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Exact Solutions of the Schröedinger Equation: Connection between Supersymmetric Quantum Mechanics and Spectrum Generating Algebras

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Using supersymmetric quantum mechanics, one can obtain analytic expressions for the eigenvalues and eigenfunctions for all nonrelativistic shape invariant Hamiltonians. These Hamiltonians also possess spectrum generating algebras and are hence solvable by an independent, group theoretical method. In this paper, we demonstrate the equivalence of the two methods of solution, and review related progress in this field.

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I. Introduction

Supersymmetric quantum mechanics (SUSYQM) [1] provides an elegant and useful prescription for obtaining closed analytic expressions for the energy eigenvalues and eigenfunctions of many one dimensional problems. It makes use of first order differential operators A and $A^\dagger$,

$$A(x,a_0) = \frac{d}{dx} + W(x,a_0), \quad A^\dagger(x,a_0) = -\frac{d}{dx} + W(x,a), \tag{1}$$

which are generalizations of the raising and lowering operators first used by Dirac for treating the harmonic oscillator. The superpotential $W(x,a_0)$ is a real function of x and a_0 is a parameter (or a set of parameters), which plays a crucial role in the SUSYQM approach. From SUSYQM, one finds that the supersymmetric partner Hamiltonians $H_- \equiv A^\dagger A$ and $H_+ \equiv AA^\dagger$ have the *same* energy eigenvalues (except for the ground state). The potentials V_- and V_+, corresponding to the Hamiltonians H_- and H_+, are related to the superpotential by

$$V_\pm = W^2(x,a_0) \pm \frac{dW(x,a_0)}{dx}. \tag{2}$$

Superpotentials $W(x,a)$ which satisfy the condition

$$V_+(x,a_0) = W^2(x,a_0) + \frac{dW(x,a_0)}{dx} = W^2(x,a_1) - \frac{dW(x,a_1)}{dx} + R(a_0) = V_-(x,a_1) + R(a_0),$$
$$a_1 = f(a_0). \tag{3}$$

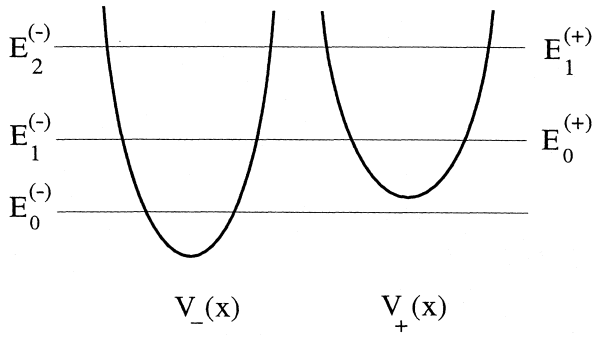

FIG. 1. A typical set of supersymmetric partner potentials with common eigenenergies.

are called "shape invariant" [2]. Here, a_1 and a_0 are parameters. Shape invariant partner potentials $V_+(x;a_0)$ and $V_i\ (x;a_1)$ have the same x-dependence. As illustrated in Figure 1, $R(a_0)$ is the energy difference of the ground states of $V_i\ (x;a_1)$ and $V_+(x;a_0)$. The functions $f(a_0)$ might include a large class of change of parameters: translations, scalings, projective transformations, as well as more complicated ones. We shall restrict our discussion to the first three.

Note that "shape invariance" is a very specialized notion. An example of two such shape invariant partners are the infinite well and the cosec^2 potential, something one would hardly guess from the name "shape invariance".

A remarkable feature of shape invariant potentials is that their entire spectrum can be determined exactly by algebraic means, without ever referring to underlying differential equations [1], analogous to the way that the one-dimensional harmonic oscillator is solved by Dirac's method of raising and lowering operators.

It has also been discovered that some of these exactly solvable systems possess a so-called "spectrum generating algebra" (SGA) [3, 5]. The Hamiltonian of these systems can be written as a linear or quadratic function of an underlying algebra, and all the quantum states of these systems can be determined by group theoretical methods.

One may therefore ask the question: Is there any connection between a general shape invariance condition within the formalism of SUSYQM, and a spectrum generating algebra? If so, then all shape invariant potentials should have such an algebra. Furthermore, we should be able to establish the connection between the SUSYQM method of solution and the group theoretical potential algebra method. Last but not least, we may be able to identify whether there are hitherto unknown potentials belonging to this family, or, on the other hand, whether the set of known potentials appears to be complete.

In this paper we discuss the work of others and ourselves, all of which lead to the conclusion that, indeed, the two methods are equivalent, and in fact, the known set of exactly solvable potentials appears to be complete.

II. Supersymmetric quantum mechanics and shape invariance

In this section, we very briefly describe supersymmetric quantum mechanics (SUSYQM), and also show how SUSYQM applied to shape invariant potentials allows one to completely determine the spectrum of a quantum system. For a more detailed description, see Ref. [1].

A quantum mechanical system described by a potential $V_-(x)$ can alternately be described by its ground state wavefunction $\psi_0^{(-)}$: from the Schrödinger equation for the ground state wavefunction, $-\psi_0'' + V(x)\psi_0 = 0$, it follows that the potential can be written as, $V_-(x) = \left(\frac{\psi_0''}{\psi_0}\right)$, where prime denotes differentiation with respect to x. Note that the potential has been adjusted to make the ground state energy $E_0 = 0$. In SUSYQM, it is customary to express the system in terms of the superpotential $W(x) = -\left(\frac{\psi_0'}{\psi_0}\right)$. V_- and W are then related by Eq. (2). The ground state wavefunction is then given by $\psi_0 \sim \exp(-\int_{x_0}^{x} W(x)dx)$, where x_0 is an arbitrarily chosen reference point. At this point it is important to point out that whenever a potential is defined in the above form in terms of the a normalizable ground state wavefunction, the zero-value for the ground state energy is assured.

Using units with $\hbar$ and $2m = 1$, the Hamiltonian H_- can now be written as

$$
\begin{aligned}
H_- &= \left(-\frac{d^2}{dx^2} + V_-(x)\right) = \left(-\frac{d^2}{dx^2} + W^2(x) - \frac{dW(x)}{dx}\right) \\
&= \left(-\frac{d}{dx} + W(x)\right)\left(\frac{d}{dx} + W(x)\right) .
\end{aligned}
\tag{4}
$$

As discussed in the Introduction, in analogy with the harmonic oscillator raising and lowering operators, we introduce operators $A = \left[\frac{d}{dx} + W(x)\right]$, and and its Hermitian conjugate $A^+ = \left[-\frac{d}{dx} + W(x)\right]$. Thus $H_- = A^\dagger A$. With these operators A and $A^\dagger$, one can construct another Hermitian operator $H_+ = AA^\dagger$. The eigenstates of H_+ are iso-spectral with excited states of H_-. The Hamiltonian H_+, with potential $V_+(x) = \left(W^2(x) + \frac{dW(x)}{dx}\right)$, is called the superpartner of the Hamiltonian H_-. To show the iso-spectrality mentioned above, let us denote the eigenfunctions of $H_\pm$ that correspond to eigenvalues $E_n^\pm$, by $\psi_n^{(\pm)}$. For $n = 1, 2, \cdots$,

$$
\begin{aligned}
H_+\left(A\psi_n^{(-)}\right) &= AA^+\left(A\psi_n^{(-)}\right) = A\left(A^+A\psi_n^{(-)}\right) \\
&= AH_-\left(\psi_n^{(-)}\right) = E_n^-\left(A\psi_n^{(-)}\right) .
\end{aligned}
\tag{5}
$$

Hence, except for the ground state which obeys $A\psi_0^{(-)} = 0$, all excited states $\psi_n^{(-)}$ of H_- have one to one correspondence with $\psi_{n-1}^{(+)} \propto A\psi_n^{(-)}$ of H_+ with exactly the same energy, i.e. $E_{n-1}^+ = E_n^-$, where $n = 1, 2, \cdots$. Conversely, one also has $A^+\psi_{n-1}^{(+)} \propto \psi_n^{(-)}$. Thus, if the eigenvalues and the eigenfunctions of H_- were known, one would automatically obtain the eigenvalues and the eigenfunctions of H_+, which is in general a completely different Hamiltonian. See Figure 2.

At this point, we could obtain the E^+'s and $\psi^{(+)}$'s from the E^-'s and $\psi^{(-)}$'s, or vice versa, but we can go no further. That is, unless we know either set *a priori*, this analysis is simply a mathematical curiosity.

