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Certain Transformations of the Apollonion Circles on the Triangle 1,W, and W2

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CERTAIN TRANSFORMATIONS OF THE APOLLONION CIRCLES ON
THE TRIANGLE 1, cJ, AND cJ*.

by

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CHAPTER I.

INTRODUCTION.

The problem to be discussed in this paper is, as the title of the thesis states, the transformation of the Apollonian circles lying on the triangle \( \triangle \alpha \varepsilon \omega \) in the complex plane. It is the author's intention to apply the linear transformation

\[
    z = k \frac{2 \omega - 1}{\omega - 2}
\]

to the above-mentioned circles and to find if any change in configuration has occurred. If there has, he will endeavor to graph this changed configuration. In conclusion, he will discuss a characteristic property which these Apollonian circles possess.

However, before attacking the problem itself, it is necessary to prove certain details which form a background for the problem.

CHAPTER II.

PRELIMINARY DISCUSSION.

In this and all the discussion to follow, we use the complex plane, as the plane of reference. Hence \( z = x + iy \)
and $\tilde{z} = x - iy$ where $\tilde{z}$ is the image of $z$ with respect to the X-axis, the axis of reals. We will also use in this discussion, a base circle, which is a unit circle about the origin, and a set of Apollonion circles.

This set consists of three circles, all passing through the base circle in such a way that the intersections of two of the circles with the base circle are images with respect to the third circle - e.g.

\[ \text{Figure I.} \]

In the above figure we have three circles intersecting the base circle in the points $a$, $b$, and $c$ respectively. Now the circle $a$ passes thru the point $a$ in such a way that the points $b$ and $c$, the intersections of the circles $b$ and $c$, respectively, with the base circle, are image points. The circle through $b$ has $a$ and $c$ as image points. The circle through $c$ has $a$ and $b$ as image points. This constitutes a set of Apollonion circles.

As stated above, we have chosen the three points $1$,
and \( w \) as the points in which the three circles intersect the base circle. We also place on these circles the further condition that they pass through the origin. Thus \( l \) lies on the X-axis. The circle through \( l \) will, therefore, make an angle of \( 0^\circ \) with the X-axis. Since any complex number \( \gamma = \rho \left( \cos \phi + i \sin \phi \right) \) the number \( \omega \) will, therefore, make an angle of \( 120^\circ \) with the X-axis and the number \( \omega^2 \) an angle of \( 240^\circ \) with the X-axis. Each of these circles, therefore, makes an angle of \( 120^\circ \) with the succeeding circle. In the figure below \( O \) is the origin, Circle A passes through the point \( l \), Circle B passes through the point \( \omega \) and circle C passes through the point \( \omega^2 \). \( a, b, \) and \( c \) are radii through the base circle. They are those parts of the circles A, B, and C which are inside of the base circle.

Now, in the triangles \( \triangle \omega \omega^2 \omega^3 \) \( \triangle \omega^2 \omega^3 \omega \) \( \triangle \omega^1 \omega \omega^2 \) we have side \( a \) = side \( b \) = side \( c \), because all these are radii of the base circle. Since the angle between two successive circles in the figure is \( 120^\circ \), then angle \( \omega \omega^1 \angle \omega^1 \omega^2 \angle \omega^0 \omega^1 = \angle \omega^1 \omega^0 \omega^2 \). We have, therefore, three triangles with two sides and the included angle of one equal respectively to the two sides and the included angle of the other. Therefore, line \( \omega \omega^1 \omega^2 \omega = \omega \omega \omega^1 \omega^2 \).
Circle A passes through \( \omega \) and has \( \omega' \) as images. Therefore, circle A bisects the line \( \omega \omega' \) since \( \angle \omega \omega' = \angle \omega \omega' \).

But \( \lambda, \omega, \) and \( R \) all lie on a straight line. Therefore, circle A is a degenerate circle, i.e. a straight line.

Circle B passes through the point \( \omega \) and has \( \omega' \) and \( \lambda \) as images. Now angle \( \omega \omega = \angle \omega \omega' \), because triangles \( \triangle \omega \omega' \omega \) and \( \triangle \omega \omega' \lambda \) were proved congruent.

Angle \( \angle \omega \omega' = \angle \omega \omega' \). Both are right angles since, by the definition of images, \( \omega \omega' \) is perpendicular to circle A and \( \omega \omega' \) is perpendicular to circle B. Therefore, angle \( \omega \omega' = \omega' \omega S \).

Now, side \( \omega \omega' = \omega' \omega' \). We have, then, two angles and the included side of triangle \( \omega \omega' \omega \) to the two respective angles and the included side of the triangle \( \omega' \omega S \). Therefore, the triangles are congruent, and side \( \omega \omega' = \omega' \omega' \). But \( \omega \omega' = \frac{1}{2} \omega \omega' \) and \( \omega' \omega' = \omega' \omega' \). Therefore, \( \omega' \omega' = \frac{1}{2} \omega' \omega' \).

Thus we see that circle B bisects the line \( \lambda \).

\( \omega' S = S' \).

But \( \omega' \omega S = \angle S \omega' \). Therefore, \( \omega S \) is a straight line. Now angle \( \omega S \omega = \angle \omega' \omega S \). But both \( \omega' \omega + \omega S \).

\( \omega S \) and \( \omega \omega' \) are a part of circle B. However, three points determine a straight line. Therefore, circle B is also a degenerate circle, or a straight line. By a similar argu-
ment, we can prove that circle $C$ is a straight line. Thus we see that the three Apollonion circles passing through the points $I, w, w'$ of the base circle degenerate into straight lines.

Figure 2.
It will prove very useful in the solution of the general problem to have the equations of these straight lines. In order to do this, we must first develop the general equations of a circle and of a straight line in \( z \bar{z} \) coordinates, where \( z \) is the complex quantity \( x + iy \) and \( \bar{z} \) is its image point \( x - iy \).

As a preliminary, we need to prove:

**THEOREM** - If a quantity is equal to its conjugate, then the quantity is real.

**PROOF** - Suppose we are given a complex quantity \( \phi = a + ib \)

and we are told that

\[ \phi = \bar{\phi} \]

then

\[ a + ib = a - ib \]

\[ 2ib = 0 \]

Now neither \( l \) nor \( b \) are equal to zero.

Therefore, \( b \) must be equal to zero.

Therefore

\[ a = a \]

or \[ \phi = \bar{\phi} \] Therefore \( \phi \) is real.

