Math Worksheet for Robot Motion Planning

1 Introduction

This worksheet focuses on analyzing the time required for a robot to complete an example motion under different schemes of operation that vary in programming difficulty. Since the time varies and other considerations may also apply, the final choice may involve a tradeoff among various considerations. But we need to establish a formal model and complete some mathematical analyses to have a basis for comparing robot navigation times.

We will work with a robot drawn as in Figure 0 that must navigate through a two-dimensional field. With two wheels that may be driven independently (forwards or backwards up to some maximum speed) and a third balance point such as a caster wheel or track ball, such a robot is generally referred to as a differential-drive robot. (Unlike wheels in a modern automobile, these wheels do not turn from side to side.)

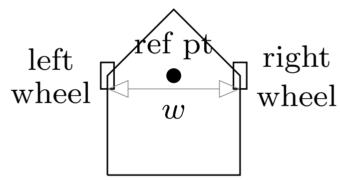

Figure 0: We will sketch our robot as shown, with a pointed front and a reference point centered between the two independently-driven wheels; there is also an assumed third balance point, for example a caster wheel in the back.

We will think about moving the reference point on the robot from a starting position at coordinates $(0, 0)$ to a target position (x, y) but keeping in mind that the movement is constrained by the use of our two wheels at separation w. We also assume that the robot should point in the same direction at the end of the motion as at the beginning. The distance w between the wheels is most often referred to as *track width*, and it constrains how tightly the robot can turn. Until the very end of the worksheet, we assume the following parameter values. The track width w is 2cm. Each wheel can turn at a maximum speed of 1cm per second, and we assume that acceleration and deceleration is instantaneous. We will use $x = 3.586$ and $y = 4.414$. Compute your answers to questions to 3 decimal places.

Students working with educational robots typically find it convenient to program the robot to make 90° turns ($\pi/2$ radians) and to go straight for a specified distance. There are two styles in which turns are typically programmed, either rotations (about the reference point as one wheel moves forward and the other backward), or swings (with one wheel moving so that the robot pivots around the fixed wheel). A more advanced programming approach might allow for turns to various angles. We will begin by considering turns of just $\pi/2$ radians and then consider more general turns.

2 Horizontal and Vertical Navigation

With all navigation along horizontal and vertical lines with turns of $\pi/2$ radians, the path of the robot will look as in Figure 1(a) or 1(b), depending whether we use rotations or swings (assuming we choose a "middling" point to make our turns. (If there are obstacles that must be avoided, the point at which the first turn is taken can be adjusted without affecting the navigation time.)

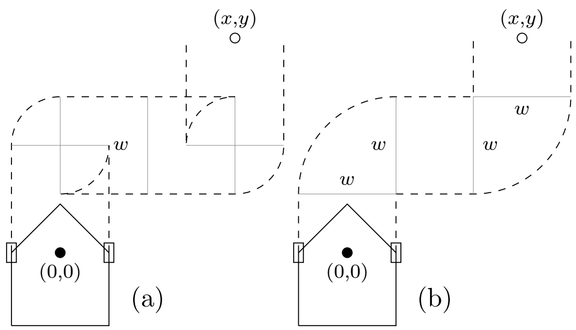

Figure 1: The paths of the robot wheels with horizontal and vertical navigation, using rotations in (a) and swings in (b).

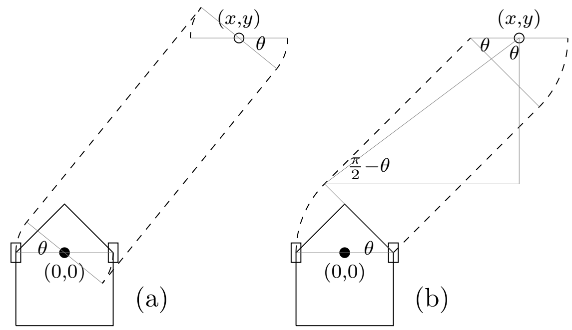

Figure 2: The paths of the robot wheels under general navigation, using rotations in (a) and swings in (b).

Exercise 1a Compute the time under the rotation approach of Figure 1(a). Note that since both wheels are always in motion, we can compute the time as being proportional to the distance traveled by either wheel. Remember the relationship that the length of a circular arc is the product of the radius and the subtended angle.

