation of understanding from the mean. The asymmetry and deviation from normal distribution is indicated by the value of skewness: either symmetrical distribution around the mean (zero), right skewed distribution with most values concentrated on the left of the mean and extreme values concentrated to the right side of the mean (a positive value) or left skewed (a negative value). Skewness between -0.5 and +0.5 is approximately symmetric; skewness less than -1 or more than +1 is highly skewed (Bulmer, 1979). Kurtosis is a sign of flattening of a distribution, a kurtosis of 3 suggests normal distribution, kurtosis less than 3 indicates distribution flatter than normal distribution, therefore less extreme values and wider spread of values around the mean (compared with normal distribution); and vice versa (Groeneveld & Meeden, 1984).
Sub question 1.2 What are the relationships among the 3 dimensions of PCK-EM? It was assumed that the 3 dimensions of PCK-EM are moderately correlated. Pearson correlation coefficients and Spearman correlation coefficients were used to examine the relationships among PCK-EM dimensions. The Pearson correlation coefficient assumes that both variables being correlated are measured with equal-interval scales, whereas the Spearman correlation coefficient assumes that both variables being correlated are ordinal. While the PCK-EM measure is designed to be an interval scale, in practice, it is closer to ranked scores; therefore, both correlation coefficients were reported to determine the direction, strength and significance of the relationships among the six dimensions of knowledge. Besides the level of statistical significance, the magnitude of the observed relationship was assessed in terms of whether correlations are small ($r \leq .10$), medium ($r = .30$), or large ($r \geq .50$) in size (Cohen, 1988).

Sub question 1.3 Are some dimension(s) of PCK-EM better developed than others? Repeated Measures of ANOVA was conducted to answer this question. The rubrics were designed to be consistent across dimensions; therefore, it is possible to compare scores directly among dimensions of PCK-EM. The analysis serves two purposes: 1) to examine whether the levels of each PCK-EM dimension is significantly different from the levels of other dimensions overall; 2) to scrutinize which dimension(s) of PCK-EM is (are) different if there is any difference. More specifically, 1) the test of equal means suggests whether there is a globally equivalent distribution across PCK-EM dimensions by $F$ value and its level of statistical significance (i.e., the main effect), as
well as effect size in terms of eta-squared; and 2) by post hoc analyses to determine which dimension(s) of PCK-EM is (are) significantly different from others.

**Sub question 1.4 Are there different groups of teachers indicated by their PCK-EM profiles?** Latent Profile Analysis (LPA), a more rigorous approach than cluster analysis, was applied to investigate whether the PCK-EM profile identified different groups of teachers and what they looked like. More specifically, it would compare models with different numbers of latent classes by the Lo-Mendell-Rubin Adjusted likelihood ratio test, and indicators such as AIC and BIC for the goodness-of-fit. When there was more than one latent class, ANOVA and Repeated ANOVA would provide information about the characteristics of each latent class.

**Question 2. The Prediction of Teaching and Learning by PCK-EM**

What are the relationships between early childhood teachers’ knowledge, practice and students’ learning gains? More specifically,

**Sub question 2.1 What is the relationship between early childhood teachers’ knowledge and their classroom teaching quality in mathematics?** Hierarchical linear modeling (HLM) was conducted to answer this question. Teachers in the sample were from 8 different schools and were therefore not independent from one another. Although propensity scores regarding characteristics of the student body in each school were applied to match paired intervention and comparison schools, overall, the differences of school characteristics were likely to impact teachers from the same school. There were also imbalanced numbers of teachers from different schools, depending on the size of the school. HLM analysis can account for the effects of the nested design. It allows for
estimating the contribution of teachers’ knowledge to the quality of teaching and provides a means of utilizing all the variance data across schools, despite the imbalanced sizes (Raudenbush & Bryk, 2002).

Level 1 HLM regression equation:

\[ \text{HIS-EM} = \beta_0 + \beta_1 (\text{PCK-EM}) + \beta_2 (\text{YEAR}) + \beta_3 (\text{CLASS}) + \beta_4 (\text{WORKSHOP}) + r \]

Level 2 HLM regression equation:

\[ \beta_0 = \gamma_{00} + \mu_0 \]
\[ \beta_1 = \gamma_{10} + \mu_1 \]
\[ \beta_2 = \gamma_{20} + \mu_2 \]
\[ \beta_3 = \gamma_{30} + \mu_3 \]
\[ \beta_4 = \gamma_{40} + \mu_4 \]

HIS-EM is the average score of teaching quality observation, YEAR, CLASS, and WORKSHOP refers to teachers’ self-reported years of teaching, number of math classes taken in pre-service trainings, and hours of math-related workshops taken during in-service time.

Of special interest was the relationship between teachers’ content knowledge (i.e., PCK-EM, level 1 predictor variable) and their teaching quality (i.e., HIS-EM, level 1 outcome variable) after controlling for school level variances (no variables at this level were specified). The current study was not interested in interpreting how and which variable(s) at school level impact the association.

It is noted that a combined PCK-EM score (i.e. mean across dimensions) was entered into the equation. According to analysis on PCK, the correlations among PCK-
EM dimensions were moderate. In the meantime, the teaching quality was measured by new assessment; and its psychometric property needs to be further investigated. Therefore, it is more appropriate to investigate whether overall, knowledge (i.e. PCK) significantly predicts teaching quality in mathematics, rather than looking at PCK-EM at the dimension level as a predictor.

Following the established standard of HLM analysis (Raudenbush & Bryk, 2002), an unconditional model (without any independent variable)

\[ \text{HIS-EM} = \beta_0 + \mu \]

\[ \beta_0 = \gamma_00 + \mu_0 \]

was conducted to calculate Intra-unit Correlation Coefficients (ICC). A significant ICC suggests there were group differences (school) in teaching quality. F test in the dependent variable (i.e., teaching quality at pre-test) indicates there were individual differences in teaching quality, in which case a conditional model at level 1 was further conducted to explore whether and to what extent teachers’ PCK explains the differences in their math teaching quality during a similar time period, after controlling for the teachers’ seniority (years of teaching), and pre-service (math courses taken) and in-service (workshops taken) math learning experience.

**Sub question 2.2 What is the relationship between early childhood teachers’ knowledge and their students’ learning gains in mathematics?** HLM was conducted to answer this question. In the current study, students were nested within classroom (teachers), and teachers were further nested within schools. Students from the same class may share some similarities regarding the relationship between teachers’ knowledge and
their learning. Similarly, teachers from the same school could be impacted by the shared characteristics of their student body in a similar way. There were also imbalanced numbers of students between classrooms, and uneven numbers of teachers between schools. HLM analysis can account for the effects of nesting using maximum likelihood estimation. It allows for estimating the contribution of teachers’ knowledge to students’ learning gains, and provides a means of utilizing all the variance data between classrooms and schools, despite the imbalanced sizes (Raudenbush & Bryk, 2002).

Hierarchical linear modeling (HLM) was conducted to analyze a data set where students (level 1) were nested within teachers (level 2), who further nested within schools (level 3). The analysis attempted to reveal whether students’ mathematics learning (level 1 outcome variable) was predicted by teachers’ content knowledge (level 2 predictor variable) after controlling for school level variances (no variables at this level were specified). Model testing proceeded in three phases: null model, random intercepts model, and random intercepts and slopes model.

Level 1 HLM regression model:

\[
\text{POST-MATH} = \pi_0 + \pi_1 (\text{PRE-MATH}) + \pi_2 (\text{GENDER}) + \pi_3 (\text{LANGUAGE}) + \pi_4 (\text{AGE}) + e
\]

Level 2 HLM regression model:

\[
\pi_0 = \beta_{00} + \beta_{01} (\text{PCK-WHAT}) + \beta_{02} (\text{PCK-WHO}) + \beta_{03} (\text{PCK-HOW}) + r_0
\]

\[
\pi_1 = \beta_{10} + \beta_{11} (\text{PCK-WHAT}) + \beta_{12} (\text{PCK-WHO}) + \beta_{13} (\text{PCK-HOW}) + r_1
\]

\[
\pi_2 = \beta_{20}
\]

\[
\pi_3 = \beta_{30}
\]
\pi_4 = \beta_{40}

Level 3 HLM regression model:

\beta_{00} = \gamma_{000} + u_{00}

\beta_{01} = \gamma_{010}

\beta_{02} = \gamma_{020}

\beta_{03} = \gamma_{030}

\beta_{10} = \gamma_{100}

\beta_{11} = \gamma_{110}

\beta_{12} = \gamma_{120}

\beta_{20} = \gamma_{200}

\beta_{30} = \gamma_{300}

\beta_{40} = \gamma_{400}

Where POST-MATH is students’ math performance by the end of the academic year, indicated by either TEAM T scores or WJ-AP standardized scores at post-test; PRE-MATH is students’ learning outcome at the beginning the school year; and PCK-WHAT, PCK-WHO, and PCK-HOW is the three dimensions of PCK-EM, indicating teachers’ knowledge. GENDER is students’ gender (1 = male, 2 = female), LANGUAGE is the language used in the assessment (1 = English, 0 = not English). AGE is students’ age in months at pre-test. When the outcome variable is WJ-AP scores, students’ age was not included because the WJ-AP score has been converted to standardized scores based on a national norm and is comparable across age spans. When the outcome variable is TEAM
T scores, students’ age in months was included in the regression equation because TEAM T score did not make age adjustment.

Similar to question 2.1, an unconditional model at level 1 was conducted following the established standard of HLM analysis (Raudenbush & Bryk, 2002). A significant ICC would suggest there were group level differences at school level (i.e., level 3) and add teachers’ level (i.e., level 2). Thereby indicators can be further added to interpret the differences in students’ performance (level 1). Pre-test scores and other level 1 indicators (see above) were entered into level 1 equation to explore whether and to what extent the performance at pre-test explains the differences in math performance at post-test. Different from the HLM analysis of math teaching quality predicted by teachers’ knowledge, the three dimensions of PCK-EM, instead of the mean of PCK-EM, were applied. The psychometric properties of the assessments at student level (i.e., WJ and TEAM) are well known (Woodcock et al, 2011; Clements & Sarama, 2011) and there were moderate correlations among the three dimensions of PCK-EM, therefore, the three dimensions of PCK-EM were entered simultaneously to investigate whether and to what extent different aspects of teachers’ PCK impacts students’ math performance.
CHAPTER IV

RESULT

The results displayed in this chapter were derived from the coded scores of open-ended responses to PCK-EM survey (content knowledge for teaching foundational mathematics to young children), HIS-EM observation scores (math teaching quality), and scores of WJ III-AP and TEAM (students’ mathematical performance). PCK-EM and HIS-EM scores were teacher level data collected at the beginning of the school year; WJ III-AP and TEAM were children’s outcome data collected at the beginning and end of the same school year. The results of the analyses are organized into two major sections: 1) the profile of PCK-EM, 2) the prediction of teaching effectiveness, including the math teaching quality and students’ learning, by PCK-EM.

A Profile of Early Childhood Teachers’ PCK in Early Mathematics

The Distribution of PCK-EM

The investigation started by exploring the profile of early childhood teachers’ PCK in foundational math. The overall PCK-EM score (mean across six dimensions) was 2.36 (on a 1-5 scale) and ranged from 1.17 to 4.50, with a standard deviation of .56. The medium level results partially supported the hypothesis that although adults have acquired knowledge in basic mathematics for their own use, the understanding from a PCK perspective was not fully unpacked for the purpose of teaching.
Teachers’ professional knowledge was further revealed by three dimensions of PCK: “what” (understanding about mathematical content), “who” (knowledge of young children learning math), and “how” (awareness of math-specific pedagogies). The score for each dimension was generated by averaging its two sub-components (i.e., “depth” and “breadth” for “what” dimension; “prior knowledge” and “misunderstanding” for “who” dimension; and “strategy” and “representation” for “how” dimension). As shown in Table 7, the average level of content understanding for mathematics (“what”) was 2.48 ($SD = .65$), the knowledge for young children learning math (“who”) was 2.27 ($SD = .70$), and the awareness of math-specific pedagogy (“how”) was 2.33 ($SD = .77$). The majority (more than 75%) of the respondents demonstrated limited understanding about content, students and pedagogy.

Table 7. Descriptive Statistics of PCK-EM Dimensions

<table>
<thead>
<tr>
<th></th>
<th>What (Knowledge of foundational math)</th>
<th>Who (Knowledge of young children learning math)</th>
<th>How (knowledge of math-specific pedagogy)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>2.48</td>
<td>2.27</td>
<td>2.33</td>
</tr>
<tr>
<td>(SD)</td>
<td>(.65)</td>
<td>(.70)</td>
<td>(.77)</td>
</tr>
<tr>
<td>Minimum</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Maximum</td>
<td>5</td>
<td>4</td>
<td>4.5</td>
</tr>
<tr>
<td>Percentiles</td>
<td>25% 2.00</td>
<td>2.00</td>
<td>1.88</td>
</tr>
<tr>
<td></td>
<td>50% 2.50</td>
<td>2.50</td>
<td>2.50</td>
</tr>
<tr>
<td></td>
<td>75% 3.00</td>
<td>2.50</td>
<td>3.00</td>
</tr>
<tr>
<td>Skewness</td>
<td>.65</td>
<td>.25</td>
<td>.41</td>
</tr>
<tr>
<td>Kurtosis</td>
<td>.83</td>
<td>-.25</td>
<td>-.10</td>
</tr>
</tbody>
</table>

Note: $N = 182$.

PCK-EM: Pedagogical Content Knowledge in Early Mathematics survey, indicating content knowledge for teaching foundational mathematics at pre-test.

What: content understanding aligned to specific age groups, i.e., understanding about foundational math.

Who: content-specific knowledge of learners and their learning, i.e., knowledge of young children learning math.

How: content-specific pedagogical knowledge, i.e., math-specific pedagogical knowledge.
The skewness of the dimensions of PCK-EM was between .25 to .65, suggesting that the distributions of each dimension were roughly symmetric but right-skewed. In other words, the majority of the scores for each dimension concentrated on the left side of the mean (around 2.5), with extreme values concentrated to the right side of the mean (between 4 and 5). The Kurtosis values were less than 3 for all dimensions. Therefore, the distribution was flatter than normal distribution and there were less extreme values and a wider spread of values around the mean (i.e., around 2.5).

The Comparisons among the dimensions of PCK-EM

An ANOVA for repeated measures (“what”, “who” and “how” dimensions of PCK-EM for the same participant) was conducted to explore: 1) whether the different aspects of knowledge developed simultaneously; and 2) if teachers’ scores did not evenly distribute across the 3 dimensions of PCK-EM, which dimension(s) of understanding was (were) better developed. The results showed that there was a significant overall difference among the three dimensions of PCK-EM ($F (2, 362) = 7.74, p < .01$) that represented a small size effect size (partial $\eta^2 = .04$). Pairwise comparison further suggested that there were significant differences in understanding between “what” and “who” ($p < .001$) and “what” and “how” ($p < .05$), but not “who” and “how”. The results indicated that teachers’ understanding about foundational math content seemed to be significantly higher than understanding about young children learning math and pedagogy of teaching early math, while there was no significant difference between the latter two.
The Relationships among the dimensions of PCK-EM

Correlational analysis was run to examine the degree to which different dimensions of PCK-EM were interrelated. It was assumed that the 3 dimensions of PCK-EM were moderately correlated. The results supported the hypothesis: significant and positive relationships were found among the three dimensions of PCK-EM. The Pearson correlational co-efficient was between .41 to .48, \( p < .001 \); and Spearman correlation revealed similar moderate correlations (between .35 to .43, \( p < .001 \)), see Table 8.

Teachers with a good understanding of math content also tended to grasp how young children learn the same concept and how to effectively present mathematical concepts.

Table 8. Pearson Correlations among PCK-EM Dimensions

<table>
<thead>
<tr>
<th>Pearson correlation</th>
<th>What: (Knowledge of foundational math)</th>
<th>Who: (Knowledge of young children learning math)</th>
<th>How: (knowledge of math-specific pedagogy)</th>
</tr>
</thead>
<tbody>
<tr>
<td>What</td>
<td>.41**</td>
<td>.48**</td>
<td></td>
</tr>
<tr>
<td>Who</td>
<td>.43**</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note: \( N = 182 \).
**The correlation is significant at .01 level (two-tailed).
What: content understanding aligned to specific age groups, i.e., understanding about foundational math
Who: content-specific knowledge of learners and their learning, i.e., knowledge of young children learning math
How: content-specific pedagogical knowledge, i.e., math-specific pedagogical knowledge

The Profiles of PCK-EM at Individual Level

To explore whether there were distinct groups of teachers regarding the profile of PCK-EM, whether the profile of PCK-EM could differentiate expert teachers from others, and what was the proportion of expert teachers, latent profile analysis was run on the total sample (182 cases) to the three dimensions of PCK-EM ("what", "who" and "how"). The results suggested that there were three clusters of teachers. More specifically, while
comparing models with two and three latent classes, the Lo-Mendell-Rubin Adjusted likelihood ratio test was 25.60 ($p < .05$), suggesting that there was a significant improvement in goodness-of-fit for a model of three latent classes over the model of two latent classes. The indicators of model fit were: $AIC = 1101.94$, $BIC = 1133.98$.

Table 9. Teachers’ Grouping Results from Latent Profile Analysis

<table>
<thead>
<tr>
<th>Cluster Mean (SD)</th>
<th>Number</th>
<th>%</th>
<th>PCK-EM (Overall)</th>
<th>What (Knowledge of foundational math)</th>
<th>Who (Knowledge of young children learning math)</th>
<th>How (Knowledge of math-specific pedagogy)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Low 87 (48%)</td>
<td>1.92</td>
<td>2.14</td>
<td>1.89</td>
<td>1.74</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Medium 88 (48%)</td>
<td>2.68</td>
<td>2.70</td>
<td>2.57</td>
<td>2.77</td>
<td></td>
<td></td>
</tr>
<tr>
<td>High 7 (4%)</td>
<td>3.88</td>
<td>4.14</td>
<td>3.36</td>
<td>4.14</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note: $N = 182$.
What: content understanding aligned to specific age groups, i.e., understanding about foundational math
Who: content-specific knowledge of learners and their learning, i.e., knowledge of young children learning math
How: content-specific pedagogical knowledge, i.e., math-specific pedagogical knowledge
SD: standardized deviation

As shown in Table 9, 87 out of 182 respondents were in cluster 1, featured by a “low” level of overall understanding; 88 teachers were in cluster 2, characterized by a “medium” level of content knowledge; the rest of the teachers (7) were in cluster 3 with relatively “high” mathematical knowledge for teaching young children. Figure 10 illustrated the comparison of PCK-EM profiles among the three groups of teachers.

To sum up, the latent profile analysis suggested that PCK-EM profiles can differentiate early childhood teachers’ content expertise and there was only a small portion of expert teachers whose PCK-EM was much higher than others. This analysis

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1 Similar results were obtained comparing models with two-cluster and one-cluster model, suggesting two-cluster model outfitted one-cluster model.
also highlighted characteristics of teachers with less professional knowledge about teaching early mathematics.

