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An Application of a Theorem of Borel on Natural Boundaries to the Theta-Zero Functions and Analogous Functions

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AN APPLICATION OF A THEOREM OF
BOREL ON NATURAL BOUNDARIES TO
THE THETA-ZERO FUNCTIONS AND
ANALOGOUS FUNCTIONS

By

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ments for the degree of Master of Arts.

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FOREWORD

In order to have the subject of Natural Boundaries in the proper perspective it will be necessary to precede it with a short treatment of the subject of analytic continuation. It will be our intention first to discuss analytic continuation from the viewpoint of infinite series and then to take up the subject of natural boundaries in relation to that method. For the sake of completeness the method of Schwarz which makes use of the principle of symmetry, has been included in this section on Analytic Continuation. We shall then record a theorem originally proven by Monsieur Emile Borel which establishes a sufficient condition upon the exponents of a power series so that there will exist a Natural Boundary.

After presenting Borel's theorem, we shall apply the results of this theorem to a class of functions known as the THETA-ZERO FUNCTIONS in an effort to demonstrate that the unit circle is the natural boundary of these functions. We shall then apply the results of his theorem to several functions which have been demonstrated to have the unit circle as a natural boundary in order to simplify the given proofs. The usefulness of this theorem in working out many of the problems on natural boundaries found in texts on the Complex Variable will be illustrated.

All footnotes will be found numbered consecutively at the end of this paper and immediately preceding the bibliography.
Section I

We shall discuss the question of Analytic Continuation by means of a Taylor Series; in the first part of its treatment we are particularly indebted to Goursat-Hedrick's work: Functions of a Complex Variable, p. 196 et seq. Let \( f(z) \) be an analytic function in a connected portion of the plane; if we know the value of \( f(z) \) and of all its successive derivatives at a point \( a \) in the region \( \mathcal{R} \) we may deduce the value of the function at some other point \( b \) in the same region.

Proof:
Join the points \( a \) and \( b \) by a path \( L \) which path lies entirely in \( \mathcal{R} \) (this path may be a broken line or any sort of curve). We shall take \( \delta \) to be the lower limit of the distance from any point of the path \( L \) to the boundary of \( \mathcal{R} \) so that a circle with a radius \( \delta \) whose center is on any point of \( L \) will lie entirely in the given region \( \mathcal{R} \).

According to the hypothesis made we know the value of the function \( f(a) \) and the values of its successive derivatives \( f'(a), f''(a), f'''(a), \ldots \) for the point \( z = a \). We can therefore write out the power series which represents the function \( f(z) \) in the neighborhood of the point \( a \):

\[
(1) \quad f(z) = f(a) + \frac{z-a}{1!} f'(a) + \frac{(z-a)^2}{2!} f''(a) + \cdots + \frac{(z-a)^n}{n!} f^{(n)}(a) + \cdots
\]

Because of the above hypothesis made concerning the distance of the path \( L \) from the boundary of \( \mathcal{R} \) the radius of the circle of convergence of this series is at least equal to \( \delta \) but it may be greater than \( \delta \) depending upon where the point \( a \) is situated. If there be a point \( b \) situated within the circle \( C_0 \) of the preceding series, we may replace \( z \) by \( b \) in order to have \( f(b) \).
If the point $b$ lies outside the circle $C_o$ we may let $\alpha$, be the point where the path $L$ leaves $C_o$. (Since it does not matter how we get to $b$ whether by one path or other as long as these paths are within the analytic region $A$ we shall suppose that $L$ cuts the circle $C_o$ in at most one point and all the succeeding circles in only two points.)

Let us select upon $L$ a point $\beta$ within $C_o$ and such that its distance from $\alpha$, be less than $\delta/2$. The series (1) and those obtained from it by successive differentiations enable us to determine the values of the function $\beta(\gamma)$ and all its successive derivatives: $\beta(\gamma), \beta'(\gamma), \ldots \ldots \beta^{(n)}(\gamma)$, \ldots, for $\gamma = \beta$. The coefficients of the series which represents the function $\beta(\gamma)$ in the neighborhood of the point $\beta$, are therefore determined if we know the coefficients of the first series (1), and we will have therefore in the neighborhood of the point $\beta$,

\[(2) \, \beta(\gamma) = \beta(\beta) + \frac{\gamma - \beta}{1} \beta'(\beta) + \ldots + \frac{(\gamma - \beta)^n}{ln} \beta^{[n]}(\beta) + \ldots \]
The radius of the circle of convergence of \( C \), of the series (2) is at least equal to \( S \); this circle contains then the point \( a \), within it, and there is also a part of the region of convergence outside of the circle \( C_0 \). If the point \( b \) lies within the circle \( C \), just found, we may put \( z = b \) in the series (2) in order to know the value of \( f(b) \).