Now we develop the general equation of a circle in coordinates.
In Cartesian coordinates, the equation of a circle is given by the following equation:

\[ A(x^2 + y^2) + 2Dx + 2Ey + F = 0 \]  

Now

\[ z = x + iy \]
\[ \bar{z} = x - iy \]

Adding

\[ \frac{z + \bar{z}}{2} = x \]

Subtracting

\[ \frac{z - \bar{z}}{2i} = y \]

Substitute these values of \( z \) and \( \bar{z} \) in the general equation of a circle where \( A, D, E, \) and \( F \) are all real.

\[ A\left(\frac{z + \bar{z}}{2}\right)^2 + \left(\frac{z - \bar{z}}{2i}\right)^2 + 2D\frac{z + \bar{z}}{2} + 2E\frac{z - \bar{z}}{2i} + F = 0 \]
\[ Az + (D + Ei)z + (D - Ei)\bar{z} + F = 0 \]  
\[ Az + (D - Ei)z + (D + iE) + F = 0 \]

is the general equation of a circle in \( z \bar{z} \) coordinates. Its conjugate is

\[ Az + (D + iE)z + (D - iE)\bar{z} + F = 0 \]  

since \( E \) is a real quantity.

Therefore, the equation is real. Therefore, it is a real curve or locus.

Now in figure 3 \( w \) is the image point of \( z \) in \( \mathbb{L} \).
and "a" the image of the base point zero in L.

Triangle \( \gamma_0 a \) is reversely similar to triangle \( w a \).

Now the condition for two triangles to be reversely similar is

\[
\begin{vmatrix}
  a & b & c \\
  \bar{p} & \bar{q} & \bar{r}
\end{vmatrix} = 0
\]

where \( a b c \) is one triangle and \( p q r \) the triangle reversely similar to triangle \( a b c \).

\[
\begin{vmatrix}
  \gamma & \bar{a} & \bar{a} \\
  \bar{w} & \bar{a} & \bar{a}
\end{vmatrix} = 0
\]
is the condition that triangle \( z o a \) and \( w a o \) be reversely similar, or
\[
\frac{z}{a} + \frac{\bar{w}}{\bar{a}} = 1 \tag{4}
\]

Let \( z = L \). Then \( \omega \), the image of \( z \) in \( L \), will also approach \( L \).

In the limit
\[
\frac{z}{a} + \frac{\bar{z}}{\bar{a}} = 1
\]

becomes the equation of \( L \), the locus of coincident images.

We have, then, the equation of a straight line in \( z \bar{z} \) coordinates.

Let us now develop the equivalent Cartesian equation of this line \( L \).

\[
\frac{x + iy}{a} + \frac{x - iy}{\bar{a}} = 1
\]
\[
a(x + iy) + \bar{a}(x - iy) = 1
\]
\[
x(\bar{a} + a) + iy(\bar{a} - a) = a\bar{a}
\]

Let \( a \) be represented by the complex number \( b + ic \) where \( b \) and \( c \) are real.

Then \( \bar{a} = a \)

Substituting
\[
2by + 2cy = b^2 - c^2
\]
\[
y = \frac{-b}{2c} x + \frac{b^2 - c^2}{2c} \tag{5}
\]
This is in the form
\[ y = mx + b \]
Therefore (5) is the equation of a straight line in Cartesian coordinates.

Thus we have verified (4) to be the equation of a straight line in \( \mathbb{R}^2 \) coordinates.

\[ \frac{z}{a} + \frac{\bar{z}}{\bar{a}} = 1 \]
\[ z + \frac{a\bar{z}}{\bar{a}} = a \]  \hspace{1cm} (6)

Let \( a \neq 0 \)

Then \( a = 0 \) which is indeterminate.

However, by Euler's formula

\[ a = \rho e^{i\phi}, \quad \bar{a} = \rho e^{-i\phi} \]

Therefore

\[ \frac{a}{\bar{a}} = \frac{\rho e^{i\phi}}{\rho e^{-i\phi}} = e^{2i\phi} \]  \hspace{1cm} (7)

Now let \( \rho = 0 \) in (6)

Therefore, \( a \neq 0 \) and (6) becomes

\[ z + e^{i2\phi} \bar{z} = 0 \]  \hspace{1cm} (8)
Write $e^{i\theta}$ (a turn)

Therefore $e^{i2\theta} = t$ 

and $z + t\bar{z} = 0$ 

Now let $\theta = 0^\circ$

$$e^{i2\theta} = e^{i0} = (\cos 0^\circ + i \sin 0^\circ) = 1$$

Therefore, (8) becomes

$$z + \bar{z} = 0$$

Changing from $z$ to Cartesian coordinates

$$x + iy + x - iy = 0$$

$$x = 0$$

Therefore $z + \bar{z} = 0$ is the equation of the X-axis.

If $\theta = \frac{\pi}{2}$

$$e^{i\theta} = e^{i\pi/2} = (\cos \pi/2 + i \sin \pi/2) = -i$$

Substitute in (8) and we obtain

$$z - \bar{z} = 0$$

Changing from $z$ to Cartesian coordinates

$$x + iy - x + iy = 0$$

$$y = 0$$

Therefore, $z - \bar{z} = 0$ is the equation of the X-axis.

However, the circle A passes through the points zero and 1, and is a straight line. Therefore, the line A is identical with the X-axis. Therefore the equation of the line A is

$$z - \bar{z} = 0$$
We may develop this equation in another way as follows:

If we take the general equation of the circle

\[ a z \bar{z} + b \bar{z} + c = 0 \tag{17} \]

we notice that in order that this equation denote a real locus, and \( c \) in the equation must be real. If they are not real, the equation is not self-conjugate, and, therefore does not represent a real locus. Since \( a \neq \bar{a} \) and \( c \neq \bar{c} \) if \( a \) and \( c \) are not real.

Now, from the theory of quadratic equations, if the coefficient of the second degree term in a quadratic equation is equal to zero, then infinity is a root of the equation, i.e., the curve passes through infinity, which means that the curve degenerates into a straight line. Also from the theory of quadratic equations, if the constant term is equal to zero, then zero is a root of the equation. But if both \( a \) and \( c \) are equal to zero, then the curve degenerates into a straight line through the origin. Therefore, from (17), the equation of a straight line through the origin is in the form

\[ \bar{z} \bar{z} + \bar{b}z = 0 \tag{18} \]

We may also use a third method as follows:

From (10)

\[ z + \bar{z} = 0 \tag{19} \]

\[ \bar{z} \bar{z} + t \bar{z} = 0 \]
But \[ \ddot{t} = \frac{1}{t} \]

Therefore (19) becomes

\[ \ddot{t} - t \dot{t} = 0 \]  \hspace{1cm} (20)

Line A passes through the point \( \ddot{z} = 1 \). Since this is a real point \( \ddot{z} = 1 \), substituting in (20) we get

\[ \ddot{t} + t = 0 \]

Therefore

\[ t = -\ddot{t} \]  \hspace{1cm} (21)

Thus, the equation of the line through the point \( \ddot{t} \) becomes

\[ \ddot{t} - \dot{t} \dot{z} = 0, \quad \text{or} \]

\[ \ddot{t} - \ddot{z} = 0 \]  \hspace{1cm} (22)

which is the same as equation (15).

We have now, therefore, a method for finding the equation of a straight line through the origin and through any point \( z \).