![Figure 10. The Profile of PCK-EM by Teachers’ Grouping.](image)

Note: N = 182.
The sample size for “Low”, “Medium” and ‘High” clusters were 87, 88, and 7 respectively.

**The Prediction of Teaching and Learning in Early Mathematics by PCK-EM**

The validity study was designed to obtain empirical evidence concerning the appropriateness of conceptualizing PCK as an indicator (more effective than traditionally defined subject matter understanding) of quality teaching and students’ learning. It was hypothesized that PCK positively predicted teaching quality and students’ learning in mathematics. This is a subset of data from previous analysis; only teachers and students from comparison schools were applied. The relationships between teachers’ knowledge and practice, as well as students’ learning gains were explored.
The Prediction of Math Teaching Quality by PCK-EM

Using HIS-EM, a 1-7 scale observation tool, teaching quality was indicated by averaging the ratings from 9 aspects for each teacher. It was suggested that overall teaching quality was about medium level ($M = 4.14$), ranging from 1.67 to 6.78, with a standard deviation of 1.30 ($N = 79$).

Hierarchical linear modeling (HLM) was conducted to analyze a data set where teachers (level 1) were nested within schools (level 2). Model testing proceeded in two phases: null model and random intercepts model. The null model revealed $\chi^2 (7) = 11.64$, $p = .11$) and an ICC of 0.051, suggesting that there were marginal significant differences at school level, about 5.1 % of the variance in teaching quality was between schools, and on average, 94.9% of the variance in teaching quality was between teachers within a given school.

Next, the average score of PCK-EM was entered into the teacher level regression equation. Both assessments involved in this analysis, PCK-EM survey and HIS-EM observation, are new measurements. Their underlying constructs and validity need to be further investigated (the analysis itself also provides evidence to validate the two assessments); therefore, it is more appropriate to explore whether there is an overall association between teachers’ knowledge and the quality of instruction than to do analysis at the dimension level.

The results indicated that average score of PCK significantly predicted the quality of teaching ($r = .81$, $p < .01$, $N = 79$), after controlling for school level differences. The further explained variance was 4.5%, indicating that PCK-EM explained 4% of the
variance in math teaching quality (within the same school). Adding other indicators of teaching experience, including the years of teaching \((p = .49)\), number of math courses taken in pre-service training \((p = .52)\), as well as hours of workshop taken during in-service time \((p = .62)\) didn’t change the significant prediction of teaching quality by PCK-EM \((r = .77, p< .05, N = 79)\). See Figure 11 for illustration and Appendix H for detailed output of HLM analysis.

Figure 11. The Prediction of Mathematics Teaching Quality by PCK-EM.

Note: \(N = 79\).
The line was made according to the estimation of intercept and regression coefficient from the regression model. The three dots represent individuals whose PCK was -1 SD, 0 SD and +1SD from the mean.

It is noted that the above results were based on a sample size of 79 teachers.

Originally, there were 82 teachers from comparison schools who answered the PCK-EM survey and observed (i.e., had teaching quality data). Scatter plot suggested there were 3 teachers with either high PCK but low teaching quality or low PCK but high teaching
quality (see Appendix I). Statistical criteria of detecting paired-data outliers such as cook’s distance and dfbeta further confirmed these teachers as outliers. Regression analysis with and without the three suspected cases revealed quite different results: conducting HLM analysis with a sample size of 82 teachers didn’t reveal any relationship between PCK and teaching quality (See Appendix I). As well, the accuracy of using PCK-EM survey to infer teachers’ knowledge can be largely impacted by teachers’ attitudes and interpretation of the survey (McCray & Chen, 2012), teaching quality observation can also be imprecise as it was a one-time evaluation (Newton, 2010). Therefore, it is likely that the three cases are outliers.

In sum, the results supported the hypothesis about the relationship between PCK-EM and math teaching quality. Because little is known about the applicability of PCK as an indicator of effective teaching in early math education, the significant relationship also partially validated the conceptualization and assessment of PCK in early mathematics. It also supported the hypothesis that PCK is a more effective indictor of content teaching competence than traditional indicators of teachers’ subject matter understanding, such as degree earned, years of teaching, courses taken in pre-service preparation, and workshops attended during in-service practice.

The Prediction of Students’ Math Learning by PCK-EM

The regression analysis investigated the association between teachers’ knowledge and students’ learning in early mathematics, which has not been fully explored in previous research. There were two different indicators of students’ learning outcome from two assessments. It was hypothesized that teachers’ PCK-EM is a reliable predictor
of students’ learning in terms of two different assessments of mathematical understanding. More specifically, the score of PCK-EM was hypothesized to predict students’ learning, and conceptual understanding (i.e., “what” dimension of PCK) was expected to be a leading predictive factor. The analysis also explored whether there was any interaction between teachers’ knowledge and students’ learning. More specifically, while students’ pre-test performances usually predict their post-test performance, this analysis examined whether the strength of the relationship depended on how much knowledge their teachers had. There was no specific hypothesis regarding which aspect of knowledge is likely to reveal an interaction or whether the interaction is negative or positive. When the impact of teachers’ knowledge on students’ learning is not big enough (to show between class differences), it is possible that only a subgroup of students get more benefit from teachers’ content expertise, thereby either enlarging (i.e., positive interaction) or decreasing (i.e., negative interaction) the learning gap between advanced and struggling students in the same class.

Table 10. Descriptive Statistics of Students’ Mathematics Performance at Pre-test

<table>
<thead>
<tr>
<th>Math Performance</th>
<th>N</th>
<th>Mean</th>
<th>SD</th>
<th>Minimum</th>
<th>Maximum</th>
</tr>
</thead>
<tbody>
<tr>
<td>WJ III Applied Problem Pretest (Standardized Score)</td>
<td>Pre</td>
<td>605</td>
<td>94.76</td>
<td>48.00</td>
<td>134.00</td>
</tr>
<tr>
<td></td>
<td>Post</td>
<td>548</td>
<td>96.57</td>
<td>49.00</td>
<td>136.00</td>
</tr>
<tr>
<td>TEAM Pretest (T Score)</td>
<td>Pre</td>
<td>585</td>
<td>25.26</td>
<td>-32.28</td>
<td>67.60</td>
</tr>
<tr>
<td></td>
<td>Post</td>
<td>547</td>
<td>32.82</td>
<td>-20.52</td>
<td>71.16</td>
</tr>
</tbody>
</table>

Note: N = 548~605.

As suggested by WJ-AP standardized score, students’ math performance was lower than the national norm \(M = 100\) and the standard deviation was slightly lower than 15 at both time points. TEAM T scores were 25.26 on average at pre-test, and 32.82 at post-test. The scores were relatively lower compared with the scaling sample of TEAM.
assessment where the mean for 4.25-year-old and 5-year-old children’s TEAM T score was 44.42 (SD = 7.85) and 56.15 (SD = 8.49) (Clements, Sarama, & Liu, 2008). See Table 10 and Table 11.

Table 11. Descriptive statistics of Students’ Mathematical Performance at Pre-test by Grade Level and Gender

<table>
<thead>
<tr>
<th>Group</th>
<th>WJ III Applied Problem Pretest (Standardized Score)</th>
<th>TEAM Pretest (T Score)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean</td>
<td>SD</td>
</tr>
<tr>
<td>Pre-K</td>
<td>99.45</td>
<td>12.34</td>
</tr>
<tr>
<td>K</td>
<td>95.25</td>
<td>11.63</td>
</tr>
<tr>
<td>1</td>
<td>93.91</td>
<td>12.64</td>
</tr>
<tr>
<td>2</td>
<td>96.53</td>
<td>16.72</td>
</tr>
<tr>
<td>3</td>
<td>90.80</td>
<td>14.53</td>
</tr>
<tr>
<td>Male</td>
<td>96.37</td>
<td>14.25</td>
</tr>
<tr>
<td>Female</td>
<td>95.51</td>
<td>11.99</td>
</tr>
</tbody>
</table>

Note: N = 548~605.

Hierarchical linear modeling (HLM) was conducted to explore whether students’ mathematics learning was predicted by teachers’ content knowledge, after controlling for school level variances. For WJ-AP test, the null model revealed $\chi^2 (7) = 10.69, p = .15$, ICC was .01, suggesting that there were no significant differences in students’ math performance at school level (1%)\(^2\). Between level 1 and level 2, $\chi^2 (65) = 144.02, p < .001$, ICC was 0.14, suggesting that there were significant differences in students’ math performances between classes (within the same school); about 14% of the variance in students’ math learning demonstrated by WJ-AP was between classrooms (i.e., teachers), and about 85% of the variance in students’ math learning demonstrated by WJ-AP was between students within a given teacher’s classroom. Therefore, predictors at the teacher

\(^2\) Although insignificant, three level analyses were kept as keeping school level differences gives a better estimation. It also made comparison with HLM analysis results using TEAM-T as outcome variable parallel.
level (PCK-EM, level 2) and student level (pre-test WJ-AP standardized score, students’
gender, language used for assessment, level 1) were added.

The three dimensions of PCK-EM, “what” “who” and “how”, were entered into
the teacher level regression equation simultaneously for several reasons: 1) It was
hypothesized that effective teaching required all three aspects of knowledge; 2)
correlation and latent profile analysis in the previous section indicated that the three
dimensions of PCK-EM were coherent but also relatively independent aspects of
knowledge; 3) and the current study was interested in exploring how content expertise
revealed by PCK predicted teaching effectiveness, with the assumption that different
aspects of knowledge may play different roles in facilitating students’ learning; 4) the
assessments for measuring students’ mathematical competence have established
reliability and validity. Given these considerations, it made sense to use the three
dimensions of PCK to predict students’ learning in mathematics

The results indicated that although the overall PCK-EM didn’t predict students’
learning, among the three dimensions of PCK, the “what” dimension significantly
predicted the intercept of the level 1 model ($r = 3.55, p< .05$), suggesting that the higher
the teachers’ score in “what” dimension of PCK (at the beginning of the school year),
the higher their students (in a class as a group) performed in WJ-AP test at the end of the
school year (after controlling for pre-test score at the beginning of the school year);
therefore it can be inferred that teachers’ score in PCK-WHAT dimension predicted
students’ math learning gains at classroom level over a year. More specifically, one
point difference (i.e., increase/decrease) of teachers’ score in “what” dimension of PCK
predicted 3.55 point (i.e., 0.2 standard deviation, because the national norm for WJ standard score suggests Mean = 100, and standard deviation = 15) difference (i.e., increase/decrease) of students’ math performance at classroom level (see Figure 12 for illustration and Appendix J for details).

![Figure 12. The Impact of Teachers’ Knowledge (indicated by PCK- WHAT Score) on Students’ Mathematics Learning (indicated by Pre- and Post- WJ-AP Standard Score).](image)

*Note*: The regression model is complex with other predictors; therefore this is only an illustration figure. Low and High PCK-WHAT refer to teachers who scored 1SD below the mean and 1SD above the mean in the “what” dimension of PCK respectively. Note that there was a significant difference of students’ post-math performance (after controlling for pre-test performance) at classroom level, categorized by teachers’ score in PCK-WHAT dimension. It can be inferred that teachers’ score in PCK-WHAT dimension predicted students’ math learning gains at classroom level over a year.

In addition, there was an interaction between the “who” dimension of PCK-EM and pre-test scores ($r = .67, p < .001$) of WJ-AP in predicting post-test scores of WJ-AP ($r = -.12, p < .10$). The regression coefficient was negative and it was marginally significant. The negative interaction suggested that while there is a positive relationship between students’ pre-test and post-test performance, teachers’ knowledge of students’ learning in mathematics decreased this relationship (See Figure 13 for illustration and
Appendix J for details). Note that teachers’ score in PCK-Who dimension didn’t predict students’ classroom level math performance, but impacted, i.e., decreased the strength of the relationship between pre- and post- math performance.

![Figure 13. The Impact of Teachers’ Knowledge (indicated by PCK-Who Score) on Students’ Mathematics Learning (indicated by Pre- and Post- WJ-AP Standard Score).](image)

Note: The regression model is complex with other predictors; therefore this is only an illustration figure. Low and High PCK-WHO refer to teachers who scored 1SD below the mean and 1SD above the mean in the “who” dimension of PCK respectively. Note that teachers’ score in PCK-WHO dimension didn’t predict students’ classroom level math performance, but impacted, i.e., decreased the strength of the relationship between pre- and post- math performance.

Similar analysis was done using students’ TEAM test T scores as outcome predictor at level 1. In the null model, \( \chi^2 (7) = 12.08 \), \( p = .097 \) for level 3, indicating that students’ math performance was marginally different between schools. ICC = .045, suggesting that school level differences explained 4.5% of the variance in students’ math performance. The null model estimation also revealed \( \chi^2 (65) = 1532.25 \), \( p < .001 \) and an ICC of 0.75 between level 1 and level 2, suggesting that there were significant differences between classes within the same school, about 71.9% of the variance in students’ math learning demonstrated by TEAM was between classrooms (i.e., teachers),
and about 23.6% of the variance in students’ math learning demonstrated by TEAM was between students within a given teacher’s classroom. Therefore, predictor at teacher level (PCK-EM, level 2) and students level (pre-test TEAM T score, age in month at pre-test, gender and language used in the assessment, level 1) were added.

Figure 14. The Impact of Teachers’ Knowledge (indicated by PCK-WHAT Score) on Students’ Mathematics Learning (indicated by Pre- and Post- TEAM T Score).

Note: The regression model is complex with other predictors; therefore this is only an illustration figure. Low and High PCK-WHAT refer to teachers who scored 1SD below the mean and 1SD above the mean in the “what” dimension of PCK respectively. Note that there was a significant difference of students’ post-math performance (after controlling for pre-test performance) at classroom level, categorized by teachers’ score in PCK-WHAT dimension. It can be inferred that teachers’ score in PCK-WHAT dimension predicted students’ math learning gains at classroom level over a year.

The three dimensions of PCK-EM, “what” “who” and “how”, were entered into the teacher level regression equation simultaneously. The results indicated that among the three dimensions of PCK, the “what” dimension significantly predicted the intercept of the level 1 model ($r = 2.94, p < .05$), suggesting that the higher the teachers’ score in the “what” dimension of PCK-EM (at the beginning of the school year), the higher their students (as a group) performed in TEAM test at the end of the school year (after controlling for pre-test score at the beginning of the school year); therefore it can be
inferred that teachers’ score in PCK-WHAT dimension predicted students’ math learning gains at classroom level over a year. More specifically, one point increase (or decrease) in teachers’ score for PCK-What dimension predicted about 2.94 points increase (or decrease) of students’ math performance in TEAM test, see Figure 14 for illustration.

![Figure 15](image)

**Figure 15. The Impact of Teachers’ Knowledge (indicated by PCK-How Score) on Students’ Mathematics Learning (indicated by Pre- Post- TEAM T Scores).**

*Note:* The regression model is complex with other predictors; therefore this is only an illustration figure. Low and High PCK-HOW refers to teachers who scored 1SD below the mean and 1SD above the mean in the “how” dimension of PCK respectively. Note that teachers’ score in PCK-HOW dimension didn’t predict students’ classroom level math performance, but impacted, i.e., increased the strength of the relationship between pre- and post- math performance.

In the meantime, there was a positive significant interaction between the “how” dimension of PCK-EM ($r = .10, p < .10$) and students’ pre-test TEAM scores ($r = .47, p < .001$) in predicting post-test TEAM T scores over a year. While the higher the pre-test score, the more likely those students will perform higher in the post-test; when students were taught by teachers with good understanding of math-specific pedagogies and representations, this relationship would be further amplified (See Appendix J for details and Figure 15 for illustration). Note that teachers’ score in PCK-How dimension didn’t
predict students’ classroom level math performance, but impacted, i.e., increased the strength of the relationship between pre- and post- math performance.

To sum up, the hypothesis was partially supported. The “what” dimension of PCK-EM significantly and reliably predicted students’ learning in mathematics (revealed by two different assessment), although a positive relationship was not found between “who” “how” dimensions of PCK-EM and students’ learning indicated by either assessment. The results confirmed the assumption that conceptual understanding (i.e., the “what” dimension of PCK) is a leading predictor of students’ learning. In addition, exploratory findings about the interaction between teachers’ knowledge (demonstrated by scores in “who” and “how” dimensions of PCK) and students’ learning enriched our understanding of the relationship between teaching and learning. Although the findings were not consistent across two assessment on students’ mathematical understanding, it was found that when teachers scored higher in the “who” dimension of PCK, students’ with limited mathematical understanding made more progress and were less likely to stay in the low rank by the end of the school year. Students’ with more advanced mathematical understanding benefited from teachers who scored higher in the “how” dimension of PCK.
The purpose of the current study was to investigate the profile of PCK-EM for early childhood teachers and explore the relationship between PCK, teaching quality and students’ learning outcomes in mathematics. Utilizing a video-elicited, open-ended online survey, it examined what teachers understood about content, students and pedagogy in teaching early mathematics. By exploring the profile of teachers’ content knowledge and its relationship to teaching and learning, it provided new insights for understanding teaching effectiveness and teacher preparation. In particular, this chapter highlights the findings through answering two major questions:

1. What characterizes early childhood teachers’ mathematical content expertise from the lens of PCK, including 1) what is the level of understanding; 2) what are the knowledge aspects that teachers know more/or less; and 3) what does expert teachers’ PCK-EM look like?

2. What are the relationships between early childhood teachers’ PCK and teaching effectiveness indicated by the quality of teaching and students’ learning gains in mathematics?

The chapter first examines the findings corresponding to these questions, and then seeks explanations and discusses implications for early math teaching and learning.
A Profile of PCK-EM in Early Childhood Teachers

The Level of Understanding

Little is known about the characteristics of early childhood professionals’ content knowledge. The responses of 182 teachers in the current study provided a profile of what early childhood teachers know and do not know about teaching foundational math. The descriptive results revealed limited understanding. The mean for PCK-EM was lower than medium level. The teachers were able to identify mathematical concepts underlying a regular classroom activity and its relevant aspects for young learners and teaching. However, their overall understanding was limited to or below basic level without conceptual sophistication and/or advanced integration.

*WHAT* do teachers know about foundational mathematics? There is no doubt that teachers need to be knowledgeable about the subject they teach. In literature, however, there is no clear definition or investigation about what knowledge is required for teaching foundational mathematics effectively. Inspired by the idea of directly assessing teachers’ knowledge used in teaching and recognition of the significance of relational understanding (i.e., understand the underlying principles and associations among concepts) over instrumental knowledge (i.e., know how to solve a math problem and memorize the number facts) (Ball, 1990; Krauss et al, 2008; Skemp, 1976), the current study investigated professionals’ conceptual sophistication of foundational mathematical ideas (i.e., depth) and their connections (i.e., breadth) through the “what” dimension of PCK.
More specifically, the survey examined whether teachers understand the part-part-whole (p-p-w) relationship beyond knowing the possible combinations of number 7. This is one of the Big Ideas that are coherent and of great significance in early mathematics (Early Math Collaborative, 2014). For young children to grasp the concepts, there are several intricate underlying principles that teachers must be aware of: 1) part-part-whole relationship, 2) there are various combinations, and 3) the total amount stays the same regardless of the combinations (Early Math Collaborative, 2014). Of equal importance is teachers’ cognition about big ideas closely related to part-part-whole relationship such as a) when you switch the two addends (parts), it is the same combination; and b) there are patterns of addends among different combinations of the same total quantity (i.e., the addends in different combinations increase and decrease by pattern, e.g. 1 and 6, 2 and 5).