Exercise 1b Compute the time under the swing approach of Figure 1(b). (We can mostly consider the distance traveled by either wheel, but when that wheel is stationary, we must account for the distance traveled by the other wheel.)

Exercise 1c When working with horizontal and vertical navigation, is the robot motion faster with rotations or swings?

3 Generalized Navigation

While the navigational approach of the prior section is simple, we would expect to be able to navigate more quickly by proceeding on a path closer to a straight line. Figures 2(a) and 2(b) show the paths under the rotation and swing approaches assuming we use the rotations or swings just to line us up for straight-line navigation.

Exercise 2a Compute the time under the rotation approach of Figure 2(a). (Since both wheels are always in motion, we can compute the time as being proportional to the distance traveled by either wheel.) You will need to compute the angle θ.

Exercise 2b Compute the time under the swing approach of Figure 2(b). (We can mostly consider the distance traveled by either wheel, but when that wheel is stationary, we must account for the distance traveled by the other wheel.) In this case, the angle θ is $\pi/4$ radians.

Exercise 2c When working with horizontal and vertical navigation, is the robot motion faster with rotations or swings?

4 Variations on Generalized Navigation

A problem that may often occur in practice with the routing of Figure 2(a) is that the robot might need to butt up against a boundary wall at the start and/or end of the path; then there will be insufficient room to perform the rotations. This motivates the following exercises.

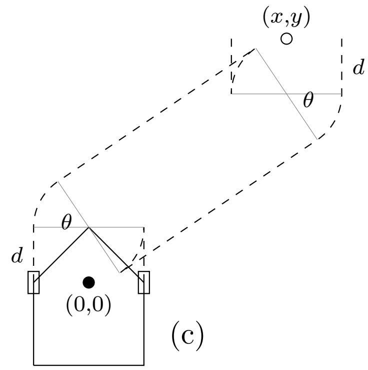

Figure 3: Here we use rotations but with straight segments of distance d at the beginning and end of the path so the rotations can occur without banging into horizontal boundaries abutting the robot's beginning and ending positions. The distance d in this picture is exaggerated for visual effect.

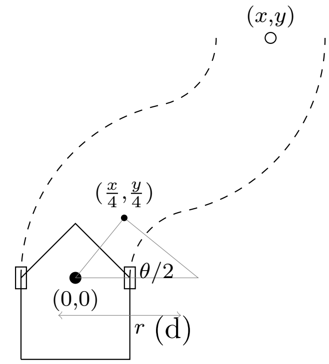

Figure 4: A path in which the reference point traverses two identical circular arcs.

Consider a variation on the path of Figure 2(a) in which the robot runs straight forward for a distance d before rotating as illustrated in Figure 3, and, similarly, rotates back at distance d before the target.

Exercise 3a If the distance from the robot's reference point to the back of the robot (bottom in the picture) is 1cm and a boundary runs immediately behind the robot, how large must d be to allow room for the rotation?

Exercise 3b Using the value of d computed in Exercise 3a, compute the time to traverse the path in Figure 3. Similarly to Exercise 2a, you will need to compute the new angle θ.

A further consideration regarding the various paths we have analyzed is that we have been making a simplifying assumption that we can change the velocities of the wheels instantaneously as long as we impose an upper limit on velocity. In reality, the robot will neither stop nor start instantaneously, and it will probably be necessary in practice to insert brief delays within the path whenever a turn begins or ends to allow a settling of robot movement before changing the motor speeds.

Exercise 4a For each of the paths considered so far (Figures 1(a), 1(b), 2(a), 2(b), and 3), how many internal delays would we need based on the discussion in the paragraph above?

We could reduce the number of internal delays to just 1 by routing the reference point along a circular arc to $(x/2, y/2)$ and then mirroring that motion to reach (x, y) as in Figure 4.

Exercise 4b Compute the time (ignoring internal delays as before) to traverse the path in Figure 4. Use the right triangle drawn for reference to compute the angle θ in this case and the radius r of each of the two arcs traversed by the reference point.

5 General x and y

Exercise 5 We have considered only one set of values for w, x, and y. For extra credit, return to the previous problems that involve computing the time for a robot motion, and find the result in terms of w, x, and y instead of using specific numbers. (Warning: In the case of Figure 2(b), it is difficult to determine the angle θ, and you should just express the result in terms of θ and determine a relationship that θ should obey.)