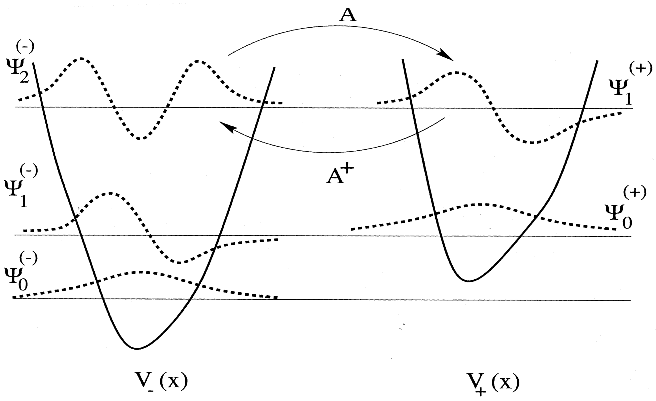

FIG. 2. Isospectrality of H_+ and H_-. Note that V_+ and V_- have different shapes, as do various ψ^+ and ψ^-.

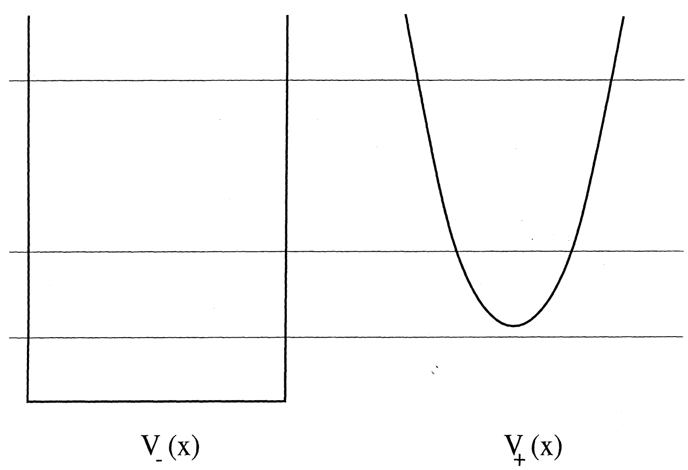

FIG. 3. Infinite Square Well and cosec$^2 x$: two shape invariant partners.

Now, let us consider the special case where $V_-(x; a_0)$ is a shape invariant potential. For such systems, potentials $V_+(x; a_0) = V_-(x; a_1) + R(a_0)$. Hence, V_- and and V_+ have the same x-dependence (although, as we shall see, this is not always obvious). Their superpotential W obeys the shape invariance condition of Eq. (3). Since potentials $V_+(x; a_0)$ and $V_-(x; a_1)$ differ by the additive constant $R(a_0)$, their respective Hamiltonians differ by that same constant. Thus, the eigenfunctions of the Schroedinger equation are the same for both potentials. In particular, they have a common ground state wavefunction, given by $\psi_0^{(+)}(x; a_0) = \psi_0^{(-)}(x; a_1) \propto \exp\left(-\int_{x_0}^{x} W(x; a_1)dx\right)$, and the ground state energy of $H_+(x; a_0)$ is $R(a_0)$, because the ground state energy of $H_-(x; a_1)$ is zero. NB: the parameter shift $a_0 \to a_1$ has an effect similar to that of a ladder operator: $\psi_1^{(-)}(x; a_0) \propto A^+(x; a_0)\, \psi_0^{(-)}(x; a_1)$. Note that ladder operators A, $A^\dagger$, like H, are also dependent on parameters a_n.

Now using SUSYQM algebra, the first excited state of $H_-(x;a_0)$ is given by $A^+(x;a_0)\psi_0^{(-)}(x;a_1)$ and the corresponding eigenvalue is $R(a_0)$. By iterating this procedure, the $(n+1)$-th excited state is given by

$$\psi_{n+1}^{(-)}(x;a_0) \propto A^+(a_0)\,A^+(a_1)\cdots A^+(a_n)\,\psi_0^{(-)}(x;a_n); \tag{6}$$

and corresponding eigenvalues are given by

$$E_0 = 0; \text{ and } E_n^{(-)} = \sum_{k=0}^{n-1} R(a_k) \text{ for } n > 0. \tag{7}$$

(To avoid notational complexity, we have suppressed the x-dependence of operators $A(x;a_0)$ and $A^+(x;a_0)$.) Thus, for a shape invariant potential, one can obtain the entire spectrum of H_- itself by the algebraic methods of SUSYQM (and of course the same is true for H_+). Now we are moving up (or down) along the ladder of a single Hamiltonian H_-, albeit the price we pay is that the $\psi_n^{(-)}$'s have different parameters a_n.

As an example let us demonstrate this method for the unlikely pair of shape invariant potentials: the infinite well and $\mathrm{cosec}^2 x$. We begin by showing that they are indeed superpotential partners. Consider a superpotential $W(x) = -b\cot x$ with $b > 0$. We restrict the domain of this potential to $(0,\pi)$. The supersymmetric partner potentials generated by this superpotential are:

$$\begin{aligned} V_-(x;b) &= W^2(x) - \frac{dW}{dx} = b(b-1)\mathrm{cosec}^2 x - b^2 \\ &\text{and} \\ V_+(x;b) &= W^2(x) + \frac{dW}{dx} = b(b+1)\mathrm{cosec}^2 x - b^2. \end{aligned} \tag{8}$$

Now for a special case of $b = 1$, the potential $V_-(x;1)$ is a trivial constant function $-b^2 = -1$, while the partner potential $V_+(x;1)$ is given by $2\,\mathrm{cosec}^2 x - 1$. Thus, in general, two supersymmetric partner potentials could be of very different shapes. $V_-(x;1)$ is just an infinite one-dimensional square well potential whose bottom is set to -1. Since we know the eigenvalues and eigenfunctions of a infinitely deep square well potential, SUSYQM allows us to determine spectrum of the very nontrivial $\mathrm{cosec}^2 x$ potential. The eigenspectrum (in simplified units) of the square-well potential $V_-(x;1)$ are given by $\psi_n^{(-)} \propto \sin(nx)$ and $E_n^{(-)} = n^2 (n = 0;1;2;\ldots)$. Hence, using $\psi_{n-1}^{(+)} \propto A\psi_n^{(-)}$ and Eq. (1) the eigenspectrum of the $\mathrm{cosec}^2 x$ potential is given by $\psi_{n-1}^{(+)} \propto \left(\frac{d}{dx} - \cot x\right)\sin(nx)$ and $E_n^{(+)} = n^2 (n = 1;2;3;\ldots)$.

As we have stated before, if one knows the spectrum of one of the partner Hamiltonians, one knows the other.

In the above example, we knew the spectrum of the infinite square well and used that to determine the spectrum of the $\mathrm{cosec}^2 x$ potential. Now we demonstrate that they are indeed shape invariant partners. One can write the potential $V_+(x;b)$ as

$$\begin{aligned} V_+(x;b) &= W^2(x) + \frac{dW}{dx} = b(b+1)\mathrm{cosec}^2 x - b^2 \\ &= (b+1)[(b+1)-1]\mathrm{cosec}^2 x - (b+1)^2 + (b+1)^2 - b^2 \\ &= V_-(x;b+1) + (b+1)^2 - b^2. \end{aligned} \tag{9}$$

So the potential $V_-(x;b)$ is a shape invariant potential as defined in Eq. (3), with $R(a_0) = (b+1)^2 - b^2$, $a_0 = b$ and $a_1 = a_0 + 1 = b + 1$.

Once this shape invariance is established we do not need a priori knowledge of the eigenvalues and eigenfunctions of a potential to determine the spectrum of a partner potential. We first used the formalism of the preceding page; Eqs. (6) and (7). Here we solve the infinitely deep potential well as an example. Setting $b = 1$ in Eq. (8), we find $V_+(x;1) = 2\mathrm{cosec}^2 x - 1$ and $V_-(x;1) = -1$. The latter represents an infinitely deep potential well in the region $0 < x < \pi$. The ground state eigenfunction and the energy of $H_-(x;1)$ are given by $\psi_0^{(+)}(x;1) \propto e^{-\int W(x;1)dx} \propto e^{\int \cot x dx} \propto \sin x$ and 0, respectively. Now, we use shape invariance to determine the excited states of this Hamiltonian. Since $V_+(x;1) = V_-(x;2) + 3$, the ground state energy $E_0^{(+)}(1)$ of $H_+(x;1)$ is equal to 3 (using the fact that the ground state energy $E_0^{(-)}(2)$ of $H_-(x;2)$ is zero.) The common ground state eigenfunction of $H_+(x;1)$ and $H_-(x;2)$ is given by $\psi_0^{(+)}(x;1) = \psi_0^{(-)}(x;2) \propto e^{-\int W(x;2)dx} \propto e^{\int 2\cot x dx} \propto \sin^2 x$. Thus the first excited state of $H_-(x;1)$ is given by $\psi_1^{(-)}(x;1) \propto A^\dagger(x;1)\sin^2 x = \left(-\frac{d}{dx} - \cot x\right)\sin^2 x \propto \sin 2x$. Thus, we have derived the energy and the eigenfunction of the first excited state of $H_-(x;1)$. By iterating this procedure, we can generate its entire spectrum. Note that our choice of $V_-(x;1) = -1$ shifts the well known infinite well spectrum: $E_n^{(-)} = n^2 - 1$.