Suppose, however, that the point \( b \) lies outside of \( C \), and let \( a_2 \) be the point where the path \( z_2, b \) cuts the circle \( C \). We select on the path \( L \) a new point \( z_2 \) within \( C \), such that the distance between the points \( z_2 \) and \( a_2 \) is less than \( S/2 \). The series (2) and those we obtain from it by successive differentiations will enable us to calculate the values of \( f(z) \) and its derivatives, \( f'(z), f''(z), f'''(z), \ldots \) at the point \( z_2 \). We shall then form the new series

\[
(3) \quad f(z) = f(z_2) + \frac{z - z_2}{1} f'(z_2) + \ldots + \frac{(z - z_2)^n}{n!} f^{(n)}(z_2) + \ldots
\]

The above series (3) represents the function \( f(z) \) in a new circle \( C_2 \), with a radius greater than or equal to \( S \). If the point \( b \) is within the circle \( C_2 \), we may replace \( z \) by \( b \) in the preceding series (3); if not, we may continue to apply the same process of continuation. At the end of a finite number of such operations we shall have a circle which contains the point \( b \) within it. (In the case of our figure the point \( b \) is found within the circle \( C_4 \).)

The reasoning just given shows that it is always possible, at least theoretically, to calculate the values of a function analytic in a region \( A \), provided that we know the sequence of values.
of the function, and its successive derivatives at a given point \( a \) of that region. From this we may conclude that a function analytic in a region \( \mathcal{A} \) is known in that region if we know it in a region, however small, about a point \( a \) of \( \mathcal{A} \), or even if we know it at all points of an arc of a curve, however short, ending in the point \( a \). We shall then say that the knowledge of the numerical terms of the sequence (4) determines an element of the function \( f(z) \). The result might be stated in the following manner: A function analytic in a region \( \mathcal{A} \) is completely determined if we know any of its elements. We can say further that two functions analytic in the same region cannot have a common element without being identical; for if they had a common element it would be possible by using that common element to cover the region by a process of analytic continuation and thus to obtain the same elements of the same function throughout the region. Further, if the function considered was not analytic throughout the whole of the region \( \mathcal{A} \), then we could employ the process of continuation as long as we were careful to select the path so that it would not include any of the singular points of the region; the above reasoning on the radius of the circle of convergence would then apply and that radius would be the distance from the path \( L \) to the nearest singular point. By so choosing the path \( L \) it is possible, at least theoretically, to continue the function \( f(z) \) so that the whole of the analytic region \( \mathcal{A} \) is covered by these overlapping circles of convergence.

From what has preceded we can conclude that it is possible to define an analytic function as soon as we know a single element of the function.
Let the region of existence of \( f_1(z) \) be \( A_1 \) and of \( f_2(z) \) be \( A_2 \), where \( A_1 \) and \( A_2 \) have the region \( A' \) in common. If in \( A' \) \( f_1(z) = f_2(z) \) then \( f_1(z) \) and \( f_2(z) \) are elements of a function, say \( F(z) \) such that \( F(z) = f_1(z) \) in \( A_1 \) and \( F(z) = f_2(z) \) in \( A_2 \) and \( f_1(z) = F(z) - f_2(z) \) in \( A' \). Moreover \( f_1(z) \) and \( f_2(z) \) are called analytic continuants of one another and \( F(z) \) is analytic in \( A_1 + A_2 \). We may also say that \( f_2(z) \) is the analytic extension into the region \( A_2 - A' \) of the analytic function \( f_1(z) \), which was given as defined only in the region \( A_1 \).

