Ideals and Centralizing Mappings in Prime Rings

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ABSTRACT. Let $R$ be a prime ring and $U$ be a nonzero ideal of $R$. If $T$ is a nontrivial automorphism or derivation of $R$ such that $uuT - uTu$ is in the center of $R$ and $uT$ is in $U$ for every $u$ in $U$, then $R$ is commutative. If $R$ does not have characteristic equal to two, then $U$ need only be a nonzero Jordan ideal.

If $R$ is a ring, a mapping $T$ of $R$ to itself is called centralizing on a subset $S$ of $R$ if $ssT - s^T s$ is in the center of $R$ for every $s$ in $S$. There has been considerable interest in centralizing automorphisms and derivations defined on rings. Miers [4] has studied these mappings defined on $C^*$-algebras. In [5] Posner proved that if a prime ring has a nontrivial centralizing derivation, then the ring must be commutative. The same result was obtained for centralizing automorphisms in [3]. In this paper it is shown that the automorphism or derivation need only be centralizing and invariant on a nonzero ideal in the prime ring in order to ensure that the ring is commutative. Also, if $R$ is of characteristic not two, then the mapping need only be centralizing and invariant on a nonzero Jordan ideal. For derivations this gives a short proof of a result related to that of Awtar [1, Theorem 3]. Awtar proved that if $R$ is a prime ring of characteristic not two with a nontrivial derivation and a nonzero Jordan ideal $U$ such that the derivation is centralizing on $U$, then $U$ is contained in the center of $R$.

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From now on assume that $R$ is a prime ring and let $Z$ be the center of $R$. Let $[x, y] = xy - yx$ and note the important identity $[x, yz] = y[x, z] + [x, y]z$. The following lemmas will be used in the proofs of the main results.

**Lemma 1.** If $b[a, r] = 0$ for all $r$ in $R$, then $b = 0$ or $a$ is in $Z$.

**Proof.** Assume that $b[a, r] = 0$ for all $r$ in $R$. Replace $r$ by $xy$ to obtain $b[a, xy] = b[x, a, y] + b[a, x]y = b[x, a, y] = 0$ for all $x$ and $y$ in $R$. Since $R$ is prime, $b = 0$ or $[a, y] = 0$ for all $x$ in $R$.

**Lemma 2.** If $D$ is a derivation of $R$ such that $uD = 0$ for all $u$ in a nonzero right ideal $U$ of $R$, then $rD = 0$ for all $r$ in $R$.

**Proof.** Let $u$ be a nonzero element in $U$ and $x$ be an element in $R$. Then $ux$ is in $U$ and $0 = (ux)^D = uDx + u(xD) = u(x^D)$. Now replace $x$ by $sr$ to obtain $0 = u(sD)r + us(rD) = us(rD)$ for all $r$ and $s$ in $R$. Since $R$ is prime and $u$ is nonzero, $rD = 0$ for all $r$ in $R$.

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LEMMA 3. If $T$ is a homomorphism of $R$ such that $u^T = u$ for all $u$ in a nonzero right ideal $U$ of $R$, then $r^T = r$ for every $r$ in $R$.

PROOF. Let $u$ be a nonzero element in $U$ and $r$, $s$ be in $R$. Since $U$ is a right ideal, $us$ and $usr$ are in $U$. Then $(usr)^T = usr = (us)^T r^T = usr^T$. Hence $us(r - r^T) = 0$ for all $s$ and $r$ in $R$. Thus $r = r^T$ for all $r$ in $R$.

LEMMA 4. If $R$ contains a nonzero commutative right ideal $U$, then $R$ must be commutative.

PROOF. Let $u$ be in $U$ and assume that $u^2$ is not zero. Such an element exists for if not, then by a variation of Levitzki's theorem [2, Lemma 1.1], $R$ has a nonzero nilpotent ideal and this is impossible in a prime ring. $U$ is a right ideal and so $ur$ and $us$ are in $U$ for every $r$ and $s$ in $R$. Since $U$ is commutative, $u^2 sr = u(us)r = us(ur) = ur(us) = u(ur)s = u^2 rs$. Hence $u^2 [r, s] = 0$ for all $r$ and $s$ in $R$. By Lemma 1, every $r$ in $R$ is in $Z$. Therefore $R$ is commutative.

THEOREM. Let $R$ be a prime ring and $U$ be a nonzero ideal of $R$. If $R$ has a nontrivial automorphism or derivation $T$ such that $uu^T = u^Tu$ is in the center of $R$ and $u^T$ is in $U$ for every $u$ in $U$, then $R$ is commutative.

PROOF. By Lemma 2 or Lemma 3, $T$ is nontrivial on $U$. Since $U$ is a nonzero ideal in a prime ring, $U$ is itself a prime ring. $U$ is then commutative by the author's result in [3] for automorphisms or by Posner's result [5] for derivations. By Lemma 4, $R$ is commutative.

COROLLARY. If $U$ is a nonzero Jordan ideal in a prime ring $R$ of characteristic not two and $T$ is a nontrivial automorphism or derivation of $R$ which is centralizing and invariant on $U$, then $R$ is commutative.

PROOF. Every nonzero Jordan ideal in a prime ring of characteristic not two contains a nonzero ideal [2, Theorem 1.1]. Apply the theorem to this ideal.

REFERENCES


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