At this point, we would like to point out that shape invariance does not always help one in determining the spectrum. There is another important ingredient necessary, and that is unbroken supersymmetry. To understand this, let us first note that the condition $E_0^{(-)} = 0$ was crucial in determining the spectrum. However, unless ψ_0 is normalizable, it is meaningless to talk about $E_0^{(-)}$. For the function ψ_0 to be normalizable, we need $\psi_0(\pm\infty) \propto \exp\left(-\int_{x_0}^{\pm\infty} W(x)dx\right) = 0$. Thus a necessary condition for this normalizability is that $\int_{x_0}^{\pm\infty} W(x)dx = \infty$. This can be accomplished if $W(x \to \infty) > 0$ and $W(x \to -\infty) < 0$ and their integrals diverge. If ψ_0 is not normalizable but $1/\psi_0^{(-)}$ is, we write $\lim_{x\to\infty} \frac{1}{\psi_0^{(-)}} \propto \exp\left(\int_{x_0}^{\pm\infty} W(x)dx\right) = \exp\left(-\int_{x_0}^{\pm\infty} -W(x)dx\right) = 0$. Thus, $W \to -W$, and the roles of V_- and V_+ are reversed in Eq. (3); i.e. $E_0^{(-)} > 0$ and $E_0^{(+)} = 0$. However, if $W(x \to \infty)$ and $W(x \to -\infty)$ both have same sign, then neither of the two functions ψ_0 and $1/\psi_0^{(-)}$ is normalizable. Systems described by superpotentials $W(x)$'s with this type of asymptotic behavior are called cases of broken supersymmetry. For this type of systems, eigenvalue spectra of H_+ and H_- are strictly identical, i.e.

$$E_n^{(-)} = E_n^{(+)}; \tag{10}$$

with ground state energies greater than zero. Extending this work to include a few select cases of broken SUSY can be done along the direction of Ref. [6]. We will, in this paper, restrict ourselves to cases of unbroken SUSY.

Most of the known exactly solvable problems possess a spectrum generating algebra (SGA) [3, 4, 5] as has been demonstrated by numerous authors, starting with Pauli [7]. We would like to find out if there is any connection between the SGA and shape invariance of these systems.

In many of these SGA approaches, the Schroedinger Equation is written as: $[\sum_i c_i T_i - \lambda]R(r) = 0$, where $rR(r)$ is the customary radial part of the wave function [Adams *et al.* [8]] and T_i's are the generators of the underlying algebra. Eigenvalues of $H = \sum_i c_i T_i$ are then given by diagonalization of these generators. For example [8], the Coulomb problem can be constructed from the generators $T_1 = \frac{1}{2}[rp_r^2 + L^2 r^{-1} - r]$, $T_2 = [rp_r]$, $T_3 = \frac{1}{2}[rp_r^2 + L^2 r^{-1} + r]$, where $p_r = -i(\partial/\partial r + 1/r); [r, p_r] = i$. The algebra is $so(2,1)$: $[T_1, T_2] = -iT_3$, $[T_2, T_3] = iT_1$, $[T_3, T_1] = iT_2$. Then $H = \frac{1}{2}p_r^2 + \frac{1}{2}L^2 r^{-2} - Zr^{-1}$ leads to the radial Schroedinger Equation reformulated as $[T_1(1+E) + T_3(1-E) - 2Z]R(r) = 0$ where E is the energy eigenvalue.

In SUSYQM, by contrast, the Hamiltonian is given in terms of the ladder operators: $H = A^\dagger A$, analogous, as we have noted earlier, to $a^\dagger$ and a in the traditional Dirac solution to the one-dimensional harmonic oscillator, or L_+ and L_- in the well-known angular momentum problems for spherically symmetric potentials. As we shall see later, the type of SGA that is most relevant to SUSYQM is known as potential algebra, studied extensively by Alhassid *et al.* [3, 4]. In potential algebra, the Hamiltonian of the system is written in terms of the Casimir operator (C_2) of the algebra, and the energy of states specified by an eigenvalue ω of C_2. This Casimir is analogous to (and often identical to) H, and will commute with a set of operators $J_\pm$ and J_3. Different states with a given ω represent eigenstates of a set of Hamiltonians that differ only in values of parameters, and share a common set of energies. This is very similar to the case of shape invariant potentials. In the next section, we will attempt to establish this connection in a more concrete fashion. In fact, for a set of solvable quantum mechanical systems we shall explicitly show that shape invariance leads to a potential algebra.

III. Potential algebra model for shape invariant potentials (SIP's)

To begin the construction of the operator algebra, let us express the shape invariance condition [Eq. (3)] in terms of A and $A^\dagger$:

$$
\begin{aligned}
V_+(x; a_0) - V_-(x; a_1) &= H_+(x; a_0) - H_-(x; a_1) \\
&= A(x; a_0)A^\dagger(x; a_0) - A^\dagger(x; a_1)A(x; a_1) = R(a_0).
\end{aligned}
\tag{11}
$$

This relation, which resembles the familiar commutator structure, but with distinct parameters a_0 and a_1, is not as exotic as it may appear. For example, we have seen such an equation in the context of angular momentum in quantum mechanics:

$$[L_+, L_-] = 2L_3. \tag{12}$$

This operator equation, when applied to spherical harmonics, gives the following result involving its eigenvalues

$$f(m-1, l)Y_l^m - f(m, l)Y_l^m = 2m\, Y_l^m.$$

We identify $a_0 = m$, $a_1 = m - 1$, and $f(m, l) \equiv l(l+1) - m(m+1)$. In a similar fashion, we would like to characterize Eq. (11) as an eigenvalue equation of operators J_+, J_- in an enlarged space, with parameters a_0, a_1 the eigenvalues of the corresponding J_3. We introduce, in analogy with 3-space, a coordinate ϕ such that J's are its "rotational" generators. After quite a bit of trial and error, we find that one such set of operators is given by

$$J_+ = e^{ip\phi} A^\dagger(x;\chi(i\partial_\phi)); \; J_- = A(x;\chi(i\partial_\phi))\, e^{-ip\phi}; \text{ and } J_3 = -\frac{i}{p}\partial_\phi. \tag{13}$$

The constant p is an arbitrary real constant that scales the spacing between eigenstates of J_3. The real function χ, as will be explained below and exemplified later, is chosen judiciously in accord with the relation among the parameters a_n. The operators $A(x;\chi(i\partial_\phi))$ and $A^\dagger(x;\chi(i\partial_\phi))$ are obtained from Eq. (1) with the substitution $a_0 \to \chi(i\partial_\phi)$. From Eq. (13), one obtains

$$[J_+, J_-] = e^{ip\phi} A^\dagger(x;\chi(i\partial_\phi)) A(x;\chi(i\partial_\phi)) e^{-ip\phi} - A(x;\chi(i\partial_\phi)) A^\dagger(x;\chi(i\partial_\phi)). \tag{14}$$

If we carry out the operation of $i\partial_\phi$ on $e^{ip\phi}$, Eq. (14) reduces to

$$[J_+, J_-] = A^\dagger(x;\chi(i\partial_\phi + p)) A(x;\chi(i\partial_\phi + p)) - A(x;\chi(i\partial_\phi)) A^\dagger(x;\chi(i\partial_\phi)). \tag{15}$$

At this point if we can judiciously choose a function $\chi(i\partial_\phi)$ such that $\chi(i\partial_\phi + p) = f[\chi(i\partial_\phi)]$, the r.h.s. of Eq. (15) becomes

$$A^\dagger(x; f[\chi(i\partial_\phi)]) A(x; f[\chi(i\partial_\phi)]) - A(x;\chi(i\partial_\phi)) A^\dagger(x;\chi(i\partial_\phi)).$$

Now using

$$a_0 \to \chi(i\partial_\phi); \quad a_1 = f(a_0) \to f[\chi(i\partial_\phi)] = \chi(i\partial_\phi + p); \tag{16}$$

and the shape invariance Eq. (11), Eq. (15) reduces to

$$[J_+, J_-] = -R(\chi(i\partial_\phi)). \tag{17}$$

As a consequence, we obtain a "deformed" Lie algebra whose generators J_+, J_- and J_3 satisfy the commutation relations

$$[J_3, J_\pm] = \pm J_\pm; \quad [J_+, J_-] = \xi(J_3); \tag{18}$$

$\xi(J_3) \equiv -R(\chi(i\partial_\phi))$ defines the deformation of the algebra from the $so(2,1)$ value of $-2J_3$. Thus we see that the shape invariance condition plays an indispensible role in the closing of this algebra.