Unfortunately, early childhood teachers in the sample did not demonstrate adequate conceptual understanding for solid mathematics teaching. The score for the “what” dimension, understanding about math concept was 2.48 out of 5. The majority of the teachers identified the mathematical ideas that appeared in the video and discussed their related concepts; however, the understanding was limited to a superficial level (i.e., behavior and procedure) without further conceptual inference. “Number composition and decomposition,” “counting,” “more or less,” and “addition” were the most frequent responses. Most teachers were unable to recognize the big ideas of mathematics in the video and/or its relevant concepts. The majority of the teaching professionals only seemed to possess instrumental understanding about number system (know how to solve a math problem and memorize the number facts), not relational understanding of
mathematics concepts (Ball, 1990; Krauss et al, 2008; Skemp, 1976). It is therefore not surprising that analysis of videotaped early math teaching revealed that teaching occurs more for the transmission of information facts than inviting for deep conceptual comprehension and high-level thinking (Bowman et al., 1982).

It is frequently stated that early childhood teachers are lacking sufficient mathematical content knowledge (e.g., Ginsburg & Ertle, 2008; Ginsburg & Golbeck, 2006) but little empirical studies exist. Although research groups on math education in upper grades (Ball, Lubienski, & Mewborn, 2001; Ma, 1999) reported a lack of profound understanding of foundational mathematics (PUFM, Ma, 1999) among elementary math teachers, it is difficult to make inferences and generalize this conclusion to early math teaching. In the meantime, among limited investigations in early childhood, little was specifically tied to mathematics. For instance, interviews with 52 early childhood educators in an urban setting revealed insufficient content understanding, but the subject domain varies across individuals (Melendez, 2008). By specifically targeting early childhood teachers’ understanding of foundational mathematics, the current study provided direct empirical findings and suggested that teachers were not well prepared to take advantage of young children’s learning potential.

**WHO** What do teachers know about the students’ cognition in mathematics? How can educators take advantage of the learning capacity, as well as the mathematical knowledge and experience that children bring to early childhood classrooms? It is necessary for teachers to know young children’s learning patterns, including prior knowledge and likely misunderstandings or learning difficulties around specific
mathematical concepts. Early childhood teachers are expert on young children’s social-emotional development and cognitive development in general; however, does understanding about child development naturally transfer into the context of math learning? The current study defined understanding of students (“who” dimension of PCK-EM) as awareness of students’ mathematical learning patterns, including prior knowledge and misunderstandings young children are likely to have around particular mathematics concepts.

More specifically tied to the study regarding the concept of part-part-whole relationship in number sense, it is necessary for teachers to know that young learners have to understand ideas that lay conceptual coherence for understanding the number system, such as: 1) there are factors hidden inside a given number; and 2) experiences with putting together two sets to make a total. Being cognizant that it is difficult for young children to grasp ideas such as “numbers can be made of and broken into groups” and “the sum stays the same regardless of the combinations” is also essential for sound teaching. These misunderstandings, although not as obvious as miscounting, are key for areas for teacher intervention to develop students’ number sense and number system knowledge.

According to the PCK-EM survey, early childhood professionals in the study did not display adequate knowledge about learners’ cognition in mathematics. The mean for “who” dimension was 2.27, and about 90% were equal to or below 3.00 (i.e., medium level understanding). Most teachers only paid attention to procedural, discrete, and explicit math ideas related to students’ mathematical cognition such as counting skills. A
few of them could more specifically name the counting principles that students need to
learn such as “one-to-one correspondence” and “understand the cardinality attribute of
numbers”. The majority of the professionals knew some common learning patterns,
however, the understanding was limited without further specification and integration into
the mathematical context. Only less than 10% of the teachers were able to identify prior
knowledge and/or misunderstandings more closely related to the p-p-w relationship.

It is generally believed that early childhood teachers know well about child
development, including their cognitive development. However, studies on pre-service
teachers revealed very limited understanding of students’ conceptual errors (Tirosh, 2000,
experience help to improve this understanding? According to the current study, in-service
teachers working with young children demonstrated vague and general understanding
about what children need to prepare in order to learn math concepts and their likely
misunderstandings.

**HOW** What do teachers know about how to transmit mathematical ideas for
young children to learn? Early childhood teachers are experts of developmentally
appropriate practice (DAP), putting young children’s developmental stage, learning styles
and individual differences within the complex social-cultural context. Would
development-driven understanding of general pedagogical strategies necessarily apply to
math teaching? In this study, “how” was defined as knowledge about math-specific
pedagogies and representations to make mathematical ideas clear. By highlighting the
pedagogies embedded in subject teaching context, as well as illustrations, examples,
models, demonstrations and analogies that can make clear mathematical concepts, it focused on professionals’ pedagogical awareness to accommodate students’ diverse needs in subject learning (i.e., mathematics).

In teaching the concepts of part-part-whole relationship for young children, it was expected that teachers could apply various pedagogical strategies and math-specific representations to facilitate students’ understanding of the big ideas. Pedagogical awareness about instructional grouping, pace, language etc. should be incorporated into teaching mathematics. In other words, the use of pedagogical strategies serves to make the math concepts comprehensible for young learners. Regarding the mathematical representations, designs such as “using blocks of various colors for children to see the two parts” and “using pizza to show the whole stays the same” well considered the underlying math concepts and young learners’ likely misunderstandings. It was also expected that teachers could come up with a variety of representations and activities to meet the diverse needs of struggling and more advanced students.

As expert practitioners of DAP, early childhood teachers in the sample didn’t demonstrate adequate pedagogical awareness in the context of mathematics subject matter teaching. The “how” dimension, understanding about math-specific pedagogy, was scored 2.33 on average, and about 90% were equal to or below 3.00 (i.e., medium level understanding). A variety of strategies were frequently mentioned to address the limited attention span and cognition of young children, including “repeat,” “slow the pace,” and “work with smaller numbers,” as well as instructional groupings such as “small group” and “individual support”. However, little attention was paid to how
pedagogies help to make the math concepts comprehensible for young learners and only a few teachers considered applying appropriate representations and/or designing an activity that can enhance students’ understanding. While teachers understood young children as concrete learners and were fond of applying manipulative, the use of manipulative was vague without considering how the manipulative would help to make the content clear.

Early childhood professionals’ pedagogical awareness was limited to a general level without integrating into math teaching. These results, although focused on early childhood math education, echo some other findings. In the comparative study of elementary teachers from U.S.A and China, Ma (1999) reported few teachers with a repertoire of mathematical representations to flexibly and appropriately present math problems. While representational fluency is one of the most important facets of students’ math competency, teachers were poorly equipped with the math-specific pedagogy and representations that would support it.

**Comparing Different Aspects of Understanding**

The level of understanding revealed by the three dimensions of PCK implied that the professionals were not well prepared to teach early math effectively. However, this does not address whether different aspects of knowledge develop simultaneously. Is there any aspect of knowledge that develops better than others? According to the repeated ANOVA, in the overall sample, the “what” dimension scored significantly higher than the “who” and “how” dimensions, while there was no significant difference between “who” and “how.” The results implied that teachers seemed to be more knowledgeable about the basic mathematical concepts that children were involved in than being aware of
what preparations young learners needed and what misunderstandings and likely learning
difficulties they may have. Their knowledge of the math content was also more advanced
than pedagogical strategies and representations to facilitate mathematical understanding.

The less developed understanding about students’ learning patterns of
mathematics was consistent with findings from other researchers. Bowman and her
colleagues’ (1982) study of early childhood educators reported that all participating
teachers demonstrated some mathematical knowledge they were expected to teach for
young children (which is similar to the basic understanding teachers show in the current
study). However, they were not able to correctly identify students’ math capabilities
(Bowman, Katz, & McNamee, 1982). In the current study, while the teachers were able
to identify the basic math concepts that appeared in the video, it seemed to be more
difficult for them to know students’ learning styles, what knowledge the children need to
have in order to participate in the learning activities and what misconceptions they may
bring into the classroom.

Pedagogical awareness was also less developed than knowledge of the content.
While most teachers were able to identify the basic mathematical concepts in the video,
they demonstrated more limited understanding about how to make the content
comprehensible. Investigations on math-specific pedagogies in early childhood and
elementary teachers revealed minimum pedagogies interwoven with a subject (Bowman
et al., 1982; Zhou et al., 2006). Similarly, the instructional designs teachers proposed in
the current study were subject-general, which seldom enhances representational fluency
in mathematics.
In sum, the features of PCK dimensions suggested that insufficient content understanding was accompanied by even more reduced knowledge of students and pedagogy. Early childhood professionals are trained as experts of child development and developmental appropriate practice; they also showed basic content understanding. However, familiarity with general child development and pedagogy, as well as a basic comprehension of math concepts didn’t seem to naturally lead to a capability of identifying and addressing what students are struggling with and/or how to successfully facilitate their understanding in the context of mathematics.

The Relative Independence of Different Aspects of Understanding

Coding on the narrative lesson analysis revealed moderate correlations between the three dimensions of PCK-EM (“what”, “who” and “how”). Both Pearson and Spearman correlations among the three dimensions were about .40, indicating that they were related but not the same aspect of knowledge. The moderate correlation suggested the existence and relevant independence of the three dimensions of PCK (i.e., “what”, “who” and “how”) proposed in PCK-EM model.

In the meantime, the lack of incorporating knowledge about content with students and pedagogy, thereby the even lower level of understanding indicated by the “who” and “how” dimensions of PCK-EM further confirmed the relative independence of the three dimensions of PCK. Although not revealed in the present study, professional trainings aiming to improve teachers’ content expertise have found that the changes in one component of PCK may not simultaneously lead to alterations in another component (Magnusson, Krajcik, & Borko, 1999). In fact, knowledge of students learning, such as
identifying students’ needs for support, can also be isolated from pedagogical awareness to address corresponding learning obstacles. Professional development training that successfully drives teachers to think from students’ perspective (i.e., “who”) found that most teachers still struggled with how to respond to students’ misconceptions (i.e., “how”) (Smith & Neal, 1989). Therefore, it makes sense to describe teachers’ knowledge through the “what” “who” and “how” dimensions of PCK as conceptualized.

**Comparing Individual Teachers**

While ANOVA test took a variable-centered approach to investigate the relative strength of different knowledge aspects in PCK, latent profile analyses were conducted to investigate the development of PCK-EM at the individual level (i.e. person-centered approach). The analyses revealed three distinctive latent clusters of individual teachers regarding their profile of PCK-EM, referred to as “high” “medium” and “low” respectively (See Table 9 and Figure 10). Significant differences were found among three clusters. More specifically, the “high” cluster featured a small portion (i.e., 4%) of teachers in the sample with relatively sophisticated understanding in all aspects of PCK. The “medium” cluster consisted of half of the teachers in the sample with relatively medium level understanding, suggesting that they had a basic understanding about content, students’ learning patterns and math instructional strategies. The “low” cluster was about half of the sample in which teachers demonstrated the most limited understanding for teaching early mathematics, revealed by an inadequate knowledge of math content, how students learn math and how to make the content comprehensible for young learners.
The features of different clusters provided a picture about expert early childhood teachers’ PCK profiles and what less knowledgeable teachers’ PCK profiles looked like. The teachers in the “high” cluster can be regarded as expert teachers. From “high” to “low” clusters, the reduced content understanding was paired with even lower levels of awareness of students’ learning patterns and instructional strategies in mathematics. Teachers in the “high” group showed much better understanding than teachers in the “medium” and “low” cluster in all of the dimensions of PCK. Expert teachers were able to see the big pictures of different concepts in a network beyond procedural fluency and discrete skills. Accompanying their deep conceptual understanding, they were cognizant of students’ learning patterns and effective pedagogies to make the content accessible. However, the percentage was small (4%); there were only 7 out of 182 teachers in the sample who revealed coherent and integrated mathematical understanding for teaching young children across the three dimensions of PCK-EM.

Other studies support this analysis regarding the characteristic of expert teachers and the percentage of experienced teachers. Only 4% of the teachers can be considered expert regarding early mathematics teaching, the finding was somewhat consistent with report about elementary math teachers, where 10% of the teachers demonstrated profound understanding of fundamental mathematics (PUFM) (Ma, 1999). Expert teachers were found to have more knowledge that is extensive, accessible, and organized for teaching (Sternberg & Horvath, 1995); while novice teachers may leave their courses taken at college with a set of disconnected facts, symbols, and language to be memorized (Hammer, 1994). Zhou and his colleagues (2006) also reported veteran mathematics
teachers in elementary schools who could discuss the content in great detail and include recommended methods.

**The Prediction of Teaching and Learning by PCK**

**The Prediction of Teaching Quality in Early Mathematics by PCK-EM**

There is no doubt that professionals’ knowledge impacts their practice; however, sound indicators of knowledge and quality of teaching remains unclear and there is insufficient evidence to confirm the assertion empirically. Investigations in elementary mathematics teaching and early math teaching reported significant prediction of teaching quality by mathematical knowledge for teaching (MKT) (e.g., Hill, et al., 2005); however, it is not clear to what extent the results can be applied to early mathematics teaching. Among the limited studies in early childhood education, McCray & Chen (2012) also reported significant relationship between PCK and teaching quality; however, the indicator of teaching effectiveness was the frequency of math-language, which may limit the scope of teaching quality. For this particular study, content expertise was highlighted in a form of integrated understanding about content, students and pedagogy aligned with foundational mathematics. High quality math teaching was demonstrated by clear learning goals, conceptual sophistication, awareness of young learners’ learning trajectories and individual differences, as well as the application of developmentally appropriate learning formats and multiple representations to engage students in a mathematics learning community.

It was hypothesized that PCK significantly predicted teaching quality over indirect indicators of teaching experience (i.e., years of teaching, mathematics courses
taken in college and hours of mathematics workshop attended). According to HLM analyses, what teachers knew significantly predicted how well they taught. There was a significant and positive relationship between the overall score of PCK-EM and math teaching quality. The higher the teachers’ understanding, the more likely they were delivering high quality math instruction (see Figure 11). In addition, mathematics learning experience of pre-service preparation (i.e., number of math courses taken) and in-service training (i.e., years of teaching and hours of math workshop taken) didn’t demonstrate any significant association with math teaching quality; and adding these extra predictors did not change the significant prediction of PCK on teaching quality.

Together, the results implied that distal indicators of academic knowledge in mathematics (i.e., number of courses taken, number of workshop taken, and years of teaching) are not rigorous indicators of teaching competence, and PCK is a more robust indicator of teaching effectiveness (i.e., math teaching quality). Substantial differences in PCK have been found among seasoned teachers around the same topic areas (Henze, 2006), suggesting that years of teaching may not be an effective indicator for teaching expertise. High level of PCK was indicated by a well-incorporated understanding of math content, how students learn math and how to effectively present the math ideas. Teaching enactment involves a decision-making process that draws upon situational understanding about content, learners and pedagogy (Heller, Daehler, Wong, Shinohara, & Miratrix, 2012; Park & Oliver, 2008a). With sophisticated understanding about content, students and pedagogy, teachers are more likely to create teachable moments to facilitate real learning and high-level thinking.
The Prediction of Students’ Learning in Early Mathematics by PCK-EM

There’s no doubt about the link between qualified teachers and improved students’ performance (Bowman, Donovan, & Burns, 2001). The study of teaching and learning, however, has been challenged by “what is the effective indicator of qualified teachers” for years with mixed findings (e.g., Begle, 1979; Monk, 1994; Rowan, Correnti, & Miller, 2002). For instance, teachers’ subject matter knowledge (measured by direct indicators such as test scores) was found to be significantly but weakly related to students’ achievement. While teachers’ experience of taking advanced mathematical courses has been reported to significantly impact students’ learning (Rowan et al., 1997), once the number reached five, there was no more extra effect (Monk, 1994). On the contrary, there was a negative association between teachers’ advanced subject understanding and a subgroup of students’ learning gains in the same subject (Begle, 1979). Elementary students taught by teachers with an advanced degree performed worse than those students who were taught by teachers without a mathematics degree (Rowan, Correnti, & Miller, 2002). The mixed findings indicated a need to further investigate effective indicators of teaching expertise empirically.

Using two different assessments to evaluate students’ mathematical understanding and the approach of PCK in defining teachers’ content expertise, the current study revealed reliable prediction of students’ mathematical learning over a year by the “what” dimension of PCK. Scores in the “what” dimension of PCK-EM significantly predicted students’ mathematical understanding at the classroom level, after controlling for the impact of the “who” and “how” dimensions of PCK and students’ pre-test scores, gender,
language used for test (and age at pre-test when math performance was not adjusted by age, i.e. TEAM) (see Figure 12 and Figure 14). The sophistication of a teacher’s mathematics understanding, including the capacity to unpack the intricate principles (i.e. Big Ideas, Early Math Collaborative, 2014) underlying seemingly simple math activities and the connections to other concepts, foretold students’ average math learning.

The findings implied the leading role of teachers’ conceptual understanding of math content (aligned with grade level) in fostering students’ mathematical competence as hypothesized. Shulman (1987) once suggested that content understanding is more fundamental in the process of pedagogical reasoning and action for subject instruction (see Table 1), and the present study provided promising evidence to support this assertion. Teaching involves a content-intensive process of knowledge transformation, which requires a solid understanding of mathematical content to be taught (Lampert, 2001; Shulman, 1987). With profound understanding of fundamental mathematics, teachers are more likely to “invite students to look beyond the surface features of procedures and concepts and see diverse aspects of knowledge as having the same underlying structure” (Baroody, Feil, and Johnson, 2007, p.26; Lloyd & Frykholm, 2000; Ma, 1999).

The hypothesis about the prediction of the “who” and “how” dimension of PCK to children’s learning was partially supported. The study found no prediction by the “who” and “how” dimension of PCK on students’ average learning gains (i.e., at a class level); however, teachers’ familiarity with students’ learning patterns and math-specific pedagogy demonstrated interactions with the relationship between students’ pre- and
post-math performance. A negative interaction was observed between teachers’ score in “who” dimension and the relationship between pre- and post-math performance (revealed by WJ-AP standard scores). While pre-test performance was highly related to post-test performance, the strength of the relationship decreased when teachers scored high in the “who” dimension of PCK. Put another way, students’ rank in mathematical performance by the end of the year was less likely to be determined by where they started (at the beginning of the year), suggesting that struggling students (who started with relatively lower mathematical understanding) made prominent progress when taught by teachers with a higher understanding about students’ learning patterns in mathematics (see Figure 13).