Since the line B passes through the point \( \ddot{w} \), we let \( \ddot{z} = \ddot{w} \).

Then \( \ddot{z} = \ddot{w} \).

Substituting in (20)

\[ \ddot{t} \ddot{w} + t \ddot{w} = 0 \]

\[ \ddot{t} = -\frac{t \ddot{w}}{\ddot{w}} \]  \hspace{1cm} (23)

Therefore (20) becomes

\[ \frac{t \ddot{w} + \ddot{t}}{\ddot{w} \ddot{z} - \ddot{w} \ddot{z}} = 0 \]  \hspace{1cm} (24)
is the equation of the line B which passes through the point \( \sigma \). The line C passes through the point \( \sigma' \). Therefore \( z = \sigma \) and \( \bar{z} = \sigma' \).

Substituting these values of \( z \) and \( \overline{z} \) in (20) we have

\[
\frac{1}{z} - \frac{t}{\sigma} = 0
\]

and thus the equation (20) becomes

\[
\frac{-t \bar{\sigma}}{\sigma} z + \bar{z} = 0
\]

\[
\bar{z} - \sigma' \bar{z} = 0
\]

which is the equation of the Apollonian circles through the points \( \sigma \).

We have now the equations of the Apollonian circles through the points \( i, \sigma \), and \( \sigma' \). Thus we are now in a position to attack the problem suggested in the title.
CHAPTER III.

TRANSFORMATIONS.

As we stated previously, the general transformation which we shall use on these circles is

\[ z = k \frac{2w - 1}{w - 2} \quad (27) \]

where \( k \) is arbitrary.

For the first transformation let \( k \) equal to 1. Then

\[ \bar{z} = k \frac{2\overline{w} - 1}{\overline{w} - 2} \quad (28) \]

or, in general

\[ z = k \frac{2w - 1}{w - 2} \quad \text{and} \quad \bar{z} = k \frac{2\overline{w} - 1}{\overline{w} - 2} \quad (29) \]

Using (28) on the line \( A \), we get

\[ z - \bar{z} = \frac{2w - 1}{w - 2} - \frac{2\overline{w} - 1}{\overline{w} - 2} \quad (30) \]

However \( z - \bar{z} = 0 \). Therefore

\[ \frac{2w - 1}{w - 2} - \frac{2\overline{w} - 1}{\overline{w} - 2} = 0 \quad (31) \]

Clearing fractions, we obtain

\[ \overline{w} - w = 0 \quad (32) \]
But this is the equation of a straight line through the origin. (cf. (18)) Therefore, the straight line \( \bar{z} - z = 0 \) has been unaltered by transformation (28).

In this and in all the work to follow let us assume that we have two complex planes - the Z-plane, and the W-plane; Let the original lines be in the Z-plane and the curves resulting from the transformations in the W-plane, i.e. let the Z-plane be mapped-on to the W-plane. Also let the coordinates in the Z-plane be given by \( x + iy \) and in the W-plane by \( u + iv \) where the U-axis is the axis of reals corresponding to the X-axis in the Z-plane.

On changing to Cartesian coordinates, \( \bar{w} - w = 0 \) becomes

\[
(w + iv) - u - iv = 0
\]

\[
\bar{w} - w = 0 \tag{33}
\]

Therefore, the line A in the Z-plane goes into the U-axis in the W-plane, i.e. a straight line goes into a straight line under (28).

Let us now transform line B by (28).

\[
\omega \bar{q} - \omega \bar{\bar{q}} = 0
\]

Substituting the values of and from (28),

\[
\frac{2w - \bar{w}}{w - 2} - \omega \frac{2\bar{w} - 1}{\bar{w} - 2} = 0 \tag{34}
\]
Clearing fractions, we have

\[2 \bar{w}(\bar{w} - w) + w(\bar{w} - 4w) + \bar{w}(4w - \bar{w}) + 2(\bar{w} - w) = 0\]

But \(\bar{w} = w\)

Therefore

\[2 \bar{w} + w \frac{1 - 4w}{\bar{w} - w} + \bar{w} \frac{4 - \bar{w}}{\bar{w} - w} + 2 = 0\]  \((35)\)

We notice that (35) is of the same form as equation (17).

But equation (17) is the general equation of a circle.

Therefore, equation (35) is the equation of a circle. Hence the straight line \(B\) in the \(Z\)-plane has been transformed under (28) into a circle in the \(W\)-plane. (See Chart I.)

Changing to Cartesian coordinates (36) becomes

\[2(w^2 + v^2) + \frac{1 - 4w}{w - 1}(w + iv) + \frac{4 - \bar{w}}{\bar{w} - 1}(w - iv) + 2 = 0\]

\[\frac{1 - 4w}{w - 1} = \frac{5 + i\sqrt{3}}{2}\]

\[\frac{4 - w}{\bar{w} - 1} = \frac{5 - i\sqrt{3}}{2}\]  \((37)\)

\[\therefore 2(w^2 + v^2) + \left(\frac{5 + i\sqrt{3}}{2}\right)(w + iv) + \left(\frac{5 - i\sqrt{3}}{2}\right)(w - iv) + 2 = 0\]  \((38)\)

\[w^2 + v^2 - \frac{5}{2}w - \frac{\sqrt{3}}{2}v + 1 = 0\]  \((39)\)
is the equation of a circle B in Cartesian coordinates. 
(See chart I.)

Now let us transform the line C by this same transformation. From (26), we have as the equation of the line C i.e. the line through the point $\omega^*$. 

$$\bar{\omega}^* - \omega^* = 0$$

Substituting the value of $\bar{\omega}$ and $\omega$ from (28), we have

$$\bar{\omega}^* - \omega^* = 0$$

$$2\bar{\omega}(\bar{\omega}^* - \omega^*) + \omega(\omega + \bar{\omega}) + \bar{\omega}(\bar{\omega} - \omega^*) + 2(\bar{\omega} - \omega^*) = 0$$

$$2\bar{\omega} + \omega \left( \frac{\omega - \bar{\omega}^*}{\bar{\omega}^* - \omega^*} \right) + \bar{\omega} \left( \frac{\bar{\omega} - \omega^*}{\omega^* - \omega} \right) + 2 = 0$$

Since $\bar{\omega} = \omega^*$

$$2\bar{\omega} + \omega \left( \frac{\omega - \bar{\omega}}{1 - \omega} \right) + \bar{\omega} \left( \frac{\bar{\omega} - \omega}{1 - \omega} \right) + 2 = 0$$

which is the equation of a circle in complex coordinates. Therefore, the line C in the Z-plane has been transformed into a circle in the W-plane by transformation (28). (See Chart I.)