Depending on the relationship between a_0 and a_1, we have different forms of the χ function in Eq. (16). This results in different deformed algebras. For example,

1. translational models: $a_1 = a_0 + p \iff \chi(z) = z$. In these models if R is a linear function of J_3 the algebra turns out to be $so(2,1)$ [11]. It is important to point out that Balantekin [12], independently, established a similar connection about the same time as us.
2. scaling models: $a_1 = e^p a_0 \equiv q a_0 \iff \chi(z) = e^z$,
3. cyclic models: $a_1 = \frac{\alpha a_0 + \beta}{\gamma a_0 + \delta} \iff \chi(z) = \frac{(\lambda_1 - \delta)\lambda_1^{z/p} + (\lambda_2 - \delta)\lambda_2^{z/p} B(z)}{\gamma[\lambda_1^{z/p} + \lambda_2^{z/p} B(z)]}$,

where $\lambda_{1,2}$ are solutions of the equation $(x-\alpha)(x-\delta)-\beta\gamma=0$ and $B(z)$ is an arbitrary periodic function of z with period p. We shall elaborate on these cases in Sec. 4. Other relations between a_0 and a_1 lead to more complicated forms for $\chi(z)$. For example, a function $\chi(z)=e^{e^z}$, is required for $a_1=f[a_0]=a_0^2$.

The operator J_+J_- corresponds to a supersymmetric Hamiltonian. From Eq. (13)

$$J_+J_-=A^\dagger(x;\chi(i\partial_\phi+p))A(x;\chi(i\partial_\phi+p))=H(x;\chi(i\partial_\phi+p)). \tag{19}$$

This is our old Hamiltonian $H_-(x;a_1)$ whose spectrum we seek; we will now suppress the subscript "-" to avoid confusion with a similar index on the generator J_-. To find the energy spectrum of H of Eq. (19), we thus need to construct the unitary representations of the operators $J_+,\ J_-$, and J_3. By definition, the action of the operators $J_+,\ J_-$ and J_3 on an arbitrary eigenstate $|h\rangle$ of J_3 is given by

$$\begin{aligned} J_3|h\rangle &= h|h\rangle, \\ J_-|h\rangle &= a(h)\,|h-1\rangle, \\ J_+|h\rangle &= a^*(h+1)\,|h+1\rangle. \end{aligned} \tag{20}$$

For determination of the representation, we now need to find the coefficients $a(h)$. Given the fact that these operators satisfy a deformed algebra [Eqs. (18)], the representation is expected to be different from our familiar $so(3)$ ($[J_+,J_-]=2\,J_3$) or its less familiar cousin $so(2,1)$ ($[J_+,J_-]=-2\,J_3$). The technique that will be followed is based on Ref. [13]. Using Eqs. (18) and (20), operating with $[J_+,J_-]$ on a state $|h\rangle$ and writing $\xi(J_3)=\xi(h)$, we obtain

$$|a(h)|^2-|a(h+1)|^2=\xi(h). \tag{21}$$

To obtain $a(h)$ from this, which involves $|a(h+1)|^2$, let us define a function $g(J_3)$ such that

$$\xi(J_3)=g(J_3)-g(J_3-1). \tag{22}$$

Thus, we have $|a(h)|^2-|a(h+1)|^2=\xi(h)=g(h)-g(h-1)$. (Note the generality of that $g(h)$; it can be changed by an additive constant or a function of unit period without affecting $\xi(h)$). The Casimir of this algebra is then given by $C_2=J_-J_++g(J_3)$ [1]. The profile of $g(h)$ determines the dimension of the unitary representation. To illustrate how this mechanism works, let us consider the two cases presented in Fig. 4.

If we label the lowest eigenstate of the operator J_3 as h_{min}, then $J_-|h\rangle=0 \implies a(h_{min})=0$. Without loss of generality we can choose the coefficients $a(h)$ to be real. Then from (21) and (22), for an arbitrary $h=h_{min}+n;\ n=0,1,2,\ldots$ one obtains by iteration

$$a^2(h)=g(h-n-1)-g(h-1). \tag{23}$$

[1] This can be verified explicitly by showing that it commutes with $J_+,\ J_-,\ J_3$

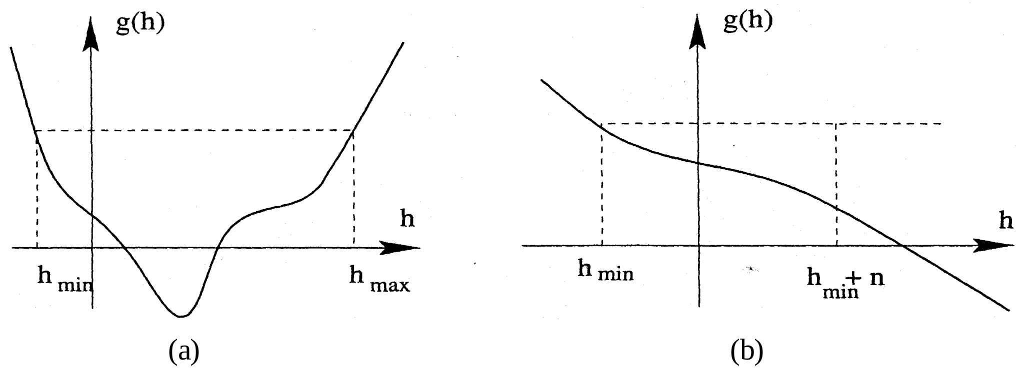

FIG. 4. Generic behaviors of $g(h)$.

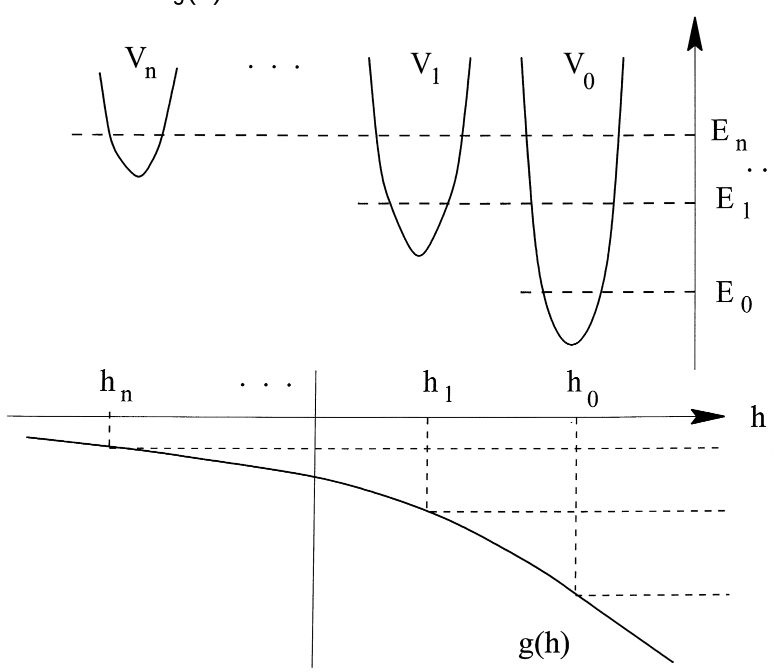

FIG. 5. Potential Algebra: Schematic of generation of SIP's by "hopping" of h.

Finite dimensional representations are represented by graphs of $g(h)$ vs. h with starting at $h = h_{min}$, then by moving in integer steps parallel to the h-axis to the point corresponding to $h = h_{max}$, as in Fig. 4a. Thus we obtain the family of partner potentials. At the end points, $a(h_{min}) = a(h_{max} + 1) = 0$, and we get a finite representation. This is the case of $su(2)$ for example, where $g(h)$ is given by the parabola $h(h+1)$. However, if $g(h)$ decreases monotonically, Fig. 4b, there exists only one end point at $h = h_{min}$. Starting from h_{min} the value of h can be increased in integer steps to infinity. In this case we have an infinite dimensional representation. As in the finite case, h_{min} labels the representation. The difference is that here h_{min} takes continuous values. Similar arguments apply for a monotonically increasing function $g(h)$ as well.

Recall, we are looking for the eigenvalue spectrum of a given V by comparing it with the partner V's with same spectrum, but sequential ground states. We can use the $J_\pm$'s properties of Eq. (20) to develop a "hopping scheme" as in Fig. 5 to move horizontally from each partner's E_0 to the E_n of our V of interest. Eq. (20) leads to either a finite representation similar to angular momentum (i.e. h's have a maximum and a minimum) or to an infinite representation (bounded from above, below, or completely unbounded).

Having established a connection between the representation of the above algebra associated with a shape invariant model, it is straightforward to obtain (using Eq. (19, 21)) the complete spectrum of the system. To illustrate how this mechanism works, we investigate a few examples in the next section.

Using a similar approach to ours, with $so(2,1)$, Balantekin and coworkers [12] have studied the cases of potentials with a positive quadratic power law in the energy eigenvalues: $E_n = \beta n^2 + \delta n + \gamma$. They have also studied the "coherent states" for shape invariant cases.