The negative interaction of “who” with the association between pre- and post-performance suggested that understanding about students’ prior knowledge, misconceptions in mathematics seemed to meet the needs of struggling children more effectively. The finding was not surprising because it is necessary for teachers to anticipate what students already know, which topics they find difficult, and what misconceptions they may have (Magnusson, Krajcik, & Borko, 1999). Cognitive disequilibrium is important in conceptual development as it helps children to accommodate and refine their schemata. Therefore, teaching builds on students’ prior ideas and aims to promote student thinking and reasoning would facilitate understanding (Kulm, Capraro, & Capraro, 2007). While the finding did not reject the importance of teachers’ knowledge for all students, it suggested that the impact of professionals’ knowledge about young children’s mathematical cognitions on students’ mathematics
performance was relatively small, and it seemed to tailor struggling students’ needs more sensitively.

In the meantime, there was a positive interaction of teachers’ scores in the “how” dimension with the relationship between pre- and post- math performance (revealed by TEAM T scores, another indicator of students’ mathematical understanding). While pre-test performance was highly related to post-test performance, the strength of the relationship between pre- and post-test scores increased when teachers had a high score in the “how” dimension of PCK. Advanced students (i.e., started at a higher level of mathematical understanding) were more likely to stay in the high rank when they had a teacher with a better understanding of math-specific pedagogy. While the finding did not reject the importance of teachers’ knowledge for all students, it suggested that the impact of professionals’ knowledge about math-specific pedagogies on students’ mathematics performance was relatively small, and it seemed to benefit advanced students more sensitively (see Figure 15).

Considering the operational definition of “how” dimension of PCK may provide some interpretations about the impact of teachers’ pedagogical content expertise on students’ math learning. A fundamental difference between an effective teacher and a regular one is not that the former has more “teaching tricks” than the latter. Rather, the “how” dimension of PCK focused on teachers’ capability of integrating pedagogies and mathematical-specific representations in lesson designs. The analogies, examples, and interpretations of particular math concepts are expected to facilitate students’ representational fluency by expanding the understanding into different levels (Copley,
According to the present study, the effect of facilitating high-level thinking appeared stronger when students were already equipped with some basic understanding. Therefore, multiple math representations and pedagogical strategies seemed to more sensitively benefit advanced students and encourage high-level thinking.

It worth noting that the interactions between teachers’ knowledge and the relationship between students’ pre- and post- math performance was not consistent across the two measures of students’ mathematical understanding. As well, no significant relationships were found between “who” or “how” dimensions of PCK and students’ mathematics learning at the classroom level as hypothesized. Together, the impact of teachers’ knowledge of students’ mathematical cognition and pedagogy on students’ learning was relatively small, compared with the conceptual sophistication revealed by the “what” dimension of PCK. In fact, the interaction suggested that while there was some impact of teachers’ knowledge on students’ learning, the impact of it was so small that it was only revealed in a subgroup of students. The even more limited level of understanding indicated by the scores in “who” and “how” dimensions of PCK (compared with “what” dimension) may also make it more difficult to capture and reveal the linkage in the whole sample of students (if there is any).

**Implications and Limitations**

**Implications**

The results from the current study suggest a need to reform teacher preparation for early math education. As a theoretical framework, PCK has shed light on our understanding of teacher knowledge and teaching as a distinct profession. Teaching is not
simply a transmission of information but a complex action that requires teachers to think from students’ perspective, identify and address students’ needs, and choose effective pedagogical strategies and mathematical representations to make content comprehensible (Shulman, 1987; Wilson, Shulman, & Richert, 1988). The insufficient content knowledge, the relative independence and imbalanced development of PCK dimensions, as well as their predictions for teaching quality and students’ learning, supported a need to unpack conceptual understanding for the purpose of teaching, and incorporate subject understanding into understanding about students learning the content, as well as content-specific pedagogies.

First, the current study provided encouraging empirical evidence to support the approach of PCK in studying teaching effectiveness. The notion of PCK highlights the prominence of content knowledge as the thread connecting other kinds of teachers’ expertise. Applying the PCK theory in studying teaching effectiveness, it revealed empirical evidence to validate the conceptualization and measurement tool of PCK in early mathematics. By considering the content expertise of teaching through the lens of PCK, teachers’ knowledge significantly predicted their math teaching quality with/without controlling for the indirect (i.e., distal) indicators of teachers’ seniority and experiences (i.e. years of teaching, number of math courses taken in pre-service preparations, and hours of math content/pedagogy workshop during in-service trainings). More knowledgeable teachers are also providing more efficient subject instructions. In addition, traditional indicators of teaching experience mentioned above didn’t predict teaching quality; and adding these indicators did not change the significant prediction of
PCK on teaching quality. Together, the results strongly supported the proposal of PCK as an effective indicator for teaching competence.

Regarding teachers’ knowledge and students’ learning, conceptual understanding (i.e., the “what” dimension of PCK) was found to be a leading factor of predicting students’ learning gains in mathematics. When instructors can unpack their subject knowledge into deconstructed ideas, students as a class made more progress in math learning. In the proposal of PCK, Shulman (1987) believed that content knowledge serves as a mediator to connect teachers’ understanding of students and pedagogy in the process of pedagogical reasoning and action proposed (see Table 1). While very few large-scale studies reported significant connections between PCK and how much students learn (i.e., Baumert, et al., 2010; Hill et al., 2005; Kersting et al., 2012; McCray & Chen, 2012), the significant finding from the current study confirmed Shulman’s assertion about the foundational role of conceptual understanding for productive teaching. The fact that only conceptual sophistication, but not knowledge of students and pedagogy, was predicting students’ learning gains as a class implied a more fundamental role of the unpacked conceptual understanding in subject learning.

The findings about the interaction between teachers’ knowledge and students’ learning gains also implied that different strategies might more sensitively accommodate students’ diverse levels of mathematical understanding. Young children have varied level of mathematical understanding; applying different aspects of content understanding can help to work within an individual child’s Zone of Proximal Development (ZPD) in specific learning areas (Rogoff, 1990; Vygotsky, 1978). In particular, content knowledge
about students and pedagogy seemed to play different roles in helping students with varied levels of mathematical understanding. To assist students with relatively limited content knowledge, figuring out what knowledge they currently have, and what learning challenges they may encounter can be more effective (Feiman-Nemser & Parker, 1990; Palinscar & Brown, 1984). When students have already obtained basic understandings, teachers could apply multiple mathematical representations to encourage high-level thinking and facilitate improvement. The predictions of teachers’ PCK on students’ learning further supported the proposal of PCK as indicator for teaching effectiveness and the validity of the PCK survey.

Second, there is a need to improve professionals’ content understandings of mathematics for the purpose of teaching. Early childhood teachers in the sample didn’t demonstrate adequate mathematics knowledge for teaching. The level of understanding was low and there were only 4% of the teachers who demonstrated sophisticated knowledge for solid mathematics teaching. For the majority of the teachers, the mathematical concepts important for young children to learn were regarded as discrete skills and facts; understanding about students’ learning patterns and effective pedagogies were also limited to a superficial level. Given the significant roles of PCK in productive teaching and the restricted level of understanding, it is necessary to boost early childhood teachers’ PCK in early mathematics.

In fact, the students in the sample made it even more urgent to improve the professionals’ PCK in foundational mathematics. While students’ from low SES families are performing poorly in mathematics as early as the entry of Kindergarten (Huttenlocher
et al., 1994; Starkey et al., 2004) and fell further behind (Starkey & Klein, 2000), the results revealed a “teaching gap” paralleled with the “learning gap” (Stigler & Hiebert, 1999). The sample in the study represented early childhood teachers serving high needs, low performance students in an urban setting. Over 90% of the students in the participating schools were enrolled in free/reduced lunch. About half of the students were not meeting the state math standards, and only one out of five students in these schools exceeded state math standards. These students are less likely to obtain sufficient support from family and rely heavily on teachers’ impact. The teachers’ lack of understanding about how to teach early mathematics can pose serious challenges for children with insufficient family resources; to close the learning gap, it is necessary to address the teaching gap.

It is also in this sense that teachers’ content expertise would contribute to addressing long-term systematic inequities in educational outcomes and there is an urgent need to improve their content understanding for teaching early mathematics. The findings implied that students benefited from knowledgeable teachers by making more progress in mathematics learning, suggesting the importance of educational resources from teachers. The results also provided specific implications about how to help less resourceful children. Differences in teachers’ conceptual understanding (i.e., “what” dimension of PCK) predicted differences in students’ math learning gains across classes; and students who had limited understanding of mathematics also were less likely to stay in the low rank when they had a teacher who knew well about children’s math learning patterns (i.e., “who” dimension of PCK). Equipping teaching professionals serving the high-needs
students with knowledge revealed by PCK, therefore, would facilitate students’ learning and help to close the learning gap.

Third, the question moves to “how to improve professionals’ PCK.” Traditional trainings didn’t seem to equip professionals well for teaching early mathematics. As certified early childhood teachers, 96% of the participants in the study had Bachelor’s degrees and about 72% of them also earned a master’s degree. On average, teachers had 13 years teaching experience, took 3 math classes in pre-service training and 9 hours of in-service math education/methods workshops. While most teachers have obtained ample pre-service training and teaching experience, the understanding revealed from PCK survey was not encouraging. Few teachers demonstrated conceptual knowledge that would benefit students’ high level thinking; and none of these indicators of teaching experience (mentioned above) predicted math-teaching quality. Together, the study implied a necessity for innovations in teacher preparations and in-service support.

The conceptualization of PCK-EM and the empirical evidence about its prediction for teaching and learning can inspire the design of teacher preparations and in-service trainings and support. The framework of PCK-EM, as well as the prompted questions in the PCK-EM survey can also be used systematically to improve teachers’ PCK. Similar to CoRe and PaP-eRs (Loughran et al., 2004) the conceptualization of PCK also has implications for teacher development and professional training. Videos are effective stimuli to elicit teachers’ reflective thinking as it provides contextualized information (Danielson, 1996). The promoted questions in PCK-EM survey can serve as guidelines for teachers to reflect, articulate and discuss their understanding of teaching
and learning a particular mathematics topic and enhancing their professional knowledge in practice. The conceptual model and the survey of PCK-EM provided a systematic way to support teacher’s professional growth.

More specifically, this study indicated the importance of addressing the nature of foundational mathematics in teacher trainings (as reflected in the “what” dimension of PCK-EM). As proposed in PCK theory (Shulman, 1986; 1987), subject matter knowledge measured by traditional indicators is not enough for productive teaching. There is a difference between the mathematical knowledge necessary to function as a common adult (i.e. how to do math) and the knowledge that one uses in teaching (i.e., explain how the principles work). Especially for foundational mathematics, the understanding is likely to get compressed and be used in more advanced problem solving without examining the fundamental principles (Ball & Bass, 2003). Adults, including teaching professionals, can use knowledge of the number system effortlessly. However, to be a productive teacher, the compressed and instrumental understanding needs to be further unpacked and deconstructed into relational understanding at a conceptual level for novices (Ball, 1990; Krauss et al, 2008; Skemp, 1976). In fact, the current study found that teachers’ unpacked conceptual understanding predicted students’ average learning gains in mathematics.

Unfortunately, the nature of mathematical concepts young children can learn in early childhood settings are not addressed in teacher preparation, professional development and teacher certification. The teacher training and preparation for early childhood educators is usually subject-general. For those who took some courses in mathematics in college or graduate study, it was usually about calculus, linear algebra,
and abstract algebra, content that is not aligned with the grade level pre-service professionals are going to teach. Teaching certificate exams, such as PRAXIS and state assessments, are also about advanced mathematics knowledge. For practicing teachers, in-service professional development usually is one-time workshops or occasional readings of articles on the topic. While advanced mathematics classes in college rely heavily on the automatic procedural fluency to solve more complex problems, teaching foundational math requires unpacking the procedural fluency and examining why it works in certain ways (Kilpatrick, Swarfford & Findel, 2001). Therefore, it is not surprising that mathematics courses and certification requirement did not lead to rigorous understanding about teaching young children (as reflected by the scores of the “what” dimension of PCK).

It is also important to improve teaching professionals’ understanding about students’ cognitions and pedagogies in the context of mathematics teaching. Beyond the lack of subject-training tradition, the application of general understanding about young children and broad approaches of pedagogies into subject teaching was insufficient. According to the study, teachers’ knowledge about how students learn mathematics and how to make the content comprehensible (as reflected by the “who” and “how” dimension) was even more limited than understanding about the math concepts. Early childhood teachers are assumed to be well prepared about child development and general pedagogy knowledge; the familiarity with young children’s cognitive development and acquaintance with general pedagogies enables teachers to sensitively accommodate students’ diverse individual needs and create an efficient learning environment. However,
general understanding about students and pedagogy didn’t guarantee a similar level of cognition about subject learning and teaching.

Unfortunately, there are even fewer opportunities for teachers to examine teaching and learning in the context of foundational mathematics. Beyond the minimum requirement for mathematics courses or workshops, even fewer courses or workshops are available to discuss how children learn a specific subject, what learning trajectories look like, and what are the most likely struggles they may have around important mathematical concepts, let alone math-specific pedagogies and representations to make the content more graspable for young children. New teachers with adequate subject matter understanding but insufficient preparations to deal with student needs were found to have difficulty making decisions about optimal instructional solutions (Grossman, 1989). While the design of the teacher preparation and in-service training may assume that isolated knowledge about subject matter, students and pedagogy can be automatically incorporated in teaching, the even lower level of knowledge about “who” and “how” compared to “what” don’t support this assumption.

To sum up, better teacher preparation and ongoing professional development opportunities should feature integration of students’ cognition and pedagogical strategies together with highlighting the nature of important foundational mathematics concepts that young children can learn. Courses that reflect a serious examination of the nature of the mathematics that teachers use in the practice of teaching do have some promise for improving students’ performance. The Big Ideas in foundational mathematics (Early Math Collaborative, 2014) can be a great resource to deepen early childhood
professionals’ conceptual understanding. Teachers need to know mathematics in ways that enable them to help students learn, professionals need to unpack their mathematical understanding and incorporate the conceptual sophistication into awareness of students’ cognitions and effective pedagogies.

In fact, well-designed professional development can make a change. Functioning relatively independently from each other, the three aspects of PCK (content understanding, knowledge of students’ vulnerabilities and pedagogical understanding to address students’ needs) were reported to become significantly related to each other through an intervention. The study was conducted among a group of early childhood teachers who participated in a one year professional training designed to increase their content understanding and how to apply that knowledge in understanding students’ learning and curricular design (Melendez, 2008). Although PCK was studied broadly across different subject areas with a limited sample size, it is suggested that the integration and coherence can be improved.

Last but not least, the importance of content expertise indicated by PCK in effective teaching does not reduce the significance of other types of knowledge. Highlighting the significance of incorporating different types of knowledge into subject teaching does not undermine the importance of knowledge about child development and general pedagogy. Insufficient preparation in learner and pedagogy may also prevent the unpacking of subject understanding and productive teaching (Grossman, 1990; Zhou et al., 2006). As well, PCK is only one type of knowledge or educational resource for students. It is important and unique for teaching professionals, but it won’t cover all the
significant aspects of knowledge required for teaching, such as teaching orientation (e.g., Grossman, 1990; Magnusson et al., 1999) and contextual knowledge (CxK) which involves one or more type of knowledge about curriculum, school and community (e.g., Grossman, 1990; Gardner & Gess-Newsome, 2011; Magnusson, Krajcik, & Borko, 1999).

**Limitations**

There is discussion about what is the best way to capture PCK. Some researchers believe that PCK is only measurable through observations of classroom teaching and nothing would replace the real classroom teaching in assessing it; others argue that knowledge is different from teaching behaviors obtained through observations since it is an internal construct (Baxter & Lederman, 1999). There are also groups that have integrated information from various resources such as interviews and classroom observations (i.e., Grossman, 1990; Loughran et al., 2008). In the current study, the prompted questions and the video stimulus are likely to elicit unique professional knowledge used in teaching by situating and confronting teachers with similar challenges as professionals who teach young children mathematics.

In the current study, professionals’ capability of unpacking mathematical knowledge for teaching was differentiated by their lesson analysis capabilities. A video was applied to elicit knowledge used for teaching foundational math since PCK can be explicit by reflection and accessible when teachers are making lesson plans (Danielson, 1996). Teachers’ video-based lesson analysis ability has also been reported to positively predict elementary students’ math learning (Kersting et al., 2010; Roth et al., 2011;
Seago, 2003). The content of the video was also carefully selected based on the suggestion of focusing on “a small number of big subject-specific ideas and concepts that may have broad consequences for teaching and learning” (Sosiak, 1999, p.146).

Significant topics of number system knowledge that impacted later school success and economy (Geary, 2013) were applied to make inferences about teachers’ understanding of mathematics used in teaching young children.

Although carefully selected, using one specific video to make inferences about teachers’ knowledge can be limited. The study analyzed responses to one particular video about number composition and decomposition in a kindergarten classroom. The other video in the survey was also about number system knowledge (i.e., fraction) from a 2nd grade classroom. Previous explorations from different cohorts of teachers have revealed positive and significant correlations between teachers’ responses to different videos. To simplify the investigation without introducing confounding factors such as grade level, content and difficulty of analyzing different videos, only one video was used. Still, the generalizability about teachers’ content expertise and its association with teaching and learning is likely to be restricted.

As well, conclusions from the present study should be cautious from topic level study (i.e., understanding about one specific math topic) to domain level inference (overall understanding of mathematics). While there are more and more research projects investigating PCK at a grain size through specific topics (Van Driel et al., 1998; Loughran et al., 2008); too much emphasis on topic specific pedagogy may be “overly specific and reach levels of details that are not responsive to the daily work of teaching”
(Sosiak, 1999, p.145). While number sense refers to “interconnected knowledge of numbers and operations” and includes “understanding concepts of quantity and relative quantity, facility with counting and ability to carry out simple operations” (Cross, 2009, p.22) this study only tested a small topic within the large domain. In the long run, it is necessary to apply multiple videos with different content for further investigation and inference making.

Still, to what extent the open-ended online survey can capture the knowledge teachers have can be questioned. Some reported mixed findings suggesting that motivation is of vital importance in eliciting content knowledge for teaching from a pencil-paper test. Compared with interviews and observations, teachers were less likely to be motivated to articulate their thoughts in a paper-pencil test (Gardner & Gess-Newsome, 2011). The current study made great effort during the data collection process by informing potential survey takers about the intervention project, relating to professionals the impact of the project on teaching and learning, and giving teachers’ incentives and technical support for completing the survey. The coding rubrics were also developed in a way to maximally reduce the impact of writing capability. In addition, the analysis was done at a group level with relatively large sample size. In fact, the level of understanding revealed from the current survey was similar to the findings from interview and observations (Bowman et al., 1982; Melendez, 2008), which to a certain extent supported the validity of the tool. Still, the author acknowledges the potential deficits of the data collection methods.
The math teaching quality observation tool used in the study has yet to be fully validated. The design of the observation tool may not cover all the important aspects of content teaching. Also, the observation was a one-time onsite visit and there is accumulated evidence about the randomness of teaching quality observations. Newton (2010) conducted a study examining the generalizability of a classroom observation tool for mathematics reform initiative. Thirty-two teachers from 2nd, 4th, 8th and 10th grade were involved; the same teacher who scored high on one occasion did not necessarily receive a high score on another occasion. The results suggested that there were considerable differences in teachers’ practice from one occasion to another. Therefore, it is possible that the teaching quality rating from one time visit from the current study may not be sensitive enough to capture the average level of individual teachers’ teaching quality.