Let us now consider an infinite sequence of real or imaginary numbers

\[ a_0, a_1, a_2, a_3, \ldots \quad a_n, \ldots \]

subject only to the condition that

\[ a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \ldots + a_n z^n + \ldots \]

converges for some value of \( z \), which value differs from zero. In supposing that (6) converges for at least one value of \( z \) different from zero it follows that there is a circle of convergence \( C \), whose radius \( R \) cannot be zero since the argument is not equal to zero. (If the radius \( R \) be infinite, then the series is convergent for every value of \( z \) and the function is said to be an integral function of the variable.) If the radius \( R \) be finite and have a value that is not zero, then the series (6) given above is an analytic function in the interior of the circle of convergence \( C \).
In the given series (6) we know only the sequence of the coefficients of the powers of \( z \) and cannot from them make any direct statements regarding the nature of the function outside the circle \( C_0 \). We cannot say without investigation whether it will be possible for us to add to the circle \( C \), an adjoining region which with the original circle forms a connected region in which we have a function analytic in \( A \) and identical with \( f(z) \) in the interior of \( C_0 \).

In order to investigate this problem we shall make use of the method previously developed: Take in \( C_0 \) a point that is different from the origin and call this point \( a \). By means of (6) and the series obtained from it by term-by-term differentiation we can calculate an element of the function \( f(z) \) which corresponds to the point \( a \), and thus we may form the power series

\[
(7) \quad f(a) + \frac{z-a}{1!} f'(a) + \cdots + \frac{(z-a)^n}{n!} f^{(n)}(a) + \cdots
\]

The preceding power series represents the function \( f(z) \) in the neighborhood of the point \( a \); and the series (7) is certainly convergent in a circle about \( a \) as center with a radius equal at least to \( R-|a| \) but it may be convergent also in a larger circle whose radius may not, however, exceed \( R+|a| \).

It may so happen that the radius of the circle of convergence of the series (7) is always equal to but never exceeds \( R-|a| \), wherever the point may be selected within the circle \( C_0 \). There would then exist no means of extending the function \( f(z) \) outside of its original circle of convergence. There does not exist therefore a function \( f(z) \) analytic in a region \( A \) containing points exterior to the circle \( C_0 \), and which coincides with the function
\( f(z) \) inside the circle \( C_0 \). The circle \( C_0 \) is then said to be the natural boundary of the function \( f(z) \).

On the other hand let us suppose that it is possible to form about \( a \) as a center a circle of convergence for (7) such that the radius of that circle is greater than \( R-|a| \) but not greater than \( R+|a| \). This new circle of convergence \( C_1 \) would then have part of its region in common with \( C_0 \), and part of its region would extend outside \( C_0 \), and thus we would have an analytic extension of the function \( f(z) \) into a new region. (The following figure illustrates the cases just enumerated. The circle \( \gamma \) is the circle whose radius is not greater than \( R-|a| \), whereas the radius of \( C_1 \) exceeds \( R+|a| \).

![Figure 3](image)

Proceeding as we did in the first part of this paper, we are able to select points upon some path \( \mathcal{L} \) in such a manner that the whole of a region \( \mathcal{A} \) in which the function is analytic can be included in circles of convergence \( \mathcal{C}_i \), \( i = 1, 2, 3, 4, \ldots, n \). It is clear from what has preceded...
that we might in general select the path $L$ in a variety of ways in order to arrive at a given point $b$. It will be advisable for us to introduce an important distinction at this point. According to MacRobert: Functions of a Complex Variable: p.208:

"A particular point $b$ can be approached usually by different continuations from the point $a$; it is possible for the function to have different values at $b$. If the values are always the same no matter how we approach $b$, then the function is said to be uniform; otherwise it is said to be multiform."

We shall confine our remarks in the work that follows to functions which are uniform because the process of analytic continuation is reversible in the case of such functions.

At this point likewise it would be advisable for us to define in somewhat general terms what we mean by a singular point. We shall quote Zoretti from his monograph: Leçons sur le Prolongement Analytique:

P.32: "Arrivons maintenant à la définition des points singuliers de la fonction. Considérons un élément quelconque de la fonction, $P(z-a)$, convergent dans le cercle $C$ (et pas dans un cercle plus grand). Prenons un point quelconque $b$ sur la circonférence de $C$; ou bien il existe une série ordonnée suivant les puissances de $(z-b)$, prolongement de la première, ou bien il n'en existe aucune; dans le second cas, le point $b$ est un point singulier."

P.41: "On peut encore préciser ainsi cette définition. Pour que $a$ soit singulier, il faut et il suffit qu'il existe un chemin $z-a$ tendant vers $a$ seul et jouissant des propriétés suivantes: 1° tout point $z$ de chemin est le centre d'un élément de fonction, ces éléments constituant dans leur ensemble une même branche de fonction, c'est-à-dire se prolongeant les uns les autres; 2° le rayon de holomorphie de chaque élément est infiniment petit avec la distance $z-a$."