To change the above equation from the complex coordinates to Cartesian coordinates, we proceed as follows:
\[ 2(w^* + w) + \left(4w - 1\right)(w - iv) + \left(w - 4\right)(w + iv) + 2 = 0 \quad (43) \]

But
\[ \frac{4w - 1}{w - iv} = \frac{5 + i\sqrt{3}}{2} \quad \frac{w - iv}{w} = \frac{5 - i\sqrt{3}}{2} \quad (44) \]

Therefore, we have
\[ 2(w^* + w) + \left(\frac{5 + i\sqrt{3}}{2}\right)(w - iv) + \left(\frac{5 - i\sqrt{3}}{2}\right)(w + iv) + 2 = 0 \quad (45) \]
\[ \therefore \quad w^* + w = \frac{5}{2} w + \frac{\sqrt{3}}{2} v + 1 = 0 \]
which is the equation of circle C in Cartesian coordinates.

To summarize, we have used the transformation (27) on the lines A, B, and C in the Z-plane and obtained a line, a circle, and a circle respectively in the W-plane. Therefore the line A is the only one which remained unchanged under transformation (27), where \( k = 1 \). It should also be noticed that the lines B and C, which are degenerate circles, have been transformed by a linear transformation into circles which agrees with the general principle that circles always go into circles under a linear transformation.

Let \( k = i \) in equation (27). Then
\[ z = i \frac{2w - 1}{w - 2} \quad \text{and} \quad \bar{z} = i \frac{2w - 1}{w - 2} \quad (46) \]

Now, \( i \) is a complex quantity, for
\[ e^{i\theta/2} = (\cos \frac{\theta}{2} + i \sin \frac{\theta}{2} = i \quad (4.7) \]
\[ i = (\cos \frac{\theta}{2} - i \sin \frac{\theta}{2} = -i \quad (4.8) \]

Therefore, transforming the line \( z - \bar{z} = 0 \), we obtain

\[ i \frac{2w - 1}{w - 2} + i \frac{2\bar{w} - 1}{\bar{w} - 2} = 0 \]
\[ 4w\bar{w} - 5w - 5\bar{w} + 4 = 0 \quad (49) \]

which is the equation of a circle in complex coordinates.

Therefore, line A is transformed into a circle by (46.)

Changing to Cartesian coordinates, we get

\[ 4(u + iv)(w - iv) - 5(u + iv) - 5(u - iv) + 4 = 0 \]
\[ u^2 + v^2 - 5/2u + 1 = 0 \quad (50) \]

which gives the Cartesian equation of circle A (Chart II).

Since the \( v \) term is missing, this shows that the circle does not cross the \( V \)-axis.

Subjecting the line B, \( w\bar{z} - w\bar{z} \) to the same transformation, we have

\[ \bar{w}i \frac{2w - 1}{w - 2} + wi \frac{2\bar{w} - 1}{\bar{w} - 2} = 0 \]
\[ 2w\bar{w} - w(\bar{\bar{w}} + 4w) - \bar{w}(4w + \bar{w}) + 2(\bar{\bar{w}} + \bar{w}) = 0 \]
\[ 2w\bar{w} - w \left( \frac{c\bar{w} + \bar{w}}{\bar{w} + c\bar{w}} \right) - \bar{w} \left( \frac{4\bar{\bar{w}} + \bar{w}}{\bar{w} + 4\bar{w}} \right) + 2 = 0 \quad (51) \]
But \( \bar{w} = w \)

Therefore
\[
2\bar{w}w - \bar{w}\left(\frac{w + 4}{w + 1}\right) - w\left(\frac{4w + 1}{w + 1}\right) + 2 = 0
\]

This is the equation of a circle in complex coordinates. Therefore, line B is the Z-plane is transformed into a circle in the W-plane. (Chart II.)

Upon changing this equation into Cartesian coordinates, we obtain the following:

Since
\[
\frac{w + 4}{w + 1} = \frac{5 - 3i\sqrt{3}}{2} \quad \frac{4w + 1}{w + 1} = \frac{5 + 3i\sqrt{3}}{2}
\]

Therefore,
\[
2(w' + w^*) - (w' + iw^*)(\frac{5 - 3i\sqrt{3}}{2}) - (w' + iw^*)\left(\frac{5 + 3i\sqrt{3}}{2}\right) + 2 = 0
\]

which is the Cartesian equation of Circle B in the W-plane (Chart II.)

Subjecting the line C to this same transformation, we have
\[
\bar{w}i\left(\frac{2w - 1}{w - 2}\right) + w'i\left(\frac{2\bar{w} - 1}{\bar{w} - 2}\right) = 0
\]

\[
2\bar{w}w - \bar{w}(w' + 4w^*) - w(4\bar{w}' + w^*) + 2 = 0
\]

\[
2\bar{w}w - \bar{w}\left(\frac{w' + 4\bar{w}^*}{\bar{w}' + \bar{w}^*}\right) - w\left(\frac{4w' + w^*}{\bar{w}' + \bar{w}^*}\right) + 2 = 0
\]
Therefore

\[ 2 \overline{w}w - \overline{w} \left( \frac{1 + 4w}{1 + \overline{w}} \right) - w \left( \frac{4 + w}{1 + \overline{w}} \right) + 2 = 0 \]  \hspace{1cm} (59)

This, however, is the equation of a circle in \( z \overline{z} \) coordinates. Therefore, the line C is transformed into a circle by transformation (46).

Its Cartesian equation is obtained as follows:

\[ \frac{1 + 4w}{1 + \overline{w}} = \frac{5 + 3i\sqrt{3}}{2} \quad \frac{4 + w}{1 + \overline{w}} = \frac{5 - 3i\sqrt{3}}{2} \]  \hspace{1cm} (60)

Therefore

\[ 2 (w + \overline{w}) - (w - i\overline{w}) \left( \frac{5 + 3i\sqrt{3}}{2} \right) (w + i\overline{w}) \left( \frac{5 - 3i\sqrt{3}}{2} \right) + 2 = 0 \]  \hspace{1cm} (61)

\[ w^2 + \overline{w}^2 - \frac{5}{2}w - \frac{3}{2}i\sqrt{3} \overline{w} + 1 = 0 \]  \hspace{1cm} (62)

is the equation of circle C in Cartesian coordinates.

We find thus that by means of transformation (46) all the three lines in the Z-plane are transformed into circles in the W-plane. (Chart II.)

Now let \( k = \omega \) in equation (27).

Then

\[ z = \omega \frac{2w - 1}{w - 2} \quad \text{and} \quad \overline{z} = \overline{\omega} \frac{2w - 1}{w - 2} \]  \hspace{1cm} (63)
If we apply this transformation to the line \( z - \bar{z} = 0 \) we obtain

\[
\frac{2\omega - 1}{\omega - 2} - \frac{2\bar{\omega} - 1}{\bar{\omega} - 2} = 0
\]

(64)

\[
2\omega\bar{\omega} - \bar{\omega}(\omega - 4\bar{\omega}) - \omega(4\omega - \bar{\omega}) + 2(\omega - \bar{\omega}) = 0
\]

(65)

But \( \bar{\omega} = \omega^* \)

Therefore

\[
2\omega\bar{\omega} - \bar{\omega}\left(\frac{1 - 4\omega}{1 - \omega}\right) - \omega\left(\frac{4 - \omega}{1 - \omega}\right) + 2 = 0
\]

(66)

Hence the line is transformed into a Circle in the W-plane by transformation (62.)

Changing to Cartesian coordinates, we have

\[
\frac{1 - 4\omega}{1 - \omega} = \frac{5 - i\sqrt{3}}{2} \quad \frac{4 - \omega}{1 - \omega} = \frac{5 + i\sqrt{3}}{2}
\]

(67)

\[
2(\omega + \nu)(\omega + i\nu)(\frac{5 - i\sqrt{3}}{2}) - (\omega + i\nu)(\frac{5 + i\sqrt{3}}{2}) + 2 = 0
\]

\[
\omega + \nu = \frac{5\omega}{2} + \frac{\sqrt{3}}{2} \nu + 1 = 0
\]

(68)

which is the equation of Circle A in the W-plane. (Chart III.)

Applying this transformation to the line B in the Z-plane,

\[
\frac{2\omega - 1}{\omega - 2} - \omega \frac{2\bar{\omega} - 1}{\bar{\omega} - 2} = 0
\]

(69)

\[
\therefore \bar{\omega} - \omega = 0
\]

Thus, the line B is transformed into a straight line in the
W-plane by (62.)

Changing to Cartesian coordinates

\[(w + iw) - (w - iw) = 0\]
\[w = 0\]  \hspace{1cm} (70)

we obtain the equation of the U-axis in the W-plane.

(Chart III). Therefore, the line B of the Z-plane is transformed into the U-axis of the W-plane.

Subjecting the line C of the Z-plane to this transformation, we have

\[\frac{2w-1}{w-2} - \frac{2\bar{w}-1}{\bar{w}-2} = 0\]  \hspace{1cm} (71)

\[2w\bar{w} (\bar{w}w - \bar{w}^2) - \bar{w}(\bar{w}^2w - 4w\bar{w}) - w.\]  \hspace{1cm} (72)

\[(4\bar{w}w - w^2\bar{w}) + 2(\bar{w}w - w\bar{w}) = 0\]

But \(\bar{w} = w\) \hspace{1cm} and \(\bar{w} = w\)

Therefore

\[2w\bar{w} - w(\frac{w+4}{w-1}) - w(\frac{4w-1}{w-1}) + 2 = 0\]  \hspace{1cm} (73)

Therefore the line C of the Z-plane is transformed into a
circle in the W-plane. (Chart III.)

Upon changing to Cartesian coordinates, we have

\[ \frac{w-4}{w-1} = \frac{5+i\sqrt{3}}{2} \quad \frac{4w-1}{w-1} = \frac{5-i\sqrt{3}}{2} \]  \hspace{1cm} (74)

\[ 2(w^2 + v^2) - (w + iv)(\frac{5+i\sqrt{3}}{2}) - (w - iv)(\frac{5-i\sqrt{3}}{2}) + 2 = 0 \]  \hspace{1cm} (75)

\[ w^2 + v^2 - \frac{5}{2}w - \frac{13}{2}v + 1 = 0 \]  \hspace{1cm} (76)

This is the Cartesian equation of the Circle C in the W-plane into which the line C of the Z-plane has been transformed by (62).

In summary, we see that our original line of the Z-plane has been transformed into a circle in the W-plane, line B went into another straight line, and line C was transformed into a circle under transformation (63).

Now let us have \( z = w^2 \) in equation (27). Then

\[ z = w^2 \left( \frac{2w-1}{w-2} \right) \quad \bar{z} = \bar{w}^2 \left( \frac{2\bar{w}-1}{\bar{w}+2} \right) \]  \hspace{1cm} (77)

Applying this transformation to the line A in the Z-plane

\[ w^2 \frac{2w-1}{w-2} - \bar{w}^2 \frac{2\bar{w}-1}{\bar{w}+2} = 0 \]  \hspace{1cm} (78)

\[ 2w\bar{w}(w^2 - \bar{w}^2) - w(w^2 - 4\bar{w}^2) - \bar{w}(4w^2 - \bar{w}^2) + 2. \]  \hspace{1cm} (79)

But \( \bar{w}^2 = \bar{w} \)

Therefore

\[ 2w\bar{w} - w \frac{w-4}{w-1} - w + 4 \frac{w-1}{w-1} + 2 = 0 \]  \hspace{1cm} (80)
This is the equation of a circle in coordinates. Therefore the line A has been transformed into a circle by (77.)

Since
\[
\frac{\omega - 4}{\omega - 1} = \frac{5 + i\sqrt{3}}{2}, \quad \frac{4\omega - 1}{\omega - 1} = \frac{5 - i\sqrt{3}}{2} \tag{81}
\]

Therefore the resulting equation in Cartesian coordinates is
\[
2(\omega + \omega^*) - (\omega + i\omega^*)\left(\frac{5 + i\sqrt{3}}{2}\right) - (\omega + i\omega^*) \cdot \omega + \omega^* \tag{82}
\]
\[
\omega^2 + \omega^* = -5/2\omega - i\sqrt{3}/2\omega + 1 = 0 \tag{83}
\]

which is the Cartesian equation of circle A in the W-plane.

(Chart IV.)

If the line B undergoes this transformation, we obtain
\[
\bar{\omega} \omega - \frac{2\omega - 1}{\omega - 2} - \bar{\omega} \omega - \frac{2\bar{\omega} - 1}{\bar{\omega} - 2} = 0 \tag{84}
\]
\[
2\omega \omega^* (\bar{\omega} \omega^* - \bar{\omega} \omega^*) - \bar{\omega} (\bar{\omega} \omega^* + 4\omega \omega^*) - \omega
\]
\[
(4\bar{\omega} \omega^* - \bar{\omega} \omega^*) + 2(\bar{\omega} \omega^* - \bar{\omega} \omega^*) = 0 \tag{85}
\]

But \(\bar{\omega} = -\omega^*\) and \(\bar{\omega}^* = \omega\)

Therefore
\[
2\omega \omega^* (\omega - \omega^*) - \bar{\omega} (\omega + 4\omega^*) - \omega (4\omega - \omega^*) + 2 = 0 \tag{86}
\]
Therefore
\[
2\omega \omega^* - \omega \frac{1 - 4\omega}{1 - \omega} - \omega \frac{4 - \omega}{1 - \omega} + 2 = 0 \tag{87}
\]
Since this is the equation of a circle, it shows that line B has also been transformed into a circle in the W-plane.

To find the Cartesian equation equivalent to the above equation, we have

$$\frac{1 - 4\omega}{1 - \omega} = \frac{5 - i\sqrt{3}}{2} \quad \frac{1 + \omega}{1 - \omega} = \frac{5 + i\sqrt{3}}{2}$$  \hspace{1cm} (88)

$$2(\omega + \nu) - (\omega + i\nu)(\frac{5 - i\sqrt{3}}{2}) - (\omega - i\nu)(\frac{5 + i\sqrt{3}}{2}) + 2 = 0$$  \hspace{1cm} (89)

$$\omega + \nu + \frac{\sqrt{3}}{2}\omega + \frac{i\sqrt{3}}{2}\nu + 1 = 0$$  \hspace{1cm} (90)

This, then, is the Cartesian equation of Circle B in the W-plane, into which the line B has been transformed under (77).

Subjecting line C to this same transformation, we have another straight line as the result of the process.

$$\frac{\bar{\omega} - 2\bar{\omega} - 1}{\omega - 2} - \frac{\omega - 2\bar{\omega} - 1}{\bar{\omega} - 2} = 0$$  \hspace{1cm} (91)

$$\omega - \bar{\omega} = 0$$  \hspace{1cm} (92)

This is the equation of a straight line in \textit{Cartesian coordinates}.

Changing to Cartesian coordinates,

$$(\omega + i\nu) - (\omega - i\nu) = 0$$  \hspace{1cm} (93)

$$\nu = 0$$  \hspace{1cm} (94)

But this is the equation of the U-axis in the W-plane.
Therefore, the line C of the Z-plane is transformed into the U-axis of the W-plane by (77).

Thus, we see that the line A is transformed into a circle, line B into a circle, and line C into another line by (77.)
CHAPTER IV.

GRAPHING

Thus far, we have been varying the value of $k$ in our transformation. Now, we shall endeavor to plot these circles. In doing this, we make use of the following information:

The three straight lines in the $Z$-plane all pass thru the same two points, zero and infinity. (All straight lines meet at infinity, which in the complex plane, is a point.) Now, in general, circles go into circles, and points on the circle go into points on the circle, under a bilinear transformation. (Cf. E. J. Townsend - "Functions Of A Complex Variable" Theorem III. p. 175) But our straight lines of the $Z$-plane are, in fact, degenerate circles. Therefore, if the straight lines pass through certain points before transformation, they will also pass through the same points transformed, after the transformation; i.e. since these lines pass through the points zero and infinity before transformation, they will also pass through the points into which zero and infinity will be transformed.

Also, the line $A$ in the $Z$-plane passes through the point $I$, the line $B$, through the point $\omega$, and the line $C$ through the point $\omega^*$. Thus we are given three points
of each circle. Since only three points are necessary to plot a given circle, we thus can plot all our circles.

First let us find the points into which zero and infinity are transformed by our general transformation

$$z = k \frac{2w-1}{w-2}$$

Set

$$z = 0$$

$$0 = \frac{2kw-k}{w-2}$$

$$w = \frac{k}{2k} = \frac{1}{2}$$

Therefore, zero goes into 1/2 in all the given transformations, since it has been shown to be independent of the value of k.

For infinity, let us substitute the letter a. Setting

$$z = a$$

we have

$$a = k \frac{2w-1}{w-2}$$

$$w - 2 = \frac{2kw-k + 2}{a}$$

$$w = \frac{2kw-k + 2}{a}$$

Let $$a = \infty$$, then since k and w are finite numbers

$$w = 0 + 2$$

$$w = 2$$

Therefore, $$\infty$$ goes into the point 2 in all the pre-
ceeding transformation, irrespective of the value of $k$. This also holds for $z = 0$.

Thus, since all the straight lines pass through the points zero and infinity before transformation, the circles or lines resulting from the respective transformations all will pass through the points $1/2$ and $2$ of the $W$-plane.

Now, the line $A$ of the $Z$-plane also passes through the point $I$. Putting $z = 1$, we have

$$l = \frac{2w - 1}{w - 2}$$

Therefore, the point $I$ in the $W$-plane corresponds to the point $I$ in the $Z$-plane. We have thus, the line $A$ of the $Z$-plane, which passes through the points $0, \infty, \text{and} \pm l$, in the $Z$-plane, passing through the points $1/2, 2, \text{and} -I$, corresponding respectively to the above points in the $W$-plane. (Chart I.) But these are all points on the real axis $U$. Hence the line $A$ of the $Z$-plane goes into the $U$-axis of the $W$-plane.

The line $B$ of the $Z$-plane passes through the point $ce$. Placing $z = ce$ we have,

$$ce = \frac{2w - 1}{w - 2}$$

and

$$w = \frac{1}{1/4} - \frac{3/4 + \sqrt{3}}{1}$$
which is the point in the W-plane, corresponding to the point \( w \) in the Z-plane. Hence, line \( \beta \) goes into a circle in the W-plane, which passes through the points \( 1/2, 2, \) and \( W \). (Chart I.)

Since line \( C \) passes through the point \( \infty \), we set \( z = \infty \) and obtain

\[
\frac{w}{w - 2} = \frac{2w - 1}{w - 2}
\]

(101)

\[
w = \frac{13/4 + 3/4 i \sqrt{3}}{1/4 + 3/4 i \sqrt{3}}
\]

(102)

Therefore, the circle \( C \) in the W-plane corresponding to the line \( C \) in the Z-plane goes through the points \( 1/2, I, \) and \( 13/4 + 3/4 i \sqrt{3} \). (Chart I.)

Set \( k = 1 \). This will give us the points in the W-plane corresponding to the points in the Z-plane after the lines of the Z-plane undergo the transformation \( z = i \frac{2w - 1}{w - 2} \).

To find the corresponding point of the point \( I \) in the Z-plane, we set \( z = I \). in the given transformation and solve for \( w \).

\[
1 = i \frac{2w - 1}{w - 2}
\]

(103)

\[
w = i \frac{2w - 1}{w - 2}
\]

(104)

\[
w = 4/5 + 3/5 i
\]

(105)
Therefore, circle A through the points $1/2, 2, \text{ and } 4/5$ plus $3/5 \, i$ corresponds to the line A of the Z-plane. (Chart II.)

The line B passes through the point $\omega$ in the Z-plane. Its corresponding point in the W-plane is

$$\omega = i \frac{2\omega - 1}{\omega - 2} \quad (106)$$

$$\omega = -\frac{5 + 3i}{3 \cdot 2^+} \quad (108)$$

Therefore, the line B of the Z-plane goes into the circle B in the W-plane, which passes through the points $1/2, 1, \text{ and } \frac{-5 + 3i}{3 \cdot 2^+}$. (Chart II.)

The line C of the Z-plane passes through the point $\omega^*$ in the Z-plane. Hence, its corresponding in the W-plane is

$$\omega^* = i \frac{2\omega - 1}{\omega - 2} \quad (109)$$

$$\omega = \frac{16.5 - 3i}{16.8} \quad (111)$$

The Circle C, therefore, passes through the points, $1/2, 2, \text{ and } \frac{16.5 - 3i}{16.8}$. (Chart II.)

In (63), $k = \omega^*$. We shall now find the corresponding
points under that transformation. The corresponding point of \( z = 1 \) is

\[
1 = \omega \frac{2\omega - 1}{\omega - 2}
\]

\[
\omega = \frac{2 - \omega}{1 - 2\omega}
\]

\[
\omega = \frac{13/4 + 3/4i\sqrt{3}}{14}
\]

Therefore, the circle \( A \) of the \( W \)-plane passes through the points \( 1/2, 2, \) and \( 13/4 + 3/4i\sqrt{3} \). The point corresponding to \( z = \omega \) is

\[
\omega = \omega \frac{2\omega - 1}{\omega - 2}
\]

\[
\omega = -1
\]

Hence the line \( B \) of the \( Z \)-plane goes into a circle which passes through the points \( 1/2, 2, \) and \(-1\). But all these points lie on the \( U \)-axis. Therefore, the line \( B \) goes into the \( U \)-axis, of the \( W \)-plane. (Chart III.)

\[
\omega' = \omega \frac{2\omega - 1}{\omega - 2}
\]

\[
\omega = 13/4 - 3/4i\sqrt{3}
\]
The point corresponding to the point \( z = \omega \), is the above. Therefore, the circle \( \mathcal{C} \) of the \( W \)-plane goes through the points \( 1/2, 2, \) and \( 13/14 - 3/14i \sqrt{3} \). (Chart III.)

Now, let \( u = \omega \). Then, the point corresponding to \( \tilde{z} = 1 \) is

\[
1 = \omega - \frac{2\omega - 1}{\omega - 2}
\]

Thus the circle \( \mathcal{A} \) of the \( W \)-plane goes through the points \( 1/2, 2, \) and \( 13/14 - 3/14i \sqrt{3} \). (Chart IV.)

The correspondent point of \( z = \omega \) is

\[
\omega = \omega + \frac{2\omega - 1}{\omega - 2}
\]

Therefore the circle \( \mathcal{B} \) of the \( W \)-plane passes through the points \( 1/2, 2, \) and \( 13/14 + 3/14i \sqrt{3} \). (Chart IV.)

The point in the \( W \)-plane corresponding to the point in the \( Z \)-plane is

\[
\omega = \omega + \frac{2\omega - 1}{\omega - 2}
\]

\[
\omega = -1
\]
Therefore, the line $C$ of the $Z$-plane goes into a circle in the $W$-plane, which passes through the points $1/2$, $2$, and $-1$. But all these points lie on the $U$-axis of the $W$-plane under transformation

$$z = w \frac{2w - 1}{w - 2} \quad (138)$$

We have now three points of every circle. The center is readily found from the Cartesian equation of the required circle. Therefore, we can easily plot the required curves. The question now is: What does the base circle go into under the various transformations? We shall answer that question in the following manner:

The equation of a unit circle in $z \bar{z}$ coordinates is

$$z \bar{z} = 1 \quad (139)$$

Taking the general transformation,

$$z = k \frac{2w - 1}{w - 2}$$

and substituting these values of $z$ and $\bar{z}$ into equation (129), we obtain

$$\frac{2}{w - 2} \frac{k \bar{w} - k}{\bar{w} - 2} = 1 \quad (130)$$

$$4k \bar{w} - 2k \bar{w} - 2k \bar{w} - k \bar{w} = 1 \quad (131)$$

But since all our $k$'s are on the unit circle.

Therefore, the above equation becomes

$$4 \bar{w} - 2(\bar{w} + w) + 1 - \bar{w} + 2(\bar{w} + w) - 3 = 0 \quad (132)$$
Hence, \[ wz = 1. \]

Therefore, the base circle is unchanged by a bilinear transformation regardless of the value of \( k \).

This concludes our transformations and graphing of the Apollonian circles. We shall now endeavor to see if we can find anything of interest about these Apollonian circles.

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CHAPTER V.
CERTAIN CHARACTERISTICS OF THESE
APOLLONION CIRCLES

Suppose we are given the equation of a unit circle about the origin. Then \( z \bar{z} = 1 \). \( (1.34) \)

Also assume that \( z_1 \) and \( z_2 \) are image points with respect to the unit circle. The equation of images then becomes \( z_1 \bar{z}_1 = 1 \). \( (1.35) \)

Thus \( z_1 = \frac{1}{z_2} \) and \( \bar{z}_1 = \frac{1}{\bar{z}_2} \). \( (1.36) \)

Hence, to find the image of a point \( z \) with respect to a unit circle we use the above relations e.g.

Suppose we want to find the image point of a given point \( \omega \), with respect to the unit circle. Then

\[
\omega \bar{z}_1 = 1
\]

\[
z_1 = \frac{1}{\omega} = \omega^{-1}
\]

But \( z_1 \) is the image point of \( z \), and not \( \bar{z}_1 \). Therefore

\[
z_1 = \omega^{-1}
\]

\[
\bar{z}_1 = \omega
\]

Thus, in the case of \( \omega \), \( z \) and \( z_1 \) are coincident images.

In general, suppose we are given a point \( z \). It is required to find its image \( z_1 \). Since
But in the case of a unit circle about the origin, for points on the circle, the equation is
\[ z \bar{z} = 1 \]
\[ z = \frac{1}{\bar{z}} \]

Since \( z \) is a point on the unit circle, it follows that
\[ \frac{1}{\bar{z}} = z \]

Substituting back in (140) we have
\[ \bar{z} = \frac{1}{z} \]

Therefore, any point on the unit circle about the origin - in this case our base circle - is its own image point with respect to the same circle. As is readily seen, the points on the outside of the circle are image points of the points on the inside of the circle. e.g.

Let
\[ z = \frac{1}{4} \]
\[ \bar{z} = 1 \]
\[ z = ? \]

Therefore, the image point of \( \frac{1}{4} \) with respect to the unit circle is \( 4i \).

Let us now consider the line A - i.e. the X-axis as the axis of reflection. Let \( z \) and \( \bar{z} \) be the image points. Since the equation of the line A is
\[ \bar{z} - z = 0 \]
then the equation of images with respect to the line A becomes

\[ y - y' = 0 \]

Suppose we are asked to find the image of the point with respect to the line A. Then

\[ \omega - y' = 0 \]
\[ \bar{z} = \omega \]
\[ \bar{z}' = \omega - \omega \]

(146)

Therefore, is the image of with respect to the line A.

If \( i \) is the point , then its image point \( \bar{z}' \) is found in the same way.

\[ i - \bar{z} = 0 \]
\[ \bar{z} = i \]
\[ \bar{z}' = i \]

(147)

Now the locus of the points on line B is given by the equation

\[ \omega z - \omega \bar{z} = 0 \]

(148)

Let us endeavor to find, if possible, a locus of the images of the points of line B with respect to line A.

Let

\[ \bar{z} = \omega z - \omega \bar{z} = 0 \]

(149)

Since

\[ z - z' = 0 \]

Since

\[ \omega z - \omega \bar{z} = \bar{z}' \]
\[ \bar{z} = \omega z - \omega \]
\[ \bar{z}' = \omega \bar{z} - \omega \bar{z} \]

(150)

\[ z' = \omega \bar{z} - \omega \bar{z} \]

(151)
But, the equation of the line of reference, the line A, is

\[ z - \tilde{z} = 0 \]

\[ \tilde{z} = \bar{z} \]

which is the locus of the images of the points on the line B. But this equation is the equation of the line C. Therefore, line C is the locus of the image points of the points on the line B with respect to the line A.

Geometrically, of course, it means that we take a point on the line B and draw a perpendicular to the line A. We extend this perpendicular as far below the X-axis as the length of the perpendicular from the point on the line B to the X-axis. The point below the X-axis from which the perpendicular distance to the X-axis is equal to the perpendicular distance from the point on the line B to the X-axis is termed the image point of the point on the line B. In this case it happens that all the points on the line B have their corresponding image points on the line C.

Now, let us take the line B as the axis of reflection. Its equation is

\[ \bar{z} \bar{z} - \omega \bar{z} = 0 \]  

Also let \( z \) and \( \tilde{z} \) again stand for the image points. The equation of the image points then becomes:

\[ \bar{\omega} \bar{z} - \omega \bar{z} = 0 \]
If we are given a point $z$ and are required to find its image. We proceed in a way exactly analogous to that used in the above discussed cases.

$$\omega - \omega z = 0$$
$$z = \overline{\omega}$$
$$z = \omega$$
$$\bar{z} = \overline{\omega} = \omega$$

Therefore, the image of the point $l$, with respect to the line $B$ is $\omega$.

In general, if we take the locus of the points on the line $A, z - \bar{z} = 0$, we shall again try to find the locus of the image points of this locus with respect to the line $B$ as the axis of reflection. Since the equation of images is

$$\omega z - \omega \bar{z} = 0$$
$$\bar{z} = \omega z$$
$$\bar{z} = \omega \bar{z}$$

Let

$$z - \bar{z} = \bar{z} (\bar{z} - z)$$

Then

$$z = \omega \bar{z} - \omega \bar{z}$$

Now - Since the line $B$ is the axis of reference

$$\omega z - \omega \bar{z} = 0$$
$$z = \omega \bar{z} = \bar{z}$$
Substituting in the previous equation
\[ z' = w' \bar{z} - w \bar{z} \]
Also from the general equation of line B
\[ \bar{z} = wz \]  \hspace{1cm} (159)
Therefore,
\[ z' = z - w \bar{z} \]  \hspace{1cm} (160)
This, the, is the locus of the images of the points on the line A with respect to the line B.

However, if we take the general equation of the line C
\[ \bar{z} = wz' \]
Since \( w' = w \) we have
\[ wz' - w \bar{z} = 0 \]
Factoring out
\[ z - w \bar{z} = 0 \]  \hspace{1cm} (161)
But this is the same as the equation for \( z' \) above.
Therefore, line C is the locus of the images of the points on the line A, with respect to the line B.

Suppose now, that the line C is the axis of reflection. Let \( z' \) and \( z' \) be the image points with respect to line C. Then the equation of images becomes
\[ \bar{z} = wz' \]  \hspace{1cm} (162)
If \( z \) is the given point, its image is
\[ \bar{z} = z / w \]
\[ z' = w \bar{z} \]  \hspace{1cm} (163)
Let represent the locus of points on the line A. Then the locus of the images of these points is


\[ z_r = \omega \bar{z} \]  
\[ z_r = \omega (\bar{z} - z) \]  
\[ z_r = \omega \bar{z} - \omega z \]  

Since line C is the axis of reflection
\[ \omega \bar{z} - \omega z = 0 \]  
\[ z = \omega \bar{z} \]  

Substituting this value of \( z \) in the above equation for \( z_r \) we have
\[ z_r = \omega \bar{z} - \omega \omega z \]  
\[ z_r = \omega \bar{z} - \omega z \]  

But \( \omega \bar{z} = z \) from (165)
Therefore
\[ z_r = z - \omega \bar{z} \]  

This is, therefore, the locus of the images of the points on the line A, with respect to the line C.

Now, taking the equation of the line B, we have
\[ \omega \bar{z} - \omega z = 0 \]  

Since
\[ \omega \bar{z} - \omega z = 0 \]  
\[ \omega \bar{z} - z = 0 \]  
Therefore
\[ z - \omega \bar{z} = 0 \]
But this is the same as equation (167). Therefore, lines A and B are loci of images with respect to line C.

Thus, in summary, we see that lines B and C are loci of image points with respect to line A, A, and C, with respect to B, and A, and B with respect to C.

Let us now take the general equation of a circle in \( z \bar{z} \) coordinates. This is:

\[
a z \bar{z} + b z + c \bar{z} + d = 0
\]

The equation of images with respect to any general circle then becomes

\[
a \bar{z} \bar{z} + b z + c \bar{z} + d = 0
\]

Now allow this circle to undergo a general linear transformation and find how it affects the equation of the image points. The general equation of transformation which we shall use is

\[
\bar{z} = \frac{q, w, + B}{\bar{w}, + B}
\]

Substituting the value of and in the general equation of images, we obtain

\[
A A w, w' + B w + B \bar{w}' + D D = 0
\]

Where

\[
A A = a q, q, + b q, \bar{q}, + b q, q, + a q, \bar{q},
\]

\[
B = a q, q, + b q, \bar{q}, + b q, q, + a q, \bar{q},
\]

\[
B = a q, q, + b q, \bar{q}, + b q, q, + a q, \bar{q},
\]

\[
D D = a q, q, + b q, \bar{q}, + b q, q, + a q, \bar{q},
\]
This is essentially the same form of an equation as the equation with which we started. Therefore, a linear transformation sends image points into image points.

Applying this knowledge to our Apollonian circles, we conclude that the circles which are loci of image points with respect to the remaining circle persist as loci of image points under the transformation

$$\frac{z}{\omega} = k \left( \frac{2\omega - 1}{\omega - 2} \right)$$

irrespective of the value of $k$ since that does not essentially change our transformation. Of course, in all this discussion we consider our straight lines as circles of infinite radius. Thus all that was said in regard to the general circle is true of straight lines.

As a side remark, it is interesting to note that as zero and infinity are image points in the $\mathbb{Z}$-plane, with respect to the unit circle, and are transformed into the points $1/2$ and 2 in the $\mathbb{W}$-plane, they are also image points in the $\mathbb{W}$-plane with respect to the unit circle.
CHART II.

I-PLANE

W-PLANE
The thesis "Certain Transformations of the Apollonian Circles on the Triangle $1, \omega, \text{ and } \omega^2$," written by Marion Joseph Kaminski, has been accepted by the Graduate School of Loyola University, with reference to form, and by the readers whose names appear below, with reference to content. It is, therefore, accepted as a partial fulfilment of the requirements for the Master's degree.

Rev. Francis J. Gerst, S.J.