As well, the assessments applied at the student level have their own drawbacks. Different measurements are also likely to capture different areas of mathematical understanding. Applying two different assessments of students’ math performance has expanded the scope of content coverage. However, many measures of student achievement, including the current study, used in research on teaching effectiveness are item-based standardized tests. While the tests are easy to administer, the high-level thinking and conceptual understanding highlighted in PCK may not be fully reflected in the students’ assessment, therefore underestimating the predictability of PCK on students’ learning.
The sample size of the profile study was 182, a fairly decent number in early childhood teaching studies; however, when it comes to the validity study, only half of the sample was involved, referring to 82 teachers and about 600 students. While the sample selection process was done rigorously, and the sample for the validity study had little difference from the overall sample, potential analysis about expert teachers’ profile of PCK, and the complex dynamic between knowledge, teaching quality and students’ learning was restricted by the sample size. In fact, the validity study of teachers’ PCK and teaching quality in mathematics, the inclusion and exclusion of 3 outliers (3.5% of the sample) revealed vastly different predictions, suggesting the sample size was not large enough to avoid the impact of few outliers.

In addition, the limited overall level of math understanding in early childhood professionals posed greater challenge to sensitively capture the complex relationships between PCK and teaching competence. In the current study, significant prediction of teachers’ knowledge (reflected by the “who” and “how” dimensions of PCK) on students’ learning at classroom level was not revealed, but favoring a specific subgroup of students. The results suggested that the effect of teacher’s knowledge on students’ learning revealed by “who” and “how” was rather small. However, the importance of teachers’ knowledge demonstrated by “who” and “how” can be underestimated due to the even less developed understanding (compared with “what”), which can limit the chance to capture the nuance dynamic between teaching and learning.

Overall, we are indeed still in infancy in terms of knowing the nuances of the construct. To a great extent, the methodology weaknesses discussed here are both a result
of and contribution to the lack of a clear understanding of the PCK concept, as well as content teaching quality and students’ learning. Studying the complexity of teaching processes has proved to be an extremely difficult process (Darling-Hammond & Bransford, 2005). The investigation on the relationship between PCK, teaching quality and student outcome has advanced our understanding about the construct, its measurement and potential revisions for the theory and assessment (Alonzo, 2007). The investigation about the relationships between teachers’ knowledge and student performance is constrained until more nuanced and valid instruments exist.

**Summary**

Foundational mathematics understanding is vital for children’s learning. Early mathematics skills such as number sense are powerful predictors of later academic achievement in mathematics, reading, science and technology (Duncan et al., 2007; Jordan, Glutting, & Ramineni, 2010; Jordan, Kaplan, Ramineni, & Locuniak, 2009; NCTM & NAEYC, 2009; Watts, Duncan, Siegler, & Davis-Kean, 2014). More important, growth in mathematical ability between 54 months and first grade has been reported to be a even stronger predictor of mathematics achievement in adolescence. Math competence (indicated by number system knowledge) in 1st grade was associated with 7th grade math performance, which in turn predicted adults’ economic status (Geary et al., 2013). Together, the evidence suggests that early math competency plays a fundamental role in learning and function of individuals, as well as the prosperity of the society in the long run.
Despite the importance of early math and the learning capability of young children, early forms of mathematical knowledge won’t naturally or fully develop without deliberate support. Learning is social and occurs with more competent others (Rogoff, 1990; Vygotsky, 1978). By participating in social interaction with more capable others, children eventually internalize the thinking or problem-solving (Bart, Yuzawa & Yuzawa, 2008) and generalize it into other settings. High-quality early mathematics teaching serves as a sound foundation for later learning in mathematics, and the single most powerful determinant of students’ learning is teachers’ knowledge (Darling-Hammond & Bransford, 2005). There is a dearth of studies to address the mathematics teaching competence in early childhood education. The majority of the investigations are small-scaled and do not link to teaching practice or students’ learning gains. To what extent the operationalized construct, the assessment tools, and the results from upper grades can be applied to early math education is open to question. Young children differ from their big brothers and sisters physically, cognitively, and emotionally. They are at distinctive developmental stages with huge individual differences, including their mathematical understanding. Their concrete learning style and developing fine motor skills also pose different challenges to representing mathematical ideas to them effectively.

Among various approaches, the notion of PCK provides a promising framework to investigate teaching competence. By exclusively focusing on academic knowledge from teacher preparation, other approaches of studying teacher knowledge assume subject matter understanding itself is sufficient for quality teaching and fail to notice the
importance of how knowledge is used in teaching a subject (Hill et al., 2005). PCK theory, on the other hand, purports that teaching requires specialized content understanding that is different from the subject knowledge possessed by common adults or content experts (Shulman, 1986, 1987). As a type of “craft knowledge” (van Driel, Verloop, & de Vos, 1998), it embodies the aspects of “content most germane to its teachability” (Shulman, 1986, p.9), including content understanding aligned with grade level of teaching, students’ cognition, and pedagogy.

By highlighting the discipline-specific nature of teaching, PCK provides a promising approach to explore early math teaching. The field of early childhood education is reluctant to specify standards for particular subject domains in order to prevent the fact-and-skill driven approach (Bowman et al., 2001). Teacher preparation is also subject-general. Therefore, the content aspect of teaching was a “missing paradigm” (Shulman, 1987, 1986). In the meantime, the investigation of early math teaching competence can enrich the theory and empirical investigations of PCK. Despite the new way of assessing by PCK and many existing studies, PCK has not been clearly defined or well-studied. These issues include but are not limited to: 1) the conceptualization of PCK and how it applies to specific subject area and grade levels; 2) the most feasible way of assessing PCK for particular subject and specific grade level.

Based on the uniqueness of foundational mathematics and the challenge of teaching young children, the current study applied PCK theory in investigating early childhood professionals’ mathematics knowledge used for teaching by proposing a conceptual model with three dimensions of knowledge, developing a survey to assess the
contextualized knowledge, and exploring the validity of the conceptualization and assessment tool. Recognizing the significance of content understanding, but also the specialized knowledge required in teaching, the PCK-EM model proposed that there are three important aspects of knowledge that are essential for teaching: 1) “what,” conceptual understanding of foundational mathematics, including the depth and breadth of unpacking mathematical knowledge; 2) “who,” knowledge about students’ learning patterns in mathematics, including prior knowledge and misunderstandings; and 3) “how,” math-specific pedagogical awareness, including pedagogical strategies and mathematical representations.

The study revealed promising evidence to support the conceptualization and assessment of PCK in early math education. PCK was a reliable indicator of teaching effectiveness (both teaching quality and students’ learning). At the teacher level, teachers’ knowledge indicated by PCK was found to significantly predict math teaching quality beyond distal indicators of teaching experience and pure academic knowledge such as years of teaching, degree earned, math courses taken and math workshops attended. At the student level, students made significantly more progress in mathematics understanding while taught by teachers with solid conceptual understanding in foundational mathematics (indicated by the “what” dimension of PCK). Smaller impacts of teachers’ understanding indicated by the “who” and “how” dimensions of PCK were found in subgroups of students. Together, these findings suggested that teachers’ knowledge as demonstrated by the PCK can have a positive impact on closing the achievement gap and thereby addressing educational equity. The findings strongly
supported PCK as an effective indicator of teaching competence and students’ learning, as well as the validity of the PCK-EM survey.

The current study also provided a starting point to improve teaching competence in foundational mathematics. According to the results, the overall level of PCK was lower than mid-range, suggesting that teachers’ understanding was more procedural and superficial. Only 4% of the teachers could be considered as experts. While there were teachers with sufficient teaching experience, as well as math learning experience from pre-service and in-service trainings, these indicators didn’t predict their content understanding used for teaching (e.g., PCK). Therefore, it is necessary to improve teachers’ content knowledge by innovations in teacher training, preparation and certification.

In addition, the proposal of PCK and the assessment tool can be applied to guide teacher preparation and trainings aiming to improve professionals’ content knowledge that matters for productive teaching and learning. First and foremost, teacher preparation, training and certification should put more emphasis on examining the nature of foundational mathematics. Conceptual understanding (i.e., the “what” dimension of PCK) has proved to be a driving force and leading factor in students’ learning. The low to medium level of “what” suggested that teachers’ mathematical knowledge has yet to be unpacked for students. Therefore, it is necessary to help teachers not only be able to execute operations (i.e., instrumental understanding), but also represent operations precisely for students (i.e., relational understanding), the latter of which is key for successful teaching (Ball, 1999; Skemp, 1976, c.f. Hutchison, 1997).
As well, it is necessary to highlight the “blending of content and pedagogy” and tailoring to students’ diverse characteristics (Shulman, 1987, p.4). PCK is a transformation of different types of knowledge bases (e.g. Wilson, Shulman, & Richert, 1988). Dewey (1902) called the transformation process “psychologizing,” Ball (1990) used the term “representation,” and Veal & MaKinster (1999) described it as “interpretation” and “specification.” The findings from the present study confirmed that “just knowing the content well was really important, just knowing general pedagogy was really important and yet when you added the two together, you didn’t get the teacher” (Berry et al., 2008). As suggested by the moderate correlation among three dimensions of PCK and the significantly lower scores in “who” and “how” dimensions of PCK (than “what”), although early childhood teachers are trained for developmentally appropriate practice, the understanding didn’t seem to transfer to content teaching automatically. Therefore, including courses and opportunities to examine students’ learning patterns and subject-specific pedagogy in mathematics would help to interweave different knowledge bases for the purpose of teaching.
APPENDIX A

PCK-EM SURVEY: PROMPTED QUESTIONS
(1) What is the **central mathematical** concept of this activity? Please justify your answer.

(2) What are other important mathematical concepts you think are related to the central mathematical concept of this activity? Please justify your answer.

(3) What prior mathematics knowledge do children need to have in order to understand the central mathematical concept of this activity?

(4) Do the children appear to understand the central mathematical concept of this activity? Please provide evidence that supports your assessment.

(5) Based on your assessment, what would you do next to reinforce or extend children’s understanding? Please justify your answer.

(6) What are some common mathematical misunderstandings children might have when learning this central mathematical concept?

(7) What has the teacher done or said to help the children understand the central mathematical concept? (Consider things like: materials, setting, lesson design, teacher language, pacing, interactions, etc.) Were those instructional choices effective? Please justify your answer.

(8) How could the teacher change this activity to meet the needs of a child who is struggling? (Consider things like: materials, setting, lesson design, teacher language, pacing, interactions, etc.) Please justify your answer.

(9) How could the teacher change this activity to meet the needs of a child who is advanced? (Consider things like: materials, setting, lesson design, teacher language, pacing, interactions, etc.) Please justify your answer.
APPENDIX B

CONCEPTUAL AND MEASUREMENT MODELS OF
PEDAGOGICAL CONTENT KNOWLEDGE (PCK)
AND PCK IN EARLY MATHEMATICS (PCK-EM)
<table>
<thead>
<tr>
<th>Dimensions of PCK Models</th>
<th>Subcomponents of PCK-EM</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>PCK</strong></td>
<td><strong>Conceptual Model</strong></td>
</tr>
<tr>
<td><strong>PCK-EM</strong></td>
<td>** WHAT **</td>
</tr>
<tr>
<td><strong>WHAT</strong></td>
<td><strong>WHAT</strong></td>
</tr>
<tr>
<td>Content understanding</td>
<td>Understanding of foundational math for teaching young children</td>
</tr>
<tr>
<td>aligned to the grade</td>
<td><strong>Depth</strong></td>
</tr>
<tr>
<td>level(s)</td>
<td><strong>Capability of “deconstructing” foundational math into its complex underlying ideas that young children need to learn</strong></td>
</tr>
<tr>
<td><strong>WHO</strong></td>
<td><strong>WHO</strong></td>
</tr>
<tr>
<td>Content-specific</td>
<td><strong>WHO</strong></td>
</tr>
<tr>
<td>knowledge of learners’</td>
<td>Understanding young children’s mathematical learning patterns</td>
</tr>
<tr>
<td>cognitions and learning</td>
<td><strong>Learning Path</strong></td>
</tr>
<tr>
<td>patterns</td>
<td>Understanding of young children’s learning progressions/trajectories of mathematical concepts</td>
</tr>
<tr>
<td><strong>HOW</strong></td>
<td><strong>HOW</strong></td>
</tr>
<tr>
<td>Content-specific</td>
<td><strong>HOW</strong></td>
</tr>
<tr>
<td>pedagogical knowledge</td>
<td>Math-specific pedagogical knowledge that can facilitate young children’s mathematical understanding</td>
</tr>
<tr>
<td><strong>HOW</strong></td>
<td><strong>Strategy</strong></td>
</tr>
<tr>
<td>Content-specific</td>
<td>Knowledge of pedagogical strategies applies for young children and learning math in general</td>
</tr>
<tr>
<td>pedagogical knowledge</td>
<td><strong>Representation</strong></td>
</tr>
<tr>
<td><strong>HOW</strong></td>
<td>Knowledge of specific representations to present math ideas/concepts (e.g. illustrations, examples, models, demonstrations and analogies)</td>
</tr>
</tbody>
</table>
APPENDIX C

PCK-EM SURVEY: GENERAL PRINCIPLES OF CODING
(1) **Seek evidence** of understanding from the *whole response* by the subcomponents of PCK:

The coder will go over the response and keep notes of evidence accordingly.

For instance, although question 1 is the only question that asks specifically about what is the central mathematical idea revealed in the video (and therefore closely related to concept understanding of the big idea(s)), to give a score for respondent’s conceptual understanding (e.g. depth), coders should go over the whole response, mark relevant evidence and make inference from answering other questions such as instructional design.

(2) **The quality of the answer**: The criteria basically are Identification & Sophistication in relation to the specific mathematical big idea(s). Because the current study is looking for teachers’ content understanding around a specific big idea or big ideas, the rating will consider to what extent the respondents were able to identify concepts, strategies etc. related to the specific big idea(s), and how appropriate it is to promote children’s understanding about the big idea(s). The highest level of answer is reflecting a repertoire of understanding related to the specific big idea(s) corresponding to each subcomponents of PCK. The quality of the answer is not solely relying on the amount of writing.

Take Number 7 video as an example, the “depth” is seeking evidence about the understanding of part-part-whole relationship, and someone who can identify “there are different combinations to reach the total” would be considered as medium high “depth.” However, to reach high level “depth”, one must show a repertoire of sophisticated understanding around the big idea, therefore other ideas such as “the total stays the same regardless of the combination” is a must.

(3) **Credit the highest level of understanding**: There can be multiple levels of understanding within one respondent’s answer. It is necessary to keep record of all the evidence; however, the final score depends on evidence of a repertoire of the highest-level understanding corresponding to particular subcomponents of PCK.

For instance, “depth” is seeking for evidence about understanding of the specific big ideas(s) and Number 7 video is about part-part-whole relationship. The same respondent
may mention number sense and operation, part-part-whole relationship, and further articulate the idea(s) somewhere else. The final score of depth will rely on the highest level of evidence. In other words, descriptions such as “number sense and operation” will only be considered as evidence for depth when nothing else was mentioned about “part-part-whole relationship.”

(4) **Downgrading is possible**: When there are some misunderstandings revealed in a response, the rating for corresponding component will be downgraded accordingly, regardless of the principle about “crediting the highest level of understanding.”

For instance, the Number 7 video was about part-part-whole relationship. A teacher may show understanding about number composition and decomposition but mistakenly believed that number zero is the central mathematical idea of the lesson. In this case, although the respondent may talk about number combinations elsewhere (which can be coded as “3” originally), the final score for “depth” will be downgraded (e.g. to “2” instead).

Another example is teachers’ knowledge of students’ misunderstanding and learning difficulties. If the respondents overlooked apparent misunderstandings/learning difficulties appeared in the video demonstrated by answering Question 4 (about assessing students’ understanding), adjustment will be made accordingly to this component (i.e. misunderstanding).

(5) **Multiple lenses**: The same response can be coded from different aspects (i.e. subcomponents of PCK).

For instance, materials can be double-coded for both strategy and representation. It is worth noting, however, strategy is more about whether particular pedagogical strategy has been identified; and for representation, it is more about how sophisticated/relevant the strategy is to the mathematical big ideas(s). In other words, the former highlights the identification and explanations (if there’s any), and the latter addresses to what degree the material is tailored to the conceptual understanding and students’ learning needs.

(6) **Anchor answers**: The coding is based on a 1 to 5 scale from low to high

<table>
<thead>
<tr>
<th>Low</th>
<th>Medium</th>
<th>High</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td></td>
</tr>
</tbody>
</table>
For a particular video from PCK-EM survey, sample answers will be provided for Low, Medium and High anchors for each subcomponents of PCK. Two possible ratings can be made between low and medium anchors, and between medium and high anchors.

For instance, anchor answers will be given for each component at level 1, level 3, and level 5. Coders should first make a rough decision about which range the answer is (low/medium/high), and then compare with the anchor answer to specify a code/level.

(7) **Rate Depth and Breadth last**: Although evidence is recorded in the designated areas for each subcomponents of PCK as the coder reads through the overall response, after finish marking evidence for each component, coders should assign a specific code to Learning Path, Misunderstanding, Strategy and Representation first, and then rate for depth and breadth. Content understanding serves as a foundation and is embedded in other components; therefore, it is more reasonable to rate Depth and Breadth last based on their own evidence and the overlapping evidence from other components.