Let us consider an element of the analytic function, a power series in $(z-a)$ which we shall denote by $P(z-a)$and let the circle of convergence of this series be $C$. Select any point $b$ on the circumference of $C$; if there exists an ordered series following the powers of $P(z-b)$which is an analytic continuation of $P(z-a)$, then $b$ is not a singular point of the function; otherwise $b$
is said to be singular. The radius of the circle of convergence of a Taylor series about a singular point is zero and thus a singular point presents an obstacle to the process of analytic continuation.

There is on the circumference of every circle of convergence of a power series at least one singular point which singular point limits the extent of that circle. (Goursat-Hedrick p.202; Zoretti p.30; and others.) (If it so happens that the singular point is at infinity, we have what is termed an integral function.) This point is such that we cannot include it in the circle of convergence of the Taylor Series (or else the series could not be said to be holomorphic inside that circle), nor can we include that point in the path which we select to go from one point inside of $C_o$ to a point exterior to $C_o$. We may easily illustrate this by means of an example.

We shall consider the known function $f(z) = \frac{1}{1-z}$; this function permits of the usual power series expansion in powers of $z$:

$$f(z) = 1 + z + z^2 + z^3 + \cdots + z^n + \cdots \quad |z| < 1$$ (a complex quantity). (There is only one such expansion according to the uniqueness theorem which concerns expansions in a series of powers — a Taylor Series — of a given variable.) If it is desired to go from the original circle of convergence which obviously has a radius of unity, to the point, say $+2$ on the axis of reals, we may, of course, choose any path we want and then test to see if the function can be continued along that path. The most direct path would apparently be to go along the axis of reals by the process as outlined in the first part of this paper until the point $+2$ was included in one of the circles. An inspection of the original function, however, will show
that the point $+1$ is a singular point of that function and hence will be an impassable barrier to continuation along the path first selected. We may, however, select a path passing as near as desired to the point $+1$ but always remaining a finite distance from it; since $+1$ is the only singular point of the function we may apply the process of analytic continuation along any path which does not include that point. In attempting to prolong the func-

![Figure 4]

tion along a path which includes the point $+1$, we have an example of a circle whose radius is never greater than $R - |a|$; if we had selected, for instance, the point $-2$, we would have an illustration of a circle of convergence whose radius was $R + |a|$. The radii of all of the circles of convergence can easily be seen by inspection of the above figure.

In a similar manner we might have considered the function $\frac{1}{1 - z}$ which has not only the point $+1$, but also $-1$ as a singular point; no paths for analytic continuation may include either of the above singular points if we were to go beyond the original circle of convergence. All of the circles of
convergence for the successive elements of the function found by the preceding method pass through the nearer of the singular points. Goursat proves on page 205 of his text that in the case of multiform functions of such nature that we get different elements at a point by following different paths, there exists at least one singular point in the interior of the region which would be swept out by one of the paths. We need not concern ourselves with that work for we are interested chiefly in uniform functions.

It may happen that there are infinitely many singular points on the circle of convergence. Two cases would then present themselves: either the infinity of singularities are all on some finite length of arc of the circle of convergence, or else they are distributed uniformly over the whole of the circumference, so that in any portion of arc, however small, there are an infinite number of them. Each of the singularities presents a barrier to analytic continuation, but in the case where they are all confined to a finite part of the circumference we can continue the function outside its original circle by choosing a path that does not include any of those singular points. In the second case this cannot be done for in any portion of the circumference, however small, there is an infinity of singularities evenly distributed. In this case the original circle of convergence is a barrier beyond which the function cannot be continued, and it constitutes what we have previously termed the natural boundary of the function.

The following are some of the definitions given by various authors on the subject of natural boundaries:

Harkness and Morley: Introduction to the Theory of Analytic Functions, p.160, mention a case of a given function with a finite radius of
convergence but on every arc of which, however small, there is found an obstacle to analytic continuation. There is then no possible continuation across this circle which constitutes what is termed the Natural Boundary of the function.

Townsend: Functions of a Complex Variable, p.258: "It is possible in the process of analytic continuation to encounter a closed curve beyond which the function cannot be analytically continued. In such a case the curve is called a natural boundary. .... A portion of the complex plane into which the function cannot be continued is called a lacunary space."