IV. Examples

IV-1. Self-similar potentials

The first example is for a scaling change of parameters $a_1 = q\,a_0 = e^p a_0$. As stated before, the function $\chi(z)$ that emulates this relationship is given by e^z. Consider the simple choice $R(a_0) = r_1, a_0$, where r_1 is a constant. This choice generates the self-similar potentials studied in Refs. [14, 15]. In this case, Eqs. (18) become:

$$[J_3, J_\pm] = \pm J_\pm; \quad [J_+, J_-] = \xi(J_3) \equiv -r_1 \exp(-pJ_3); \tag{24}$$

which is a deformation of the standard $so(2,1)$ Lie algebra. For this case, from Eqs. (24) and (22) one gets [2]

$$g(h) = \frac{r_1}{e^p - 1} e^{-ph} = -\frac{r_1}{1-q} q^{-h}; \quad q = e^p. \tag{25}$$

Note that for scaling problems [15], one requires $0 < q < 1$, which leads to $p < 0$. From the monotonically decreasing profile of the function $g(h)$, it follows that the unitary representations of this algebra are infinite dimensional. Then from Eq. (23),

$$a^2(h) = g(h-n-1) - g(h-1) = r_1 \frac{q^n - 1}{q - 1} q^{1-h}. \tag{26}$$

To determine the energy eigenvalues, we find the expectation value of H in Eq. (19) in an arbitrary eigenstate $|h\rangle$ of J_3. This leads to the spectrum of the Hamiltonian $H_-(x, a_1)$ from

$$H_-|h\rangle = a^2(h)|h\rangle = r_1 \frac{q^n - 1}{q - 1} q^{1-h}|h\rangle. \tag{27}$$

Therefore, the eigenenergies are

$$E_n(h) = r_1 \alpha(h) \frac{q^n - 1}{q - 1}; \quad \alpha(h) \equiv q^{1-h}. \tag{28}$$

---

[2] To obtain a solution of Eq. (22), we have been guided by solutions of the differential equation $\xi(u) = \frac{dg(u)}{du}$.

To compare the above spectrum obtained using the group theoretic method with the results obtained from SUSYQM [16], we go to the coordinate representation. Here $|h\rangle \propto e^{iph\phi}\psi_{h_{min};n}(x)$ [3] and hence, the Schrödinger equation for the Hamiltonian H_- reads

$$
\begin{aligned}
\left\{-\frac{d^2}{dx^2} + W^2(x;\chi(i\partial_\phi + p)) - W'(x;\chi(i\partial_\phi + p)) - E\right\} e^{ip\phi h}\psi_{h_{min};n}(x) = 0;\\
\left\{-\frac{d^2}{dx^2} + W^2(x;e^{i\partial_\phi + p}) - W'(x;e^{i\partial_\phi + p}) - E\right\} e^{iph\phi}\psi_{h_{min};n}(x) = 0;\\
\left\{-\frac{d^2}{dx^2} + W^2(x;\alpha(h)) - W'(x;\alpha(h)) - E\right\} \psi_{h_{min};n}(x) = 0;
\end{aligned}
\tag{29}
$$

which is exactly the Schrödinger equation appearing in Ref. [15], with eigenenergies given by Eq. (28). The elegant correspondence that exists between potential algebra and supersymmetric quantum mechanics for shape invariant potentials is further described in Ref. [16].

For a more general case [15], we assume $R(a_0) = \sum_{j=1}^{\infty} R_j a_0^j$. In this case

$$
g(h) = \sum_{j=1}^{\infty} \frac{R_j}{1-e^{jp}} e^{-jph}; \tag{30}
$$

and one gets

$$
\begin{aligned}
a^2(h) &= g(h-n-1) - g(h-1)\\
&= \sum_{j=1}^{\infty} \alpha_j(h)\frac{1-q^{jn}}{1-q^j};
\end{aligned}
\tag{31}
$$

where $\alpha_j(h) = R_j e^{-j(h-1)}$. These results agree with those obtained in Ref. [15].

IV-2. Cyclic potentials

Let us consider a particular change of parameters given by the following cycle (or chain):

$$
a_0;\ a_1 = f(a_0);\ a_2 = f(a_1);\ \ldots;\ a_{k-1} = f(a_{k-2});\ a_k = f(a_{k-1}) = a_0; \tag{32}
$$

and choose $R(a_i) = a_i \equiv \omega_i$. This choice generates the cyclic potentials studied in Ref. [9].

Cyclic potentials form a series of shape invariant potentials; the series repeats after a cycle of k iterations. In Fig. 6 we show the first potential $V(x;a_0)$ from a 3-chain ($k=3$) of cyclic potentials, corresponding to $\omega_0 = 0.15$, $\omega_1 = 0.25$, $\omega_2 = 0.60$.

Such potentials have an infinite number of periodically spaced eigenvalues. More precisely, the level spacings are given by $\omega_0; \omega_1; \ldots; \omega_{k-1}; \omega_0; \omega_1; \ldots; \omega_{k-1}; \omega_0; \omega_1; \ldots$.

---

[3] $J_3|h\rangle \to -\frac{i}{p}\partial_\phi e^{iph\phi} = h e^{iph\phi}$

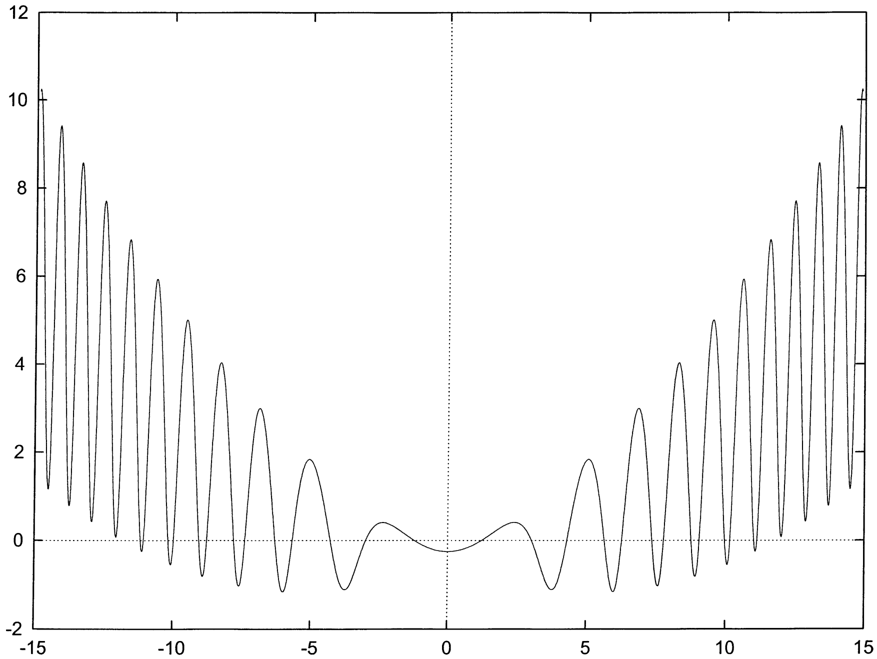

FIG. 6. First potential $V(x; a_0)$ from a 3-chain ($k = 3$).

In order to generate the change of parameters (32) the function f should satisfy $f(f(\ldots f(x)\ldots)) \equiv f^k(x) = x$. The equation (a projective map)

$$f(y) = \frac{\alpha y + \beta}{\gamma y + \delta}, \tag{33}$$

with specific constraints on the parameters $\alpha, \beta, \gamma, \delta$, satisfies such a condition [9].

The next step is to identify the Lie algebra behind this model. For this, we need to find the function χ satisfying the equation

$$\chi(z + p) = f(\chi(z)) \equiv \frac{\alpha\chi(z) + \beta}{\gamma\chi(z) + \delta}. \tag{34}$$

It is a difference equation and its general solution is given by

$$\chi(z) = \frac{(\lambda_1 - \delta)\lambda_1^{z/p} + (\lambda_2 - \delta)\lambda_2^{z/p} B(z)}{\gamma\left[\lambda_1^{z/p} + \lambda_2^{z/p} B(z)\right]}, \tag{35}$$

where $\lambda_{1,2}$ are solutions of the equation $(x - \alpha)(x - \delta) - \beta\gamma = 0$. For simplicity $B(z)$ can be chosen to be an arbitrary constant. Plugging this expression in Eqs. (18) we obtain:

$$\begin{aligned} &[J_3, J_\pm] = \pm J_\pm, \\ &[J_+, J_-] = \xi(J_3) \equiv -\frac{1}{c}\,\frac{A(\lambda_1 - \delta)\lambda_1^{-J_3} + B(\lambda_2 - \delta)\lambda_2^{-J_3}}{A\lambda_1^{-J_3} + B\lambda_2^{-J_3}}. \end{aligned} \tag{36}$$

Applying our standard procedure to find the spectrum of the Hamiltonian $H_- = J_+J_-$ we find that the ground state is at zero energy; the next $(k-1)$ eigenvalues are $E_l = \sum_{j=0}^{l}\omega_j$; $l = 0,1,\ldots,(k-2)$; and all other eigenvalues are obtained by adding arbitrary multiples of the quantity $\Omega_k \equiv \omega_0 + \omega_1 + \cdots + \omega_{k-1}$. This result is in complete agreement with [9].