(8) **Rate each component independently**: Although double coding is applied, coders should look for evidence for each subcomponents of PCK and assign a score based on the coding rubrics. The scoring of one subcomponents of PCK should not impact the scoring of the other components.
APPENDIX D

PCK-EM SURVEY: CODING RUBRICS

& ANCHOR ANSWERS FOR VIDEO “NUMBER 7”
<table>
<thead>
<tr>
<th>WHAT_Depth</th>
<th>Little or unrelated understanding</th>
<th>Basic and related, understanding</th>
<th>Specific, related and advanced understanding</th>
</tr>
</thead>
<tbody>
<tr>
<td>Score</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td><strong>Definition</strong></td>
<td>Understanding of a specific big idea or big ideas, demonstrated by the capability of “deconstructing” a foundational math concept into its complex underlying ideas that young children need to learn.</td>
<td>General mathematical ideas limited to behavioral or procedural level; No evidence of critical inference at a conceptual level</td>
<td>Basic understanding about the specific big idea(s); Limited evidence of critical inference at a conceptual level</td>
</tr>
<tr>
<td><strong>Example</strong></td>
<td>General description at behavioral or procedural level without making inference about the underlying mathematical ideas (i.e. Part-Part-Whole/number composition and decomposition for this video). E.g. Problem solving; number sense; counting</td>
<td>Basic understanding about the big idea(s) of Part-Part-Whole (P-P-W). E.g. There is more than one way to show a number; the same number of objects can be arranged in different ways; two parts equal a whole.</td>
<td></td>
</tr>
<tr>
<td>WHAT_Breadth</td>
<td>Little or unrelated understanding</td>
<td>Basic and related, understanding</td>
<td>Specific, related and advanced understanding</td>
</tr>
<tr>
<td>-------------------------------------------------</td>
<td>-----------------------------------</td>
<td>---------------------------------</td>
<td>-----------------------------------------------</td>
</tr>
<tr>
<td><strong>Score</strong></td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td><strong>Definition</strong></td>
<td>General or broad level mathematical ideas related to the big idea(s)</td>
<td>Basic and relevant mathematical concepts to the big idea(s)</td>
<td>A repertoire of specific and relevant mathematical ideas to the big idea(s)</td>
</tr>
<tr>
<td></td>
<td>No evidence of critical inference at a conceptual level</td>
<td>Limited evidence of critical inference at a conceptual level</td>
<td>Strong evidence of critical thinking at a conceptual level</td>
</tr>
<tr>
<td><strong>Example</strong></td>
<td>Misunderstanding about relevant mathematical ideas, OR mention relevant mathematical ideas in a vague way</td>
<td>Mathematical ideas related to P-P-W described in a vague way</td>
<td>A repertoire of mathematical ideas related to the big ideas of P-P-W, including but not limited to:</td>
</tr>
<tr>
<td></td>
<td><em>E.g. Number sense, counting</em></td>
<td><em>E.g.</em> Addition/subtraction More or Less</td>
<td>(1) Missing-addend word problem;</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>(2) There are patterns in the combination;</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>(3) Commutative law of addends and the sum;</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>(4) Three parts of decomposition;</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>(5) More or less related to the quantity of two sets</td>
</tr>
<tr>
<td>WHO_Prior Knowledge</td>
<td>Little or unrelated understanding</td>
<td>Basic and related, understanding</td>
<td>Specific, related and advanced understanding</td>
</tr>
<tr>
<td>---------------------</td>
<td>----------------------------------</td>
<td>---------------------------------</td>
<td>---------------------------------------------</td>
</tr>
<tr>
<td><strong>Score</strong></td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td><strong>Definition</strong></td>
<td>Vague, simple list of prior knowledge No evidence of critical inference at a conceptual level</td>
<td>Basic and relevant prior knowledge Limited evidence of critical inference at a conceptual level</td>
<td>A repertoire of specific and relevant prior knowledge Strong evidence of critical thinking at a conceptual level</td>
</tr>
<tr>
<td><strong>Example</strong></td>
<td>Simple list of vague prior knowledge related to P-P-W: <em>E.g.</em> Counting; number sense; quantify</td>
<td>Experience related to P-P-W mentioned in a vague way OR specific aspects of counting and number sense <em>E.g.</em> know the concept of addition; one to one correspondence; cardinality attribute of numbers</td>
<td>A repertoire of prior knowledge closely related to the big idea(s) including but not limited to: (1) There are factors hidden inside a given number; (2) Experience with putting together two sets to make a total (3) Specific aspects of counting/number sense</td>
</tr>
<tr>
<td>Score</td>
<td>Definition</td>
<td>Example</td>
<td></td>
</tr>
<tr>
<td>-------</td>
<td>------------</td>
<td>---------</td>
<td></td>
</tr>
</tbody>
</table>
| 1     | Misunderstanding and difficulties with little relation to the big idea(s) of mathematics | Behavioral or procedural level mistakes or learning difficulties with little critical inference.  
*E.g.* Be afraid of math; miscounting (in general) |
| 2     | Basic misunderstanding and difficulties related to the big idea(s) | Clear aspects of miscounting including connections between different representations OR P-P-W related misunderstanding that appeared explicitly in the video  
*E.g.* Cannot understand that there are more than one combination or two parts equal a whole. |
| 3     | Limited evidence of critical inference at a conceptual level | A repertoire of misunderstanding/learning difficulties young children may have around the P-P-W OR very sophisticated understanding of counting, including but not limited to: (1) two different sets cannot make one number;  
(2) there are factors hidden inside a number; number can be made of and broken into group;  
and (3) the sum stays the same. |
<p>| 4     | Strong evidence of critical thinking at a conceptual level | |
| 5     | A repertoire of specific and relevant misunderstanding and learning difficulties to the big idea(s) | |</p>
<table>
<thead>
<tr>
<th>HOW_Pedagogy</th>
<th>Little or unrelated understanding</th>
<th>Basic and related, understanding</th>
<th>Specific, related and advanced understanding</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Score</strong></td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td><strong>Definition</strong></td>
<td>Pedagogies in general with little or no explanation&lt;br&gt;No evidence of critical inference</td>
<td>Pedagogical strategies with extensive clarifications&lt;br&gt;Limited evidence of critical thinking about incorporating pedagogies into the math content</td>
<td>A repertoire of specific pedagogical strategies and extensive clarifications&lt;br&gt;Strong evidence of critical thinking about incorporating pedagogies into the math content</td>
</tr>
<tr>
<td><strong>Example</strong></td>
<td>List a few general pedagogical strategies such as small group.</td>
<td>Identify a variety of general pedagogies and/or discuss applications of pedagogies extensively with/out integration into the math content.</td>
<td>Identify specific teaching strategies and provide extensive clarification AND discuss applications of teaching strategies extensively into the math content.</td>
</tr>
<tr>
<td>HOW_Representation</td>
<td>Little or unrelated understanding</td>
<td>Basic and related, understanding</td>
<td>Specific, related and advanced understanding</td>
</tr>
<tr>
<td>---------------------</td>
<td>----------------------------------</td>
<td>---------------------------------</td>
<td>-----------------------------------------------</td>
</tr>
<tr>
<td><strong>Score</strong></td>
<td>1</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td><strong>Definition</strong></td>
<td>Knowledge of specific representations (illustrations, examples, models, demonstrations and analogies) that can make clear a specific big idea or big ideas</td>
<td>Representation/activities with little relation to the big idea(s) of mathematics No evidence of critical inference at a conceptual level</td>
<td>Basic and related representations/activities to accommodate students’ needs Limited evidence of critical inference at a conceptual level</td>
</tr>
<tr>
<td><strong>Example</strong></td>
<td>Consider the use of representations without integrating into the specific mathematical big idea E.g. concrete materials, and meaningful items for young children</td>
<td>Apply appropriate representations or design an activity that can enhance students’ understanding around the big ideas with some clarifications, the design is extended from the video in the survey. E.g. use blocks of various colors for children to see the two parts, use pizza to show the whole stays the same</td>
<td>A repertoire of representations and activities that can be flexibly used in different scenarios and accommodate students (in general, struggling and more advanced)’ diverse needs, the ideas are innovative compared with the video in the survey</td>
</tr>
</tbody>
</table>
Appendix E

PCK-EM SURVEY: CODING FORM
<table>
<thead>
<tr>
<th>PCK Subcomponents</th>
<th>Score</th>
<th>NA</th>
<th>Low</th>
<th>Medium</th>
<th>High</th>
</tr>
</thead>
<tbody>
<tr>
<td>WHAT_Depth</td>
<td>Score</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>Understanding of a specific big idea or big ideas, demonstrated by the capability of “deconstructing” a foundational math concept into its complex underlying ideas that young children need to learn</td>
<td>Evidence</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>WHAT_Breadth</td>
<td>Score</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>Awareness of mathematical ideas and concepts related to a specific big idea or big ideas</td>
<td>Evidence</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>WHO_Prior Knowledge</td>
<td>Score</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>Understanding of young children’s prior knowledge in learning a specific big idea or big ideas</td>
<td>Evidence</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>WHO_Misunderstanding</td>
<td>Score</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>Knowledge of students’ likely misunderstandings and learning difficulties around a specific big idea or big ideas</td>
<td>Evidence</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>HOW_Pedagogy</td>
<td>Score</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>Knowledge of pedagogical strategies (from the video or for own teaching) that can facilitate students’ understanding of a specific big idea or big ideas</td>
<td>Evidence</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>HOW_Representation</td>
<td>Score</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>Knowledge of specific representations (illustrations, examples, models, demonstrations and analogies) that can make clear a specific big idea or big ideas</td>
<td>Evidence</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
APPENDIX F

DEMOGRAPHIC INFORMATION SURVEY:

ABOUT MY TEACHING (FALL, 2011)
1. **Respondent ID**  What is the confidential ID number assigned to you?

2. **How many years have you been teaching?**

3. **About how many pre-service math education/methods classes have you taken (excluding all college-level math classes such as calculus and statistics)?**

4. **About how many hours of in-service math education workshops have you taken in the last two years?**

5. **Please check all of the teaching certificates/endorsements you have earned (Check all that apply).**
   - Type 04 early childhood teacher certificate [Birth- Grade 3]
   - Type 03 elementary education certificate [Grades K-9]
   - Early childhood special education certificate
   - Bilingual/ESL endorsement
   - Special Teaching Certificate [Grades K-12]
   - Other (please specify)

6. **Do you have a bachelor's (BA/BS) degree?**
   If yes, in what subject area (major) did you earn your bachelor's degree?

7. **Do you have a Master's degree (i.e. M.A., M.S., M.Ed., etc.)?**
   If yes, in what field or discipline (major) did you earn your Master's degree?

8. **Do you speak any language(s) other than English?**
   Language 1: 
   Which language(s)? Language 2: 
   Language 3:

9. **How would you rate your speaking fluency in each of these languages?**

10. **When you were school age, was the instructional language at school different from the primary language spoken at your home?**

11. **Have you ever taken any pre-service or professional development courses specifically targeted for teaching ELL students?**

12. **How many pre-service and professional development courses have you taken that provide training for teaching ELL students?**

13. **How many years of experience do you have working with ELL students in a classroom setting?**

14. **Does your school have any formal policies about supporting students' home language?**

15. **Does your school provide bi-lingual instruction for students?**
16. Which of the following bi-lingual instructional practices, if any, does your school support?  
   My school supports some other bi-lingual instructional practice. (What?)

17. How many students are in your class?

18. How many of them speak English as their primary or only language?

19. How many of them speak English as a second language or are English Language Learners (ELLs)?
APPENDIX G

DEMOGRAPHIC INFORMATION SURVEY:
ABOUT MY TEACHING (SPRING, 2013)
The purpose of this questionnaire is to gather information to improve professional development. Please answer all of the questions. We appreciate your time.

1. Respondent ID
   What is the confidential ID number assigned to you?

2. How old are you?
   24 and under, 25-34,
   35-44, 45-54,
   55-64, 65 and over

3. Are you
   Female Male

4. What is your race or ethnicity?
   African-American or Black
   American Indian or Alaska Native
   Asian
   Caucasian or White
   Hispanic or Latino
   Native Hawaiian or other Pacific Islander
   Other (Please specify)

5. How many of each type of math class did you take in High School (if any)?
   0 1 2 3 or more
   Algebra
   Trigonometry
   Geometry
   Calculus
   Statistics
   Other

6. How many of each type of math class did you take in college and graduate school (if any)?
   0 1 2
   3 or more
   Math Concepts for Teachers
   Math teaching Methods
   Algebra
   Trigonometry
   Geometry
   Calculus
   Statistics
   Other

7. How many years have you taught the grade you are teaching now?
   Less than 1 year 1-2 years more than 2 years
APPENDIX H

TWO LEVEL HLM ANALYSIS RESULTS: THE PREDICTION OF TEACHING QUALITY IN MATHEMATICS BY PCK-EM
Table H1. Descriptive Statistics at Teacher Level for HLM Analysis

<table>
<thead>
<tr>
<th></th>
<th>N</th>
<th>Mean</th>
<th>SD</th>
<th>Minimum</th>
<th>Maximum</th>
</tr>
</thead>
<tbody>
<tr>
<td>PCK-EM</td>
<td>82</td>
<td>2.24</td>
<td>0.47</td>
<td>1.17</td>
<td>3.50</td>
</tr>
<tr>
<td>HIS-EM</td>
<td>79</td>
<td>4.14</td>
<td>1.30</td>
<td>1.67</td>
<td>6.78</td>
</tr>
<tr>
<td>Year</td>
<td>82</td>
<td>13.60</td>
<td>9.66</td>
<td>1.00</td>
<td>41.00</td>
</tr>
<tr>
<td>Class</td>
<td>82</td>
<td>3.24</td>
<td>2.77</td>
<td>0.00</td>
<td>15.00</td>
</tr>
<tr>
<td>Workshop</td>
<td>82</td>
<td>11.45</td>
<td>16.02</td>
<td>0.00</td>
<td>80.00</td>
</tr>
</tbody>
</table>

Note:
PCK: pedagogical content knowledge in early mathematics, indicating content knowledge for teaching foundational mathematics, it is the mean for PCK-What, PCK-Who and PCK-How at a 1-5 scale.
HIS-EM: high impact strategies in early mathematics, indicating the quality of teaching mathematics at a 1-7 scale.
Year: Years of teaching
Class: Number of mathematical classes taken in pre-service trainings
Workshop: hours of workshop taken during in-service work.

- **Unconditional Model**

  **Level-1 Model**

  \[ HIS-EM_{ij} = \beta_{0j} + r_{ij} \]

  **Level-2 Model**

  \[ \beta_{0j} = \gamma_{00} + u_{0j} \]

  **Mixed Model**

  \[ HIS-EM_{ij} = \gamma_{00} + u_{0j} + r_{ij} \]

Table H2. Final estimation of variance components

<table>
<thead>
<tr>
<th>Random Effect</th>
<th>Standard Deviation</th>
<th>Variance Component</th>
<th>d.f.</th>
<th>( \chi^2 )</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>INTRCPT1, ( u_0 )</td>
<td>0.29500</td>
<td>0.08702</td>
<td>7</td>
<td>11.64492</td>
<td>0.112</td>
</tr>
<tr>
<td>level-1, ( r )</td>
<td>1.26771</td>
<td>1.60709</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- **Using PCK to predict Teaching Quality Estimates**

  **Level-1 Model**

  \[ HIS-EM_{ij} = \beta_{0j} + \beta_{ij}(PCK-EM_{ij}) + r_{ij} \]
**Level-2 Model**

\[ \beta_{0j} = \gamma_{00} + u_{0j} \]
\[ \beta_{1j} = \gamma_{10} \]

**Mixed Model**

\[ \text{HIS-EM}_{ij} = \gamma_{00} + \gamma_{10} \times \text{PCK-EM}_{ij} + u_{0j} + r_{ij} \]

Table H3. Final estimation of fixed effects

<table>
<thead>
<tr>
<th>Fixed Effect</th>
<th>Coefficient</th>
<th>Standard Error</th>
<th>t-ratio</th>
<th>Approx. d.f.</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>For INTRCPT1, ( \beta_0 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>INTRCPT2, ( \gamma_{00} )</td>
<td>2.314568</td>
<td>0.655289</td>
<td>3.532</td>
<td>7</td>
<td>0.010</td>
</tr>
<tr>
<td>For PCKMEAN slope, ( \beta_1 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>INTRCPT2, ( \gamma_{10} )</td>
<td>0.807780</td>
<td>0.283411</td>
<td>2.850</td>
<td>70</td>
<td>0.006</td>
</tr>
</tbody>
</table>

Table H4. Final estimation of variance components

<table>
<thead>
<tr>
<th>Random Effect</th>
<th>Standard Deviation</th>
<th>Variance Component</th>
<th>d.f.</th>
<th>( \chi^2 )</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>INTRCPT1, ( u_{0j} )</td>
<td>0.06038</td>
<td>0.00365</td>
<td>7</td>
<td>8.44658</td>
<td>0.294</td>
</tr>
<tr>
<td>level-1, ( r )</td>
<td>1.23866</td>
<td>1.53428</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- *Adding Teachers’ Background Information Estimates*

**Level-1 Model**

\[ \text{HIS-EM}_{ij} = \beta_{0j} + \beta_{1j} \times (\text{YEAR}_{ij}) + \beta_{2j} \times (\text{CLASS}_{ij}) + \beta_{3j} \times (\text{WORKSHOP}_{ij}) + \beta_{4j} \times (\text{PCK-EM}_{ij}) + r_{ij} \]

**Level-2 Model**

\[ \beta_{0j} = \gamma_{00} + u_{0j} \]
\[ \beta_{1j} = \gamma_{10} \]
\[ \beta_{2j} = \gamma_{20} \]
\[ \beta_{3j} = \gamma_{30} \]
\[ \beta_{4j} = \gamma_{40} \]
Mixed Model

\[ \text{HIS-EM}_{ij} = \gamma_{00} + \gamma_{10} \times \text{YEAR}_{ij} + \gamma_{20} \times \text{CLASS}_{ij} + \gamma_{30} \times \text{WORKSHOP}_{ij} + \gamma_{40} \times \text{PCK-EM}_{ij} + u_0j + r_{ij} \]

Table H5. Final estimation of fixed effects

<table>
<thead>
<tr>
<th>Fixed Effect</th>
<th>Coefficient</th>
<th>Standard error</th>
<th>t-ratio</th>
<th>Approx. d.f.</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>For INTRCPT1, ( \beta_0 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>INTRCPT2, ( \gamma_{00} )</td>
<td>2.375547</td>
<td>0.764726</td>
<td>3.106</td>
<td>7</td>
<td>0.017</td>
</tr>
<tr>
<td>For YR_TEA slope, ( \beta_1 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>INTRCPT2, ( \gamma_{10} )</td>
<td>-0.011119</td>
<td>0.015919</td>
<td>-0.698</td>
<td>67</td>
<td>0.487</td>
</tr>
<tr>
<td>For NUM_CLAS slope, ( \beta_2 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>INTRCPT2, ( \gamma_{20} )</td>
<td>0.036882</td>
<td>0.056543</td>
<td>0.652</td>
<td>67</td>
<td>0.516</td>
</tr>
<tr>
<td>For HR_WORKS slope, ( \beta_3 )</td>
<td></td>
<td></td>
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</tr>
<tr>
<td>INTRCPT2, ( \gamma_{30} )</td>
<td>0.004474</td>
<td>0.008890</td>
<td>0.503</td>
<td>67</td>
<td>0.616</td>
</tr>
<tr>
<td>For PCKMEAN slope, ( \beta_4 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>INTRCPT2, ( \gamma_{40} )</td>
<td>0.772689</td>
<td>0.296320</td>
<td>2.608</td>
<td>67</td>
<td>0.011</td>
</tr>
</tbody>
</table>

Table H6. Final estimation of variance components

<table>
<thead>
<tr>
<th>Random Effect</th>
<th>Standard Deviation</th>
<th>Variance Component</th>
<th>d.f.</th>
<th>( \chi^2 )</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>INTRCPT1, ( u_0 )</td>
<td>0.04864</td>
<td>0.00237</td>
<td>7</td>
<td>7.47685</td>
<td>0.381</td>
</tr>
<tr>
<td>level-1, ( r )</td>
<td>1.25683</td>
<td>1.57963</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
APPENDIX I

OUTLIER DETECTION BASED ON THE RELATIONSHIP BETWEEN PCK-EM AND THE QUALITY OF TEACHING MATHEMATICS
Figure K1. Scatter Plot between PCK and Teaching Quality in Mathematics.

Table I1. Descriptive information about outliers

<table>
<thead>
<tr>
<th>ID</th>
<th>PCK</th>
<th>HIS-EM</th>
<th>YEAR</th>
<th>COURSE</th>
<th>WORKSHOP</th>
</tr>
</thead>
<tbody>
<tr>
<td>211</td>
<td>1.17</td>
<td>4.67</td>
<td>11</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>257</td>
<td>3</td>
<td>2.22</td>
<td>28</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>302</td>
<td>3.83</td>
<td>2.89</td>
<td>10</td>
<td>2</td>
<td>0</td>
</tr>
</tbody>
</table>
Table I2. Statistical information about outliers

<table>
<thead>
<tr>
<th>ID</th>
<th>Studentized Residual</th>
<th>Centered Leverage</th>
<th>Standard Dfbeta Intercept</th>
<th>Standard Dfbeta Slope</th>
</tr>
</thead>
<tbody>
<tr>
<td>211</td>
<td>.70902</td>
<td>0.02561</td>
<td>0.12477</td>
<td>-0.11512</td>
</tr>
<tr>
<td>257</td>
<td>-1.4698</td>
<td>0.00735</td>
<td>0.09722</td>
<td>-0.12728</td>
</tr>
<tr>
<td>302</td>
<td>-1.1856</td>
<td>0.03898</td>
<td>0.21142</td>
<td>-0.23979</td>
</tr>
</tbody>
</table>

Note:
Three types of measures were applied to identify outliers, studentized residual for discrepancy, centered leverage for leverage, and cook’s distance and standard dfbetas for influence. Based on the sample size, the threshold value for those three indicators are: 3 for absolute value of studentized residual (regardless of sample size); .002 for centered leverage (3k/n, k is the number of predictor and n is the number of observation); and .16 for standard Dfbeta (2/Sqrt(N), N is the number of observations). Values larger than the threshold may be problematic and should be checked as deserving special attention. Please see Williams, R. (2014) for reference.

Table I3. Descriptive information for the Sample with/out Outliers

<table>
<thead>
<tr>
<th>Teachers’ level Descriptive with Outliers</th>
<th>N</th>
<th>Mean</th>
<th>SD</th>
<th>Minimum</th>
<th>Maximum</th>
</tr>
</thead>
<tbody>
<tr>
<td>PCK-EM</td>
<td>85</td>
<td>2.27</td>
<td>0.53</td>
<td>1.17</td>
<td>3.83</td>
</tr>
<tr>
<td>HIS-EM</td>
<td>81</td>
<td>4.11</td>
<td>1.30</td>
<td>1.67</td>
<td>6.78</td>
</tr>
<tr>
<td>Year</td>
<td>85</td>
<td>13.69</td>
<td>9.62</td>
<td>1.00</td>
<td>41.00</td>
</tr>
<tr>
<td>Class</td>
<td>85</td>
<td>3.22</td>
<td>2.73</td>
<td>0.00</td>
<td>15.00</td>
</tr>
<tr>
<td>Workshop</td>
<td>85</td>
<td>11.09</td>
<td>15.84</td>
<td>0.00</td>
<td>80.00</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Teachers’ level Descriptive without Outliers</th>
<th>N</th>
<th>Mean</th>
<th>SD</th>
<th>Minimum</th>
<th>Maximum</th>
</tr>
</thead>
<tbody>
<tr>
<td>PCK-EM</td>
<td>82</td>
<td>2.24</td>
<td>0.47</td>
<td>1.17</td>
<td>3.50</td>
</tr>
<tr>
<td>HIS-EM</td>
<td>79</td>
<td>4.14</td>
<td>1.30</td>
<td>1.67</td>
<td>6.78</td>
</tr>
<tr>
<td>Year</td>
<td>82</td>
<td>13.60</td>
<td>9.66</td>
<td>1.00</td>
<td>41.00</td>
</tr>
<tr>
<td>Class</td>
<td>82</td>
<td>3.24</td>
<td>2.77</td>
<td>0.00</td>
<td>15.00</td>
</tr>
<tr>
<td>Workshop</td>
<td>82</td>
<td>11.45</td>
<td>16.02</td>
<td>0.00</td>
<td>80.00</td>
</tr>
</tbody>
</table>

Note:
PCK: pedagogical content knowledge in early mathematics, indicating content knowledge for teaching foundational mathematics, it is the mean for PCK-What, PCK-Who and PCK-How at a 1-5 scale.
HIS-EM: high impact strategies in early mathematics, indicating the quality of teaching mathematics at a 1-7 scale.
Year: Years of teaching
Class: Number of mathematical classes taken in pre-service trainings
Workshop: hours of workshop taken during in-service work.
HLM Analysis Results for PCK Predicting Teaching Quality with Outliers

- **Unconditional Model**

Table I4. Final estimation of variance components

<table>
<thead>
<tr>
<th>Random Effect</th>
<th>Standard Deviation</th>
<th>Variance Component</th>
<th>d.f.</th>
<th>$\chi^2$</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>INTRCPT1, $u_0$</td>
<td>0.31626</td>
<td>0.10002</td>
<td>7</td>
<td>12.27865</td>
<td>0.091</td>
</tr>
<tr>
<td>level-1, $r$</td>
<td>1.26539</td>
<td>1.60120</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- **Using PCK to predict Teaching Quality Estimates**

Table I5. Final estimation of fixed effects

<table>
<thead>
<tr>
<th>Fixed Effect</th>
<th>Coefficient</th>
<th>Standard error</th>
<th>t-ratio</th>
<th>Approx. d.f.</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>For INTRCPT1, $\beta_0$</td>
<td>3.049910</td>
<td>0.620162</td>
<td>4.918</td>
<td>7</td>
<td>0.002</td>
</tr>
<tr>
<td>INTRCPT2, $\gamma_{00}$</td>
<td>0.467379</td>
<td>0.262601</td>
<td>1.780</td>
<td>73</td>
<td>0.079</td>
</tr>
<tr>
<td>For PCK-EM slope, $\beta_1$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>INTRCPT2, $\gamma_{10}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table I6. Final estimation of variance components

<table>
<thead>
<tr>
<th>Random Effect</th>
<th>Standard Deviation</th>
<th>Variance Component</th>
<th>d.f.</th>
<th>$\chi^2$</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>INTRCPT1, $u_0$</td>
<td>0.24277</td>
<td>0.05894</td>
<td>7</td>
<td>10.67753</td>
<td>0.153</td>
</tr>
<tr>
<td>level-1, $r$</td>
<td>1.25829</td>
<td>1.58330</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table I7. Final estimation of fixed effects

<table>
<thead>
<tr>
<th>Fixed Effect</th>
<th>Coefficient</th>
<th>Standard error</th>
<th>t-ratio</th>
<th>Approx. d.f.</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>For INTRCPT1, $\beta_0$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>INTRCPT2, $\gamma_{00}$</td>
<td>3.183926</td>
<td>0.714333</td>
<td>4.457</td>
<td>7</td>
<td>0.003</td>
</tr>
<tr>
<td>For YEAR slope, $\beta_1$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>INTRCPT2, $\gamma_{10}$</td>
<td>-0.018033</td>
<td>0.016021</td>
<td>-1.126</td>
<td>70</td>
<td>0.264</td>
</tr>
<tr>
<td>For CLASS slope, $\beta_2$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>INTRCPT2, $\gamma_{20}$</td>
<td>0.037012</td>
<td>0.057867</td>
<td>0.640</td>
<td>70</td>
<td>0.525</td>
</tr>
<tr>
<td>For WORKSHOP slope, $\beta_3$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>INTRCPT2, $\gamma_{30}$</td>
<td>0.006821</td>
<td>0.009055</td>
<td>0.753</td>
<td>70</td>
<td>0.454</td>
</tr>
<tr>
<td>For PCK-EM slope, $\beta_4$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>INTRCPT2, $\gamma_{40}$</td>
<td>0.430892</td>
<td>0.270189</td>
<td>1.595</td>
<td>70</td>
<td>0.115</td>
</tr>
</tbody>
</table>

Table I8. Final estimation of variance components

<table>
<thead>
<tr>
<th>Random Effect</th>
<th>Standard Deviation</th>
<th>Variance Component</th>
<th>d.f.</th>
<th>$\chi^2$</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>INTRCPT1, $u_0$</td>
<td>0.19091</td>
<td>0.03645</td>
<td>7</td>
<td>8.93849</td>
<td>0.256</td>
</tr>
<tr>
<td>level-1, $r$</td>
<td>1.27503</td>
<td>1.62571</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
APPENDIX J

THREE LEVEL HLM RESULTS: THE PREDICTION OF STUDENTS’ MATHEMATICAL PERFORMANCE BY TEACHERS’ PCK-EM
Table J1. Descriptive information about teachers and students

<table>
<thead>
<tr>
<th>Teachers’ level Descriptive</th>
<th>N</th>
<th>Mean</th>
<th>SD</th>
<th>Minimum</th>
<th>Maximum</th>
</tr>
</thead>
<tbody>
<tr>
<td>PCK-What</td>
<td>81</td>
<td>2.37</td>
<td>0.54</td>
<td>1.00</td>
<td>4.00</td>
</tr>
<tr>
<td>PCK-Who</td>
<td>81</td>
<td>2.12</td>
<td>0.64</td>
<td>1.00</td>
<td>4.00</td>
</tr>
<tr>
<td>PCK-How</td>
<td>81</td>
<td>2.22</td>
<td>0.70</td>
<td>1.00</td>
<td>4.00</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Students’ level Descriptive</th>
<th>N</th>
<th>Mean</th>
<th>SD</th>
<th>Minimum</th>
<th>Maximum</th>
</tr>
</thead>
<tbody>
<tr>
<td>WJ-Pre</td>
<td>605</td>
<td>94.76</td>
<td>12.91</td>
<td>48.00</td>
<td>134.00</td>
</tr>
<tr>
<td>WJ-Post</td>
<td>548</td>
<td>96.57</td>
<td>12.87</td>
<td>49.00</td>
<td>136.00</td>
</tr>
<tr>
<td>TEAM-Pre</td>
<td>585</td>
<td>25.26</td>
<td>19.35</td>
<td>-32.28</td>
<td>67.60</td>
</tr>
<tr>
<td>TEAM-Post</td>
<td>547</td>
<td>32.82</td>
<td>18.43</td>
<td>-20.52</td>
<td>71.16</td>
</tr>
<tr>
<td>Gender</td>
<td>607</td>
<td>1.52</td>
<td>.50</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>Age-Pre</td>
<td>607</td>
<td>76.38</td>
<td>17.31</td>
<td>46</td>
<td>119</td>
</tr>
<tr>
<td>Language</td>
<td>607</td>
<td>.85</td>
<td>.36</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

Note:
WJ: standardized score of Woodcock Johnson-III Applied Problem, indicating students’ mathematics performance, with a mean of 100 and standard deviation of 15, adjusted by age and national norm.
TEAM: T score of Tools for Early Assessment in Math, mathematics performance, indicating young children’s math performance, scaled by rasch model, not adjusted by age or national norm.
Gender: 1- male, 2 - female
Language: 1- English, 2 - Spanish

HLM Analysis Based on WJ-AP Test

- Unconditional Model

Level-1 Model

$$WJSS-POS_{ijk} = \pi_{0jk} + e_{ijk}$$

Level-2 Model

$$\pi_{0jk} = \beta_{00k} + r_{0jk}$$

Level-3 Model

$$\beta_{00k} = \gamma_{000} + u_{00k}$$

Mixed Model

$$WJSS-POS_{ijk} = \gamma_{000} + r_{0jk} + u_{00k} + e_{ijk}$$
Table J2. Final estimation of level-1 and level-2 variance components

<table>
<thead>
<tr>
<th>Random Effect</th>
<th>Standard Deviation</th>
<th>Variance</th>
<th>d.f.</th>
<th>( \chi^2 )</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>INTRCPT1, ( r_0 )</td>
<td>4.87711</td>
<td>23.78618</td>
<td>65</td>
<td>144.02037</td>
<td>&lt;0.001</td>
</tr>
<tr>
<td>level-1, ( e )</td>
<td>11.86036</td>
<td>140.66809</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table J3. Final estimation of level-3 variance components

<table>
<thead>
<tr>
<th>Random Effect</th>
<th>Standard Deviation</th>
<th>Variance</th>
<th>d.f.</th>
<th>( \chi^2 )</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>INTRCPT1/INTRCPT2, ( u_{00} )</td>
<td>1.29822</td>
<td>1.68538</td>
<td>7</td>
<td>10.69445</td>
<td>0.152</td>
</tr>
</tbody>
</table>

- **Random Intercept Model**

**Level-1 Model**

\[
WJ-POST_{ijk} = \pi_0 + \pi_1(GENDER_{ijk}) + \pi_2(WJ-PRE_{ijk}) + \pi_3(LANGUAGE_{ijk}) + e_{ijk}
\]

**Level-2 Model**

\[
\begin{align*}
\pi_0 & = \beta_{00k} + r_{0jk} \\
\pi_1 & = \beta_{10k} \\
\pi_2 & = \beta_{20k} \\
\pi_3 & = \beta_{30k}
\end{align*}
\]

**Level-3 Model**

\[
\begin{align*}
\beta_{00k} & = \gamma_{000} + u_{00k} \\
\beta_{10k} & = \gamma_{100} \\
\beta_{20k} & = \gamma_{200} \\
\beta_{30k} & = \gamma_{300}
\end{align*}
\]

WJ-PRE has been centered around the group mean.

**Mixed Model**

\[
WJ-POST_{ijk} = \gamma_{000} + \gamma_{100} GENDER_{ijk} + \gamma_{200} WJ-PRE_{ijk} + \gamma_{300} LANGUAGE_{ijk} + r_{0jk} + u_{00k} + e_{ijk}
\]
Table J4. Final estimation of fixed effects

<table>
<thead>
<tr>
<th>Fixed Effect</th>
<th>Coefficient</th>
<th>Standard Error</th>
<th>t-ratio</th>
<th>Approx. d.f.</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>For INTRCPT1, $\pi_0$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>For INTRCPT2, $\beta_{00}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>INTRCPT3, $\gamma_{000}$</td>
<td>100.177468</td>
<td>2.132996</td>
<td>46.966</td>
<td>7</td>
<td>&lt;0.001</td>
</tr>
<tr>
<td>For IC-GENDE slope, $\pi_1$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>For INTRCPT2, $\beta_{10}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>INTRCPT3, $\gamma_{100}$</td>
<td>-1.421381</td>
<td>0.780234</td>
<td>-1.822</td>
<td>462</td>
<td>0.069</td>
</tr>
<tr>
<td>For WJSS-PRE slope, $\pi_2$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>For INTRCPT2, $\beta_{20}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>INTRCPT3, $\gamma_{200}$</td>
<td>0.671017</td>
<td>0.032683</td>
<td>20.531</td>
<td>462</td>
<td>&lt;0.001</td>
</tr>
<tr>
<td>For D-LANGUA slope, $\pi_3$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>For INTRCPT2, $\beta_{30}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>INTRCPT3, $\gamma_{300}$</td>
<td>-1.625558</td>
<td>1.630310</td>
<td>-0.997</td>
<td>462</td>
<td>0.319</td>
</tr>
</tbody>
</table>

Table J5. Final estimation of level-1 and level-2 variance components

<table>
<thead>
<tr>
<th>Random Effect</th>
<th>Standard Deviation</th>
<th>Variance Component</th>
<th>d.f.</th>
<th>$\chi^2$</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>INTRCPT1, $r_0$</td>
<td>5.78091</td>
<td>33.41891</td>
<td>65</td>
<td>276.32261</td>
<td>&lt;0.001</td>
</tr>
<tr>
<td>level-1, $e$</td>
<td>8.57381</td>
<td>73.51029</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table J6. Final estimation of level-3 variance components

<table>
<thead>
<tr>
<th>Random Effect</th>
<th>Standard Deviation</th>
<th>Variance Component</th>
<th>d.f.</th>
<th>$\chi^2$</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>INTRCPT1/INTRCPT2, $u_{00}$</td>
<td>1.70069</td>
<td>2.89235</td>
<td>7</td>
<td>12.71487</td>
<td>0.079</td>
</tr>
</tbody>
</table>

- Random Intercept and Slope Model

Level-1 Model

$$WJ-POST_{ijk} = \pi_{0jk} + \pi_{1jk}*(GENDER_{ijk}) + \pi_{2jk}*(WJ-PRE_{ijk}) + \pi_{3jk}*(LANGUAGE_{ijk}) + e_{ijk}$$

Level-2 Model

$$\pi_{0jk} = \beta_{00k} + \beta_{01k}*(PCK-WHAT_{jk}) + \beta_{02k}*(PCK-WHO_{jk}) + \beta_{03k}*(PCK-HOW_{jk}) + r_{0jk}$$

$$\pi_{1jk} = \beta_{10k}$$

$$\pi_{2jk} = \beta_{20k} + \beta_{21k}*(PCK-WHAT_{jk}) + \beta_{22k}*(PCK-WHO_{jk}) + \beta_{23k}*(PCK-HOW_{jk})$$

$$\pi_{3jk} = \beta_{30k}$$
Level-3 Model

\begin{align*}
\beta_{00k} &= \gamma_{000} + u_{00k} \\
\beta_{01k} &= \gamma_{010} \\
\beta_{02k} &= \gamma_{020} \\
\beta_{03k} &= \gamma_{030} \\
\beta_{10k} &= \gamma_{100} \\
\beta_{20k} &= \gamma_{200} \\
\beta_{21k} &= \gamma_{210} \\
\beta_{22k} &= \gamma_{220} \\
\beta_{23k} &= \gamma_{230} \\
\beta_{30k} &= \gamma_{300} 
\end{align*}

WJSS-PRE has been centered around the group mean.
PCK-WHAT PCK-WHO PCK-HOW have been centered around the grand mean.