Forsyth: Theory of Functions (2nd Edit.), p.144: "If a function be defined within the continuous region of a plane by an aggregate of elements in the form of power-series, which are continuations of one another, and if the power-series cannot be continued across the boundary of that region, the aggregate of elements in the region is a complete representation of a single uniform monogenic function which exists only for values of the variable in that region. "The boundary of the region of continuity of the function is, in the latter case, called the natural limit of the function, as it is a line beyond which the function cannot be continued."

Burkhardt-Rasor: Theory of Functions of a Complex Variable, p.391: "A closed line of singularities of the function is a natural boundary of the function."

For a curve to be a natural boundary it is not strictly essential that it be closed, although the definitions given by some authors would at first leave that impression. The essential feature of a natural boundary is that there be an unbroken sequence of singularities on the curve, that is, that any portion of arc, howsoever small, shall contain an infinite number of singular points of the function. This unbroken sequence of singularities presents an impassable barrier for analytic continuation, and thus it is a natural boundary. If the curve is not closed, then theoretically at least, the area on the other side of the curve may be covered with a series of overlapping circles by so choosing a path that will pass around the line of singularities. If, on the other hand, the curve is closed and contains
an infinite number of singularities evenly distributed over its length, there would be no path that could be chosen which would not contain at least one singular point as a barrier to the process of continuation.

Let us suppose that a curve satisfying the above requirement, namely that it contains an infinite number of singularities upon any portion of it, no matter how small, contains some point \( \beta \) which appears not to be a singular point. On either side of \( \beta \) and within any finite distance from it, however small, there will be an infinite number of singularities according to the hypothesis made about the distribution of the singularities. Hence the point \( \beta \) will be a limiting point of singularities, and hence must itself be a singular point. Therefore on any arc of curve which meets our above requirement there will be no points other than singular points, and we must have a natural boundary of the function inasmuch as it is impossible to continue the function over that portion of arc.

In his book, *Functions of a Complex Variable*, Goursat discusses the problem of building up a function with an arbitrary curve \( L \) as its natural boundary. His discussion is to be found on pages 210 et seq. of his text, and in the matter which follows his method of treatment will be more or less adhered to. We will consider a curve \( L \), closed or not, with the restriction that it have a definite radius of curvature at each point; let us have given likewise, a series of absolutely convergent terms \( \sum c_r \). In addition we shall take a sequence of points \( a_1, a_2, a_3, \ldots, a_r, \ldots \) such that they are all on the curve \( L \) and are distributed in such a manner that on any finite arc of this curve there will be an infinite number of
such points. The series \( F(z) = \sum_{r=1}^{\infty} \frac{cr}{a_r - z} \)
is convergent for every point \( z_0 \) not belonging to the curve \( L \) and represents an analytic function in the neighborhood of that point.

If the curve \( L \) is not closed and does not have any double points, then the series \( F(z) \) represents an analytic function over the whole extent of the plane except for the points of the curve \( L \). We are not yet able to conclude that the curve \( L \) is a natural boundary of the function for we have yet to show that analytic extension of \( F(z) \) is not possible across any portion of the curve \( L \). In order to show that it is not possible analytically to continue the function \( F(z) \) across the given curve it is proven that the circle of convergence of the power series representing \( F(z) \) in the neighborhood of \( z_0 \) can never enclose an arc of the curve \( L \), no matter how small the arc may be. His proof is as follows:

"Suppose that the circle \( C \), with the center \( z_0 \), actually encloses an arc \( \Delta \phi \), and on the normal to this arc at \( a_i \) let us take the point \( \zeta \)' so close to the point \( a_i \) that the circle \( C_i \), described about the point \( \zeta \)' as center with the radius \( |z' - a_i| \), shall lie entirely in the interior of \( C \) and not have any point in common with the arc \( \Delta \phi \) other than the point \( a_i \) itself. By the theorem which has just been demonstrated, the circle \( C_i \) is the circle of convergence for the power series which represents \( F(z) \) in the neighborhood of the point \( \zeta \)' But this is in contradiction to the general properties of power series, that the circle of convergence be smaller than the circle with the center \( \zeta \)' which is tangent internally to the circle \( C \)."  Loc. cit. p.210.