IV-3. Scarf potential with $a_n = a_{n-1} + \delta$

As a concrete example of translational algebra, we will examine the Scarf potential, which is related to the Pöschl-Teller II potential by a redefinition of the independent variable. We will show that the shape invariance of the Scarf potential automatically leads to its potential algebra: $so(2,1)$. (Exactly similar analyses can be carried out for the Morse, the Rosen-Morse, and the Pöschl-Teller potentials.) The Scarf potential is described by its superpotential $W(x;a_0;B) = a_0 \tanh x + B\,\mathrm{sech}\,x$. The potential $V_-(x;a_0;B) = W^2(x;a_0;B) - W'(x;a_0;B)$ is then given by

$$V_-(x;a_0;B) = \left[B^2 - a_0(a_0+1)\right]\mathrm{sech}^2 x + B(2a_0+1)\,\mathrm{sech}\,x\tanh x + a_0^2. \tag{37}$$

The eigenvalues of this system are given by ([1])

$$E_n = a_0^2 - (a_0-n)^2. \tag{38}$$

The partner potential $V_+(x;a_0;B) = W^2(x;a_0;B) + W'(x;a_0;B)$ is given by

$$\begin{aligned} V_+(x;a_0;B) &= [B^2 - a_0(a_0-1)]\,\mathrm{sech}^2 x + B(2a_0-1)\,\mathrm{sech}\,x\tanh x + a_0^2. \\ &= V_-(x;a_1;B) + a_0^2 - a_1^2, \end{aligned} \tag{39}$$

where $a_1 = a_0 - 1$. Thus, $R(a_0)$ for this case is $a_0^2 - a_1^2 = 2a_0 - 1$, linear in a_0.

Now, consider a set of operators $J^\pm$ which are given by

$$J^\pm = e^{\pm i\phi}\left[\pm\frac{\partial}{\partial x} - \left\{\left(-i\frac{\partial}{\partial\phi} \pm \frac{1}{2}\right)\tanh x + B\,\mathrm{sech}\,x\right\}\right]. \tag{40}$$

It can be explicitly checked that the commutator of the $J^\pm$ operators, as defined above, is indeed given by $-2J_3$, thus forming a closed $so(2,1)$ algebra. Moreover, the operator J^+J^-, acting on the basis $|j,m\rangle$ gives:

$$\begin{aligned} J^+J^- \equiv &\left[B^2 - \left(m^2 - \frac{1}{4}\right)\right]\mathrm{sech}^2 x \\ &+ B\left(2\left(m - \frac{1}{2}\right) + 1\right)\mathrm{sech}\,x\tanh x + \left(m - \frac{1}{2}\right)^2; \end{aligned} \tag{41}$$

which is just the $H_{scarf}(x; m-\frac{1}{2}; B)$, i.e. the Scarf Hamiltonian with a_0 replaced by $m - \frac{1}{2}$. Thus the energy eigenvalues of the Hamiltonian will be the same as that of the operator $J^+J^- = J_3^2 - J_3 - J^2$. Hence, the energy is given by $E = m^2 - m - j(j+1)$. In this example, the quantum number j plays the role of h_{min} defined in the previous section. Substituting $j = n - m$, one gets

$$\begin{aligned} E_n &= m^2 - m - (n-m)(n-m+1) \\ &= \left(m - \frac{1}{2}\right)^2 - \left[n - \left(m - \frac{1}{2}\right)\right]^2. \end{aligned} \tag{42}$$

which is the same as Eq. (38), with a_0 replaced by $(m - \frac{1}{2})$. Note that the parameter B of the Hamiltonian does not show up in the expression for the energy, and thus plays the role of a "spectator". As we shall show later, this property is shared by all known exactly solvable models with two independent parameters.

Thus for this potential, (as well as for the Morse, Rosen-Morse and Pöschl-Teller potentials mentioned above), there are actually an infinite number of partner potentials, each characterized by an allowed value of the parameter m, that correspond to the same value of $j = n - m$ and thus to the same energy E. Hence the name "potential algebra" ([3, 4]).

V. Natanzon potentials

In the Section 4.3, we noted that for SIP's with translationally related parameters (i.e. $a_n = a_{n-1} + \delta$), the shape invariance condition led to the closing of the algebra to the familiar $so(3)$ or $so(2,1)$, provided that $R(a_0)$ was linear in a_0 [11]. Several SIP's belong to this category; among them are the Morse, Scarf I, Scarf II, and generalized Pöschl-Teller potentials. However, there are many important SIP's (e.g., Coulomb), whose associated $R(a_0)$'s are not linear in a_0. Our method of the previous section would lead to deformed potential algebras for these systems. While we now know how to get deformed representations of such algebras, in this section we shall take a different approach. We choose to generalize the *structure* of the operators $J_\pm$ such that their *algebra* still remains linear. In fact, in this section, we reverse the scheme of the last section: rather than showing the algebraic structure hidden in a shape invariant system, we generate shape invariant potentials from an underlying potential algebra. To do this we take advantage of the properties of a generalized quadratic potential discussed by Natanzon [10].

Alhassid *et al.* [3] have shown that the algebra associated with the *general* potential of the Natanzon class is $so(2,2)$. The Schrödinger equation for these potentials reduces in general to the hypergeometric equation.

We will briefly examine the properties of the $so(2,2)$ algebra in this section, and show its connection to the Natanzon potentials [10]. We shall propose an additional constraint to select a shape invariant subset of the Natanzon potentials. We shall then show that this constraint indeed produces all known SIP's of the translational type. We shall find in fact that this subset of Natanzon potentials associated with the translational SIP's has the simpler $so(2,1)$ algebra.

We begin by describing Alhassid *et al.*'s representation of the $so(2,2)$ algebra in terms of differential operators. For consistency, we use the formalism and the notations of Ref. [3]. Our program here is to take the Alhassid *et al.* $so(2,2)$ operators, which they call A and B, and see how these can be related to the previous section's J's; i.e., the operators we associated with shape invariance.

The differential operators of Alhassid can be written explicitly as

$$
\begin{aligned}
A_\pm \equiv A_1 \pm A_2 &= \frac{1}{2} e^{\pm i(\phi+\theta)} \left[\mp \frac{\partial}{\partial\chi} + \tanh\chi\,(-i\partial_\phi) + \coth\chi\,(-i\partial_\theta) \right]; \\
A_3 &= -\frac{i}{2}(\partial_\phi + \partial_\theta); \\
B_\pm \equiv B_1 \pm B_2 &= \frac{1}{2} e^{\pm i(\phi-\theta)} \left[\mp \frac{\partial}{\partial\chi} + \tanh\chi(-i\partial_\phi) + \coth\chi(+i\partial_\theta) \right]; \\
B_3 &= -\frac{i}{2}(\partial_\phi - \partial_\theta).
\end{aligned}
\tag{43}
$$

The A's and B's separately form an $so(2;1)$ algebra:

$$[A_3; A_\pm] = \pm A_\pm ; \qquad [A_+; A_-] = -2A_3;$$

and similarly for the B's. The Casimir operator C_2; i.e., the operator which commutes with all of the above (cf. the ordinary angular momentum operators L^2 vis-a-vis $L_\pm$, L_z) is given by

$$\begin{aligned} C_2 &= 2(A_3^2 - A_+A_- - A_3) + 2(B_3^2 - B_+B_- - B_3) \\ &= \left[\frac{\partial^2}{\partial\chi^2} + (\tanh\chi + \coth\chi)\frac{\partial}{\partial\chi} + \mathrm{sech}^2\chi(-i\partial_\phi)^2 - \mathrm{cosech}^2\chi(-i\partial_\theta)^2 \right] . \end{aligned} \tag{44}$$

Operators A_3, B_3 and C_2 commute, and can therefore be simultaneously diagonalized, and their actions on their common eigenstate are given by

$$\begin{aligned} C_2|\omega; m_1; m_2\rangle &= \omega(\omega+2)\,|\omega; m_1; m_2\rangle; \\ A_3|\omega; m_1; m_2\rangle &= m_1\,|\omega; m_1; m_2\rangle; \\ B_3|\omega; m_1; m_2\rangle &= m_2\,|\omega; m_1; m_2\rangle . \end{aligned} \tag{45}$$

(It is important to note that the Casimir operator given above is indeed self-adjoint, once we recognize that the appropriate "measure"; viz., the volume element over which it is integrated, is $\sinh\chi\cosh\chi\, d\chi\, d\phi\, d\theta$. This is compurable to the more familiar "3-space" algebra $so(4)$, for which $[A_3; A_\pm] = \pm A_\pm$, $[A_+; A_-] = +2A_3$, and the measure is $\sin\chi\cos\chi\, d\chi\, d\phi\, d\theta$.)

We thus have the eigenvalues and eigenfunctions of C_2, A_3, and B_3. The problem resembles the familiar 2-particle angular momentum case for H, L_{1z}, L_{2z}. A_3 and B_3 certainly have differential forms $(-i\partial_\phi \pm i\partial_\theta)$ analogous to L_z. However, our C_2 cannot, in its present form, be a Schrödinger Hamiltonian, since it has a first order derivative term. When we seek to eliminate this term, we discover that this constrains the allowed potentials to the special family, discovered by Natanzon [10].

To connect the Casimir operator C_2 of the $so(2;2)$ algebra [Eq. (44)] to the general Natanzon potential, we try the standard set of operations to transform both coordinate system and variables: first we perform a similarity transformation on C_2 by a function F and then follow that up by an appropriate change of variable $\chi \to g(r)$ [4]. Under the similarity transformation,

$$\frac{d}{d\chi} \longrightarrow F\frac{d}{d\chi}F^{-1} = \left(\frac{d}{d\chi} - \frac{\dot F}{F}\right); \quad \frac{d^2}{d\chi^2} \longrightarrow \left(\frac{d^2}{d\chi^2} - \frac{2\dot F}{F}\frac{d}{d\chi} + \frac{2\dot F^2}{F^2} - \frac{\ddot F}{F}\right);$$

where dots represent derivatives with respect to χ. The Casimir operator C_2 of Eq. (44) transforms as:

$$\begin{aligned} C_2 \longrightarrow \tilde C_2 = \Bigg[& \frac{d^2}{d\chi^2} + \left(\tanh\chi + \coth\chi - \frac{2\dot F}{F}\right)\frac{d}{d\chi} + \frac{2\dot F^2}{F^2} - \frac{\ddot F}{F} \\ & - (\tanh\chi + \coth\chi)\frac{\dot F}{F} + \mathrm{sech}^2\chi(-i\partial_\phi)^2 - \mathrm{cosech}^2\chi(-i\partial_\theta)^2 \Bigg] . \end{aligned} \tag{46}$$

[4] No connection to the $g(h)$ discussed in the previous section.

Now, let us carry out a change of variable from χ to r via $\chi = g(r)$. We are going to denote differentiation with respect to r by a prime. The operators $\frac{d}{d\chi}$ and $\frac{d^2}{d\chi^2}$ transform as

$$\frac{d}{d\chi} = \frac{1}{g'}\frac{d}{dr}; \qquad \frac{d^2}{d\chi^2} = \frac{1}{g'^2}\left[\frac{d^2}{dr^2} - \frac{g''}{g'}\frac{d}{dr}\right].$$

The operator $\mathcal{C}_2$ now transforms into

$$\begin{aligned}\mathcal{C}_2 = \frac{1}{g'^2}\Bigg[\frac{d^2}{dr^2} &+ \left\{-\frac{g''}{g'} - \frac{2F'}{F} + g'(\tanh g + \coth g)\right\}\frac{d}{dr} \\ &+ \frac{2F'^2}{F^2} - \frac{F''}{F} + \frac{F'\,g''}{F\,g'} \\ &- \frac{F'\,g'}{F}(\tanh g + \coth g) + g'^2\left(\mathrm{sech}^2 g(-i\partial_\phi)^2 - \mathrm{cosech}^2 g(-i\partial_\psi)^2\right)\Bigg].\end{aligned} \tag{47}$$

In order for $g'^2\mathcal{C}_2$ to be a Schrödinger Hamiltonian, we require the "coefficient" of the first order derivative $\frac{d}{dr}$; viz the expression inside the curly brackets in Eq. (47), to vanish. This constrains the relationship between the two functions F and g to be

$$-\frac{g''}{g'} - \frac{2F'}{F} + g'(\tanh g + \coth g) = 0; \tag{48}$$

which yields

$$F \sim \left(\frac{\sinh(2g)}{g'}\right)^{\frac{1}{2}}. \tag{49}$$

Thus, the operator $\mathcal{C}_2$, transforms into

$$\begin{aligned}\mathcal{C}_2 = \frac{1}{g'^2}\Bigg[\frac{d^2}{dr^2} &+ g'^2\left(\frac{(1-\tanh^2 g)^2 - 4\tanh^2 g}{4\tanh^2 g}\right) \\ &+ \frac{1}{2}\{g; r\} + g'^2\left(\mathrm{sech}^2 g(-i\partial_\phi)^2 - \mathrm{cosech}^2 g(-i\partial_\psi)^2\right)\Bigg].\end{aligned} \tag{50}$$

This Casimir operator now has the form

$$\mathcal{C}_2 = -\frac{1}{g'^2}H;$$

where H is a one-dimensional Hamiltonian with the potential $U(r)$ given by

$$\begin{aligned}E - U(r) = g'^2&\left(\frac{(1-\tanh^2 g)^2 - 4\tanh^2 g}{4\tanh^2 g}\right) + \frac{1}{2}\{g; r\} \\ &+ g'^2\left[\mathrm{sech}^2 g(-i\partial_\phi)^2 - \mathrm{cosech}^2 g(-i\partial_\psi)^2\right].\end{aligned} \tag{51}$$

Following Alhassid, we now must relate these $so(2,2)$ operators– in particular the transformed Casimir– to the Natanzon potentials. A general Natanzon potential $U(r)$ is implicitly defined by [10]

$$U[z(r)] = \frac{-fz(1-z) + h_0(1-z) + h_1 z}{Q(z)} - \frac{1}{2}\{z, r\}, \tag{52}$$

with $Q(z)$ quadratic in z : $Q(z) = az^2 + b_0 z + c_0 = a(1-z)^2 - b_1(1-z) + c_1$ and $f, h_0, h_1, a, b_0, b_1, c_0, c_1$ are constants. The Schwarzian derivative $\{z, r\}$ is defined by

$$\{z, r\} \equiv \frac{d^3z/dr^3}{dz/dr} - \frac{3}{2}\left[\frac{d^2z/dr^2}{dz/dr}\right]^2. \tag{53}$$

The relationship between the variables z $(0 < z < 1)$ and r is implicitly given by

$$\left(\frac{dz}{dr}\right) = \frac{2z(1-z)}{\sqrt{Q(z)}}. \tag{54}$$

Now, for our potential [Eq. (51)] to take the form of a general Natanzon potential, we have to relate the variables g and z in such a way that the potential in terms of z is given by Eq. (52). Since the potential has to be a ratio of two quadratic functions of z, we find, after some work, that this can be accomplished with the identification $z = \tanh^2 g$, which leads to

$$\begin{aligned} U(z(r)) &= \frac{E\, Q + [-\frac{7}{4} + \frac{5}{2}z - \frac{7}{4}z^2] - z(1-z)(-i\partial_\phi)^2 + (1-z)(-i\partial_\theta)^2}{Q} - \frac{1}{2}\{z, r\} \\ &= \Big[-\left(aE - \frac{7}{4} + (-i\partial_\phi)^2\right) z(1-z) + \left(c_0 E - \frac{7}{4} + (-i\partial_\theta)^2\right)(1-z). \\ &\quad + ((a + b_0 + c_0)E - 1)\Big]/Q(z) - \frac{1}{2}\{z, r\}. \end{aligned} \tag{55}$$

Here we have used

$$g' = \frac{dg}{dz} z' = \frac{1}{2\sqrt{z(1-z)}} \frac{2z(1-z)}{Q} = \sqrt{\frac{z}{Q}}.$$

Now, with the following identification

$$\begin{aligned} f &= aE - \frac{7}{4} + (-i\partial_\phi)^2, \\ h_0 &= c_0 E - \frac{7}{4} + (-i\partial_\theta)^2, \\ h_1 &= (a + b_0 + c_0)E - 1, \end{aligned} \tag{56}$$

the potential of Eq. (55) indeed has the form of a general Natanzon potential [Eq. (52)].

We are finally ready to explicitly demonstrate the connection between the potential algebra based on Natanzon potentials, viz., $so(2,2)$, and the shape invariant potentials of supersymmetric quantum mechanics. We now note that the similarity transformation can be rewritten: since $g = \tanh^{-1}\sqrt{z}$, and $g' = \sqrt{\frac{z}{Q}}$, Eq. (54) yields $\frac{\sinh(2g)}{g'} = \frac{z}{z'}$.

At this point we go back to the operators $A_\pm$ [Eq. (43)] and ask how these operators transform under the similarity transformation given by $F \sim \left(\frac{\sinh(2g)}{g'}\right)^{\frac{1}{2}} \sim \sqrt{\frac{z}{z'}}$. This transformation carries the operators $A_\pm$ to

$$
\begin{aligned}
A_\pm \to \tilde{A}_\pm = \frac{e^{\pm i(\phi+\theta)}}{2}\Bigg[\mp\left(\frac{d}{d\chi} + \frac{1}{2z'}\frac{dz'}{d\chi} - \frac{1}{2z}\frac{dz}{d\chi}\right)\\
+ \tanh\chi\,(-i\partial_\phi) + \coth\chi\,(-i\partial_\theta)\Bigg]. \qquad (57)
\end{aligned}
$$

Except for the expression $\left(\frac{1}{2z'}\frac{dz'}{d\chi} - \frac{1}{2z}\frac{dz}{d\chi}\right)$, this looks very much like Eq. (43), which gives in fact the $A_\pm$ of the shape invariant Pöschl-Teller potential [1]. Thus, if $\left(\frac{1}{2z'}\frac{dz'}{d\chi} - \frac{1}{2z}\frac{dz}{d\chi}\right)$ were to be a linear combination of $\tanh\chi$ and $\coth\chi$, the operators $\tilde{A}_\pm$ could be cast in a form similar to the operators $A_\pm$ of Eq. (43), and we would get $A_\pm$'s that generate shape invariant Hamiltonians.

Hence to get shape invariant potentials, we require

$$
\left(\frac{1}{2z'}\frac{dz'}{d\chi} - \frac{1}{2z}\frac{dz}{d\chi}\right) = \alpha\tanh\chi + \beta\coth\chi. \qquad (58)
$$

This leads to

$$
z' = z^{1+\beta}(1-z)^{-\alpha-\beta}, \qquad (59)
$$

which is a constraint on the relationship between the variables z and r. Since these variables are already constrained by Eq. (54), only a handful of solutions would be compatible with both restrictions. The $z(r)$'s that are compatible with both Eqs. (54) and (59) are given by

$$
z^{1+\beta}(1-z)^{-\alpha-\beta} = \frac{2z(1-z)}{\sqrt{Q(z)}}, \qquad (60)
$$

where $Q(z)$ is a quadratic function of z. After some computation, we find that there is only a finite number of values of α, β which satisfy Eq. (60). These values are listed in Table I, and they exhaust all known shape invariant potentials that lead to the hypergeometric equation. Thus, if the requirement of Eq. (58) is, as we conjecture, the most general possibility, then the family of known shape-invariant potentials is the complete set of such potentials.

TABLE I. All allowed values of α, β and the superpotentials that they generate. Note that all known solvable potentials can be reached from these by special limits of μ_1 and μ_2 [19].

α	β	$z(r)$	Superpotential	Potential
0	0	$z = e^{-r}$	$\mu_1 \coth \frac{r}{2} + \mu_2$	Eckart
0	$-\frac{1}{2}$	$z = \sin^2 \frac{r}{2}$	$\mu_1 \mathrm{cosec}\, r + \mu_2 \cot r$	Gen. Pöschl-Teller trigonometric
0	-1	$z = 1 - e^{-r}$	$\mu_1 \coth \frac{r}{2} + \mu_2$	Eckart
$-\frac{1}{2}$	0	$z = \mathrm{sech}^2 \frac{r}{2}$	$\mu_1 \mathrm{cosech}\, r + \mu_2 \coth r$	Pöschl-Teller II
$-\frac{1}{2}$	$-\frac{1}{2}$	$z = \tanh^2 \frac{r}{2}$	$\mu_1 \tanh \frac{r}{2} + \mu_2 \coth \frac{r}{2}$	Gen. Pöschl-Teller
-1	0	$z = 1 + \tanh \frac{r}{2}$	$\mu_1 \tanh \frac{r}{2} + \mu_2$	Rosen -Morse

Interestingly, while the potential algebra of a general Natanzon system is $so(2,2)$, and requires two sets of raising and lowering operators $A_\pm$ and $B_\pm$, all translational shape invariant potentials turn out to need only one such set. For all SIPs of Table 4.1 of Ref. [1], one finds that all partner potentials are connected by change of just one independent parameter. Other parameters, if present, do not change from case to case. Thus there is a series of potentials that only differ in one parameter.

For example, the two shape invariant partner potentials of Rosen-Morse I form are given by

$$\begin{aligned} V_- &= a(a-1)\mathrm{cosec}^2 x + 2b \cot x - a^2 + \frac{b^2}{a^2}; \\ V_+ &= a(a+1)\mathrm{cosec}^2 x + 2b \cot x - a^2 + \frac{b^2}{a^2}. \end{aligned} \tag{61}$$

These two potentials are related by the transformation $a \longrightarrow a+1$, while b is merely a "spectator". This suggests a lower symmetry than $so(2,2)$. From the potential algebra perspective, all these potentials differ only by the eigenvalue of a *single* operator (a linear combination of A_3 and B_3), and all are characterized by a common eigenvalue of C_2. Thus, these shape invariant potentials can be associated with a $so(2,1)$ potential algebra generated by operators A_+, A_- and the linear combination of A_3 and B_3 mentioned above.

VI. Conclusions

In this paper, we have reviewed the topic of solvable shape invariant Hamiltonians from supersymmetric quantum mechanics. We have summarized the apparently unrelated topic of group symmetries known as potential algebras. We have then shown the relationship between the two. We have derived the potential algebras for shape invariant systems, where hierarchies of supersymmetric potentials are characterized by changes of parameters that are related by translational $[a_n = a_{n-1} + \delta]$, scaling $[a_n = q\, a_{n-1}]$, and mapping of the form $a_n = \frac{\alpha a_{n-1} + \beta}{\gamma a_{n-1} + \delta}$. (The last map leads to cyclic potentials.) In general, one finds deformations of the $so(2,1)$ Lie algebra. We have discussed these deformations, but then showed that for the translational case, they may be avoided by generalizing the operator structure to keep the resulting algebra linear. This led to the identification with Natanzon potentials.

Our approach therefore has linked the group theoretic (potential algebra) approach and the supersymmetric quantum mechanics approach for treating shape invariant potentials. Its application has led to the conclusion that the known family of exactly solvable SIP's is complete.

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References

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[1] F. Cooper, A. Khare and U. Sukhatme, Phys. Rep. **251**, 268 (1995) and references therein.

[2] L. Infeld and T. E. Hull, Rev. Mod. Phys., **23**, 21 (1951); L. E. Gendenshtein, Pismah. Eksp. Teor. Fiz. **38**, 299 (1983) [ETP Lett., **38**, 356 (1983)]; L. E. Gendenshtein and I. V. Krive, Sov. Phys. Usp. **28**, 645 (1985).

[3] J. Wu, Y. Alhassid and F. Gürsey, Ann. Phys. **196**, 163 (1989).

[4] J. Wu and Y. Alhassid, Phys. Rev. A **31**, 557 (1990).

[5] A. O. Barut, A. Inomata and R. Wilson, J. Phys. A: Math. Gen. **20**, 4075 (1987); J. Phys. A: Math. Gen. **20**, 4083 (1987); M. J. Englefield, J. Math. Phys. **28**, 827 (1987); M. J. Englefield and C. Quesne, J. Phys. A: Math. Gen. **24**, 3557 (1987); R. D. Tangerman and J. A. Tjon, Phys. Rev. A **48**, 1089 (1993).

[6] R. Dutt *et al*., Phys. Lett. **174A**, 2854 (1993).

[7] W. Pauli, Z. Physik **36**, 336 (1926); V. Fock, Z. Physik **98**, 145 (1935).

[8] B. G. Adams, J. Cizeka and J. Paldus, *Lie Algebraic Methods And Their Applications to Simple Quantum Systems*. (the Advances in Quantum Chemistry, Vol. 19), ed., Per-Olov Löwdin. Academic Press 1987.

[9] Asim Gangopadhyaya and Uday P. Sukhatme, Phys. Lett. **A224**, 5 (1996); U.P. Sukhatme, C. Rasinariu and A. Khare, Phys. Lett. **A234**, 401 (1997), (hep-ph/9706282).

[10] G. A. Natanzon, Teor. Mat. Fiz., **38**, 219 (1979).

[11] A. Gangopadhyaya, J. V. Mallow and U. P. Sukhatme, in *Proceedings of Workshop on Supersymmetric Quantum Mechanics and Integrable Models, June 1997*, eds. Henrik Aratyn *et al.*, (Springer-Verlag, Berlin, 1997).

[12] A. B. Balantekin, Phys. Rev. A **57**, 4188 (1998); (quant-ph/9712018).

[13] M. Rocek, Phys. Lett. B **255** 554 (1991); T. Curtright and C. Zachos, PhysLett **243B**, 237 (1990); T. Curtright, G. Ghandour and C. Zachos, J. Math. Phys. **32** 676 (1991).

[14] A. Shabat, Inverse Prob. **8**, 303 (1992); V. P. Spiridonov, Phys. Rev. Lett. **69**, 398 (1992).

[15] D. Barclay *et al*., Phys. Rev. **A48**, 2786 (1993).

[16] S. Chaturvedi *et al.*, Phys. Lett. **A248**, 109 (1998).

[17] A. Gangopadhyaya, J. V. Mallow and U. P. Sukhatme, Phys. Rev. **A 58**, 4287 (1998).

[18] A. Gangopadhyaya, P. Panigrahi and U. Sukhatme, Helv. Phys. Acta **67**, 363 (1994).