Mixed Model

\begin{align*}
WJSS-POST_{ijk} &= \gamma_{000} + \gamma_{010} \cdot \text{PCK-WHAT}_{jk} + \gamma_{020} \cdot \text{PCK-WHO}_{jk} + \gamma_{030} \cdot \text{PCK-HOW}_{jk} \\
&+ \gamma_{100} \cdot \text{GENDER}_{ijk} + \gamma_{200} \cdot WJSS-PRE_{ijk} + \gamma_{210} \cdot WJSS-PRE_{ijk} \cdot \text{PCK-WHAT}_{jk} + \gamma_{220} \cdot WJSS-PRE_{ijk} \cdot \text{PCK-WHO}_{jk} \\
&+ \gamma_{230} \cdot WJSS-PRE_{ijk} \cdot \text{PCK-HOW}_{jk} + \gamma_{300} \cdot LANGUAGE_{ijk} + r_{0jk} + u_{00k} + e_{ijk}
\end{align*}
### Table J7. Final estimation of fixed effects

<table>
<thead>
<tr>
<th>Fixed Effect</th>
<th>Coefficient</th>
<th>Standard error</th>
<th>t-ratio</th>
<th>Approx. d.f.</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>For INTRCPT1, $\pi_0$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>For INTRCPT2, $\beta_{00}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>INTRCPT3, $\gamma_{000}$</td>
<td>99.979575</td>
<td>2.117329</td>
<td>47.220</td>
<td>7</td>
<td>&lt;0.001</td>
</tr>
<tr>
<td>For PCK-WHAT, $\beta_{01}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>INTRCPT3, $\gamma_{010}$</td>
<td>3.580519</td>
<td>1.633370</td>
<td>2.192</td>
<td>62</td>
<td>0.032</td>
</tr>
<tr>
<td>For PCK-WHO, $\beta_{02}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>INTRCPT3, $\gamma_{020}$</td>
<td>-0.748346</td>
<td>1.425113</td>
<td>-0.525</td>
<td>62</td>
<td>0.601</td>
</tr>
<tr>
<td>For PCK-HOW, $\beta_{03}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>INTRCPT3, $\gamma_{030}$</td>
<td>-1.108014</td>
<td>1.271935</td>
<td>-0.871</td>
<td>62</td>
<td>0.387</td>
</tr>
<tr>
<td>For GENDER slope, $\pi_1$</td>
<td></td>
<td></td>
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<td></td>
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</tr>
<tr>
<td>For INTRCPT2, $\beta_{10}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>INTRCPT3, $\gamma_{100}$</td>
<td>-1.459914</td>
<td>0.776423</td>
<td>-1.880</td>
<td>459</td>
<td>0.061</td>
</tr>
<tr>
<td>For WJSS-PRE slope, $\pi_2$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>For INTRCPT2, $\beta_{20}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>INTRCPT3, $\gamma_{200}$</td>
<td>0.665973</td>
<td>0.032748</td>
<td>20.336</td>
<td>459</td>
<td>&lt;0.001</td>
</tr>
<tr>
<td>For PCK-WHAT, $\beta_{21}$</td>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>INTRCPT3, $\gamma_{210}$</td>
<td>0.095096</td>
<td>0.077312</td>
<td>1.230</td>
<td>459</td>
<td>0.219</td>
</tr>
<tr>
<td>For PCK-WHO, $\beta_{22}$</td>
<td></td>
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<td></td>
</tr>
<tr>
<td>INTRCPT3, $\gamma_{220}$</td>
<td>-0.108255</td>
<td>0.061085</td>
<td>-1.772</td>
<td>459</td>
<td>0.077</td>
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<tr>
<td>For PCK-HOW, $\beta_{23}$</td>
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</tr>
<tr>
<td>INTRCPT3, $\gamma_{230}$</td>
<td>-0.039600</td>
<td>0.056348</td>
<td>-0.703</td>
<td>459</td>
<td>0.483</td>
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<tr>
<td>For LANGUAGE slope, $\pi_3$</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>For INTRCPT2, $\beta_{30}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>INTRCPT3, $\gamma_{300}$</td>
<td>-1.363465</td>
<td>1.617312</td>
<td>-0.843</td>
<td>459</td>
<td>0.400</td>
</tr>
</tbody>
</table>

### Table J8. Final estimation of level-1 and level-2 variance components

<table>
<thead>
<tr>
<th>Random Effect</th>
<th>Standard Deviation</th>
<th>Variance Component d.f.</th>
<th>$\chi^2$</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>INTRCPT1, $r_0$</td>
<td>5.49881</td>
<td>30.23690</td>
<td>62</td>
<td>256.42206</td>
</tr>
<tr>
<td>level-1, $e$</td>
<td>8.53402</td>
<td>72.82944</td>
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<td></td>
</tr>
</tbody>
</table>

### Table J9. Final estimation of level-3 variance components

<table>
<thead>
<tr>
<th>Random Effect</th>
<th>Standard Deviation</th>
<th>Variance Component d.f.</th>
<th>$\chi^2$</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>INTRCPT1/INTRCPT2, $u_{00}$</td>
<td>1.76209</td>
<td>3.10496</td>
<td>7</td>
<td>13.68483</td>
</tr>
</tbody>
</table>
HLM Analysis Based on TEAM Test

- **Unconditional Model**

**Level-1 Model**

\[ \text{TEAM-POST}_{ijk} = \pi_{0jk} + e_{ijk} \]

**Level-2 Model**

\[ \pi_{0jk} = \beta_{00k} + r_{0jk} \]

**Level-3 Model**

\[ \beta_{00k} = \gamma_{000} + u_{00k} \]

**Mixed Model**

\[ \text{TEAM-POST}_{ijk} = \gamma_{000} + r_{0jk} + u_{00k} + e_{ijk} \]

Table J10. Final estimation of level-1 and level-2 variance components

<table>
<thead>
<tr>
<th>Random Effect</th>
<th>Standard Deviation</th>
<th>Variance Component</th>
<th>d.f.</th>
<th>( \chi^2 )</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>INTRCPT1, r__1</td>
<td>15.40458</td>
<td>237.30120</td>
<td>65</td>
<td>1532.25088</td>
<td>&lt;0.001</td>
</tr>
<tr>
<td>level-1, e</td>
<td>8.82152</td>
<td>77.81919</td>
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<td></td>
</tr>
</tbody>
</table>

Table J11. Final estimation of level-3 variance components

<table>
<thead>
<tr>
<th>Random Effect</th>
<th>Standard Deviation</th>
<th>Variance Component</th>
<th>d.f.</th>
<th>( \chi^2 )</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>INTRCPT1/INTRCPT2</td>
<td>3.85750</td>
<td>14.88031</td>
<td>7</td>
<td>12.07505</td>
<td>0.097</td>
</tr>
</tbody>
</table>

- **Random Intercept Model**

**Level-1 Model**

\[ \text{TEAM-POST}_{ijk} = \pi_{0jk} + \pi_{1jk}(\text{GENDER}_{ijk}) + \pi_{2jk}(\text{TEAM-PRE}_{ijk}) + \pi_{3jk}(\text{AGE}_{ijk}) + \pi_{4jk}(\text{LANGUAGE}_{ijk}) + e_{ijk} \]
Level-2 Model

\[ \pi_{0jk} = \beta_{00k} + r_{0jk} \]
\[ \pi_{1jk} = \beta_{10k} \]
\[ \pi_{2jk} = \beta_{20k} \]
\[ \pi_{3jk} = \beta_{30k} \]
\[ \pi_{4jk} = \beta_{40k} \]

Level-3 Model

\[ \beta_{00k} = \gamma_{000} + u_{00k} \]
\[ \beta_{10k} = \gamma_{100} \]
\[ \beta_{20k} = \gamma_{200} \]
\[ \beta_{30k} = \gamma_{300} \]
\[ \beta_{40k} = \gamma_{400} \]

TEAM_T_P has been centered around the group mean.

Mixed Model

\[ TEAM-POST_{ijk} = \gamma_{000} + \gamma_{100} \times GENDER_{ijk} + \gamma_{200} \times TEAM-PRE_{ijk} + \gamma_{300} \times AGE_{ijk} + \gamma_{400} \times LANGUAGE_{ijk} + r_{0jk} + u_{00k} + e_{ijk} \]

Table J12. Final estimation of fixed effects

<table>
<thead>
<tr>
<th>Fixed Effect</th>
<th>Coefficient</th>
<th>Standard error</th>
<th>t-ratio</th>
<th>Approx. d.f.</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>For INTRCPT1, ( \pi_0 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>For INTRCPT2, ( \beta_{00} )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>INTRCPT3, ( \gamma_{000} )</td>
<td>-26.693381</td>
<td>3.288861</td>
<td>-8.116</td>
<td>7</td>
<td>&lt;0.001</td>
</tr>
<tr>
<td>For GENDER slope, ( \pi_1 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>For INTRCPT2, ( \beta_{10} )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>INTRCPT3, ( \gamma_{100} )</td>
<td>0.125630</td>
<td>0.740771</td>
<td>0.170</td>
<td>443</td>
<td>0.865</td>
</tr>
<tr>
<td>For TEAM-PRE slope, ( \pi_2 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>For INTRCPT2, ( \beta_{20} )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>INTRCPT3, ( \gamma_{200} )</td>
<td>0.452602</td>
<td>0.039514</td>
<td>11.454</td>
<td>443</td>
<td>&lt;0.001</td>
</tr>
<tr>
<td>For AGE slope, ( \pi_3 )</td>
<td></td>
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<td></td>
</tr>
<tr>
<td>For INTRCPT2, ( \beta_{30} )</td>
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<td></td>
</tr>
<tr>
<td>INTRCPT3, ( \gamma_{300} )</td>
<td>0.774443</td>
<td>0.036944</td>
<td>20.963</td>
<td>443</td>
<td>&lt;0.001</td>
</tr>
<tr>
<td>For LANGUAGE slope, ( \pi_4 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>For INTRCPT2, ( \beta_{40} )</td>
<td></td>
<td></td>
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</tr>
<tr>
<td>INTRCPT3, ( \gamma_{400} )</td>
<td>0.702338</td>
<td>1.515986</td>
<td>0.463</td>
<td>443</td>
<td>0.643</td>
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</tbody>
</table>
Table J13. Final estimation of level-1 and level-2 variance components

<table>
<thead>
<tr>
<th>Random Effect</th>
<th>Standard Deviation</th>
<th>Variance Component</th>
<th>d.f.</th>
<th>$\chi^2$</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>INTRCPT1, $r_0$</td>
<td>4.51789</td>
<td>20.41132</td>
<td>65</td>
<td>209.70900</td>
<td>&lt;0.001</td>
</tr>
<tr>
<td>level-1, $e$</td>
<td>8.02449</td>
<td>64.39240</td>
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</tr>
</tbody>
</table>

Table J14. Final estimation of level-3 variance components

<table>
<thead>
<tr>
<th>Random Effect</th>
<th>Standard Deviation</th>
<th>Variance Component</th>
<th>d.f.</th>
<th>$\chi^2$</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>INTRCPT1/INTRCPT2, $u_{00}$</td>
<td>1.76094</td>
<td>3.10091</td>
<td>7</td>
<td>14.94295</td>
<td>0.036</td>
</tr>
</tbody>
</table>

- **Random Intercept and Slope Model**

**Level-1 Model**

TEAM-POST$_{ijk} = \pi_{0jk} + \pi_{1jk} (GENDER_{ijk}) + \pi_{2jk} (TEAM-PRE_{ijk}) + \pi_{3jk} (AGE_{ijk}) + \pi_{4jk} (LANGUAGE_{ijk}) + e_{ijk}$

**Level-2 Model**

\[
\begin{align*}
\pi_{0jk} &= \beta_{00k} + \beta_{01k} (PCK-WHAT_{jk}) + \beta_{02k} (PCK-WHO_{jk}) + \beta_{03k} (PCK-HOW_{jk}) + r_{0jk} \\
\pi_{1jk} &= \beta_{10k} \\
\pi_{2jk} &= \beta_{20k} + \beta_{21k} (PCK-WHAT_{jk}) + \beta_{22k} (PCK-WHO_{jk}) + \beta_{23k} (PCK-HOW_{jk}) \\
\pi_{3jk} &= \beta_{30k} \\
\pi_{4jk} &= \beta_{40k}
\end{align*}
\]

**Level-3 Model**

\[
\begin{align*}
\beta_{00k} &= \gamma_{000} + u_{00k} \\
\beta_{01k} &= \gamma_{010} \\
\beta_{02k} &= \gamma_{020} \\
\beta_{03k} &= \gamma_{030} \\
\beta_{10k} &= \gamma_{100} \\
\beta_{20k} &= \gamma_{200} \\
\beta_{21k} &= \gamma_{210} \\
\beta_{22k} &= \gamma_{220} \\
\beta_{23k} &= \gamma_{230} \\
\beta_{30k} &= \gamma_{300} \\
\beta_{40k} &= \gamma_{400}
\end{align*}
\]

TEAM-T-P has been centered around the group mean.

PCK-WHAT PCK-WHO PCK-HOW have been centered around the grand mean.
Mixed Model

\[ TEAM-POST_{ijk} = \gamma_{000} + \gamma_{010} \cdot PCK-WHAT_{jk} + \gamma_{020} \cdot PCK-WHO_{jk} + \gamma_{030} \cdot PCK-HOW_{jk} + \gamma_{100} \cdot GENDER_{jk} + \gamma_{200} \cdot TEAM-PRE_{ijk} + \gamma_{210} \cdot TEAM-PRE_{ijk} \cdot PCK-WHAT_{jk} + \gamma_{220} \cdot TEAM-PRE_{ijk} \cdot PCK-WHO_{jk} + \gamma_{230} \cdot TEAM-PRE_{ijk} \cdot PCK-HOW_{jk} + \gamma_{300} \cdot AG_{ijk} + \gamma_{400} \cdot LANGUAGE_{ijk} + r_{0jk} + u_{00k} + e_{ijk} \]

Table J15. Final estimation of fixed effects

<table>
<thead>
<tr>
<th>Fixed Effect</th>
<th>Coefficient</th>
<th>Standard error</th>
<th>t-ratio</th>
<th>Approx. d.f.</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>For INTRCPT1, ( \pi_0 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>For INTRCPT2, ( \beta_{00} )</td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>INTRCPT3, ( \gamma_{000} )</td>
<td>-26.057782</td>
<td>3.267015</td>
<td>-7.976</td>
<td>7</td>
<td>&lt;0.001</td>
</tr>
<tr>
<td>For PCK-WHAT, ( \beta_{01} )</td>
<td></td>
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</tr>
<tr>
<td>INTRCPT3, ( \gamma_{010} )</td>
<td>2.944084</td>
<td>1.379133</td>
<td>2.135</td>
<td>62</td>
<td>0.037</td>
</tr>
<tr>
<td>For PCK-WHO, ( \beta_{02} )</td>
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</tr>
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<td>INTRCPT3, ( \gamma_{020} )</td>
<td>0.320626</td>
<td>1.211536</td>
<td>0.265</td>
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<td>0.792</td>
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<tr>
<td>For PCK-HOW, ( \beta_{03} )</td>
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</tr>
<tr>
<td>INTRCPT3, ( \gamma_{030} )</td>
<td>-1.445375</td>
<td>1.085483</td>
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<td>0.188</td>
</tr>
<tr>
<td>For GENDER slope, ( \pi_1 )</td>
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<td></td>
</tr>
<tr>
<td>For INTRCPT2, ( \beta_{10} )</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>INTRCPT3, ( \gamma_{100} )</td>
<td>0.223976</td>
<td>0.735702</td>
<td>0.304</td>
<td>440</td>
<td>0.761</td>
</tr>
<tr>
<td>For TEAM-PRE slope, ( \pi_2 )</td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>For INTRCPT2, ( \beta_{20} )</td>
<td></td>
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</tr>
<tr>
<td>INTRCPT3, ( \gamma_{200} )</td>
<td>0.469746</td>
<td>0.040811</td>
<td>11.510</td>
<td>440</td>
<td>&lt;0.001</td>
</tr>
<tr>
<td>For PCK-WHAT, ( \beta_{21} )</td>
<td></td>
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<td></td>
<td></td>
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</tr>
<tr>
<td>INTRCPT3, ( \gamma_{210} )</td>
<td>-0.027961</td>
<td>0.080413</td>
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<td>0.728</td>
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<tr>
<td>For PCK-WHO, ( \beta_{22} )</td>
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</tr>
<tr>
<td>INTRCPT3, ( \gamma_{220} )</td>
<td>0.061997</td>
<td>0.081530</td>
<td>0.760</td>
<td>440</td>
<td>0.447</td>
</tr>
<tr>
<td>For PCK-HOW, ( \beta_{23} )</td>
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<td></td>
</tr>
<tr>
<td>INTRCPT3, ( \gamma_{230} )</td>
<td>0.100586</td>
<td>0.059960</td>
<td>1.678</td>
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<td>0.094</td>
</tr>
<tr>
<td>For AGE slope, ( \pi_3 )</td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>For INTRCPT2, ( \beta_{30} )</td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>INTRCPT3, ( \gamma_{300} )</td>
<td>0.764334</td>
<td>0.036828</td>
<td>20.754</td>
<td>440</td>
<td>&lt;0.001</td>
</tr>
<tr>
<td>For LANGUAGE slope, ( \pi_4 )</td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>For INTRCPT2, ( \beta_{40} )</td>
<td></td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>INTRCPT3, ( \gamma_{400} )</td>
<td>0.661875</td>
<td>1.506500</td>
<td>0.439</td>
<td>440</td>
<td>0.661</td>
</tr>
</tbody>
</table>

Table J16. Final estimation of level-1 and level-2 variance components

<table>
<thead>
<tr>
<th>Random Effect</th>
<th>Standard Deviation</th>
<th>Variance Componenent</th>
<th>d.f.</th>
<th>( \chi^2 )</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>INTRCPT1, ( \eta_0 )</td>
<td>4.37298</td>
<td>19.12928</td>
<td>62</td>
<td>203.14506</td>
<td>&lt;0.001</td>
</tr>
<tr>
<td>level-1, ( e )</td>
<td>7.96027</td>
<td>63.36586</td>
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</tr>
</tbody>
</table>
Table J17. Final estimation of level-3 variance components

<table>
<thead>
<tr>
<th>Random Effect</th>
<th>Standard Deviation</th>
<th>Variance Component</th>
<th>d.f.</th>
<th>$\chi^2$</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>INTRCPT1/INTRCPT2, $u_{i0}$</td>
<td>1.81417</td>
<td>3.29120</td>
<td>7</td>
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VITA

Yinna Zhang was born and raised in Lijiang, Yunnan, China, as a proud member of the minority ethnicity group, Naxi. Before attending Loyola University Chicago & Erikson Institute, she studied at Beijing Normal University and received a Master of Arts in Developmental Psychology. She was admitted by Tsinghua University, Beijing in 2002 and earned a Bachelor of Science in Biological Science in 2006.

Zhang worked as research assistant at Early Math Collaborative during her graduate study and presented her collaborative work in national and international conferences such as AERA, IERC, NAEYC, NCTM, and SRCD. During the summer of 2011, she got a scholarship from Interuniversity Consortium of Political and Social Science Research (ICPSR) at University of Michigan, which supported her to further strengthen understanding and skills for quantitative analysis. She worked as an intern at Educational Testing Service (ETS) during the summer of 2012 on a project designing assessment to measure content knowledge for teaching mathematics in middle school; and later served as a consultant. She volunteered in a nursery class from 2009 to 2014 and taught mathematics for kindergarten and first grade in summer, 2013 at Lab School of University of Chicago. From 2013 to 2014, she served as co-chair for the doctoral student association at Erikson. She also worked as a consultant for evaluating an early intervention project in summer 2014. Currently, Zhang is working as a research analyst at Erikson Institute. She lives in Chicago, Illinois.