If the curve \( L \) is closed then \( F(z) \) represents two distinct functions, one of these functions being defined for the interior of the region bounded by \( L \) and the other for the exterior of the curve. Both of these functions would have the curve \( L \) for their natural boundary for that curve would represent an impassable barrier to analytic continuation of the function. An
example of such a function has been discussed by Weierstrass and his function is found in Forsyth: Theory of Functions, p.164, as well as in numerous other texts. His function is not of the exact form as the one given above but it is interesting to note that it has the unit circle as a natural boundary for the two parts of the function, one of which exists in the region of the plane exterior to that circle and the other in the interior of that circle. The function considered is

\[ f(z) = \sum_{n=0}^{\infty} \frac{1}{3^n + 3^{-n}} \]

In an example which follows his general discussion Goursat builds up an arbitrary function having a straight line as a natural boundary. No figure is given but one might be put in the following form:

![Figure 5](image)

His discussion follows:

"Let AB be a segment of a straight line, and \(a, \beta\) the complex quantities representing the extremities \(A, B\). All the points \(y = \frac{m+n\beta}{m+n}\) where \(m\) and \(n\) are two positive integers varying from 1 to \(\infty\) are on the segment AB, and on a finite portion of this segment there are always an infinite number of points of that kind, since the point \(y\) divides the segment AB in the ratio \(m/n\). On the other hand, let \(C_{m,n}\) be the general term of an absolutely convergent double series. The double series

\[ F(z) = \sum_{m+n\beta} \frac{C_{m,n}}{m+n} - z \]
represents an analytic function having the segment $AB$ for a natural boundary. We can, in fact, transform this series into a simple series with a single index in an infinite number of ways. It is clear that by adding several series of this kind it will be possible to form an analytic function having the perimeter of any given polygon as a natural boundary." **Loc. cit.,** p.211.

Thus it is possible to start out with an arbitrary line or curve and by means of a suitable calculation build up a function having that line as a natural boundary. Theoretically, at least, this method has much to recommend it; we will, however, center our attention upon a more practical condition that can be placed upon the coefficients of a power series. This condition will enable us to determine with very little calculation whether a given unit-circle is the natural boundary of a function.

In the preceding paragraphs we have considered the method of analytic continuation by means of power series; the importance of that method is limited only by the difficulties and the labor of applying it. It is of unlimited theoretical importance but in actual practice it is frequently replaced by some method easier of application and involving less labor. These other methods are not so valuable in theoretical discussions but they are of more practical importance in application. We shall now consider a method introduced by Schwarz which makes use of the principle of symmetry. (In the discussion which follows we have depended almost entirely upon the presentation of Schwarz’s method by Townsend: **Functions of a Complex Variable,** p.252 et seq.)

Let us consider a uniform function $\phi(z)$ which is holomorphic in a region $S$, lying in the upper half-plane and having a segment $AB$ of the axis
of reals as a part of its boundary. We shall allow \( z \) to approach any point \( \alpha \) of \( AB \) along any path that lies entirely in \( S \); then \( \phi_{\alpha}(z) \) approaches a definite real value \( \phi_{\alpha}(\alpha) \). Now \( \phi_{\alpha}(\alpha) \) is a continuous function of \( \alpha \) because it satisfies the usual conditions of continuity, namely 1st that it is defined at the point \( \alpha \), 2nd that it has a unique limit as the variable approaches \( \alpha \), and 3rd the value of the limit is equal to the value of the function at the point \( \alpha \).

Let \( \overline{\alpha} \) be the image of \( \alpha \) in the axis of reals. The assemblage of the \( \overline{\alpha} \) points constitutes a region \( S_2 \) which is symmetrical to the region \( S \), with respect to \( AB \), the real axis. Associate with each value of \( \overline{\alpha} \) a functional value which is the conjugate imaginary of \( \phi_{\alpha}(z) \). The assemblage of these values defines a function \( \phi_{\overline{\alpha}}(\overline{\alpha}) \) which function is holomorphic in \( S_2 \) and converges to the real values \( \phi_{\alpha}(\alpha) = \phi_{\overline{\alpha}}(\overline{\alpha}) \) along the axis of reals. (Cf. Figure 6)

![Figure 6](image)

In the continuous region \( S_1 + S_2 \) and the points along the axis of reals between the points \( A \) and \( B \), the functions satisfy the following condition:

**Theorem II.** Given two functions \( \phi_{\alpha}(\alpha) + \phi_{\overline{\alpha}}(\overline{\alpha}) \) which are holomorphic respectively in the adjacent regions \( S_1 + S_2 \) having an arc \( C \) of an ordinary curve as that portion of their boundaries common to the two. The necessary and sufficient condition that each of these functions is an analytic continuation of the other is that they converge uniformly to equal values on \( C \). (ibid. p.250)
since we say that $\phi_1(z)$ is an analytic continuation of $\phi_i(z)$.

Each of the functions $\phi_1(z) + \phi_2(z)$ are then elements of some function $f(z)$ which is holomorphic in the region $S$ which consists of the sum of the partial regions $S_1 + S_2$. Moreover, $f(z)$ takes the common values of the elements $\phi_1(z), \phi_2(z)$ along the axis of reals between $A$ and $B$. This method presents obvious advantages from the standpoint of ease of application for all that is necessary to effect an analytic continuation is to reflect the given region upon the X-axis, and then associate with the reflected region a function which is the conjugate imaginary function of $\phi_i(z)$.

It is possible to generalize the particular form of the expansion as given before by means of the following method: we may let the points of the portion $AB$ of the real axis correspond to the points of a regular arc $C$ of an analytic curve. To any point $t_0$ of $AB$ there corresponds a point $z = (x_0, y_0)$ of $C$, and since the two functions by which we can express $C$ are analytic, namely $x = \psi_i(t)$, $y = \psi_2(t)$, we may expand each in powers of $(t - t_0)$. The resulting series converge for all values of the variable within their circles of convergence, and hence we may give complex as well as real values to $t$. The letter $\tau$ may be used to denote these real and complex values:

$$z = \tau + i\eta = \psi_i(\tau) + i \psi_2(\tau) = \psi(\tau)$$

which is holomorphic and which has a derivative different from zero for all points of $AB$ with the exception possibly of the end points. The function $z = \psi(\tau)$ is then defined for a region $S$ of the $\tau$-plane which consists of the inner points of $AB$ and certain regions $S_1$ and $S_2$ lying symmetrically with
The region $S$ can then be said to have corresponding to it a region $S'$ which region consists of the points of $C$ and of the regions $S_1'$ and $S_2'$ lying on either side of $C$, in which the inverse function $\tau = \phi(\zeta)$ is uniquely determined and holomorphic. We may so restrict the region $S$ that the function $\zeta = \psi(\tau)$ and its inverse function $\tau = \phi(\zeta)$ map the regions $S, S'$ upon each other. (Cf. Figure 7)

![Diagram showing regions $S, S', S_1, S_2, S_1', S_2'$ and the mapping between them.]

Figure 7

To any conjugate complex points $\tau_1$ and $\tau_2$, lying respectively in $S_1$ and $S_2$ we have two associated corresponding points $\zeta_1$ and $\zeta_2$, lying in $S_1'$ and $S_2'$ and conversely. It is to be noted that the particular values of $\zeta$ thus associated depend upon the form of the curve $C$ and not upon the form of the parametric equations of the curve. We might replace $\tau$ in the preceding parametric equations by any other analytic function of a real variable, say $\varphi$. If then, we permit $\varphi$ to take complex values, conjugate complex points in the $\tau$-plane correspond to conjugate complex points in the $\varphi$-plane, and thus we get the same corresponding values of $\zeta$. 
Of the two \( \mathbf{z} \)-points corresponding to conjugate complex values of a parameter \( \tau \), either may be said to be a reflection or image of the other with respect to the curve \( C \). Since the regions \( S'_1 \) and \( S'_2 \) consist of these \( \mathbf{z} \)-points, then we may say that the region \( S'_1 \) is a reflection of the region \( S'_2 \) with respect to the curve \( C \).

This definition of reflection in a curve may be used to extend the idea of "analytic continuation by reflection in the real axis". Let \( S'_1 \) and \( S'_2 \) be two adjacent regions such that \( S'_2 \) is the reflection of \( S'_1 \) with respect to the regular arc \( C \) of an analytic curve with parametric equations

\[
\chi = \psi_1(t), \quad \mathbf{z} = \psi_2(t)
\]

and also given \( \phi_1(\mathbf{z}) \) which is a function holomorphic in \( S'_1 \) and defined for values of \( \mathbf{z} \) along the arc of the curve \( C \) which values are approached by any path lying wholly in \( S'_1 \). The following theorem gives the necessary and sufficient condition that \( \phi_1(\mathbf{z}) \) may be continued by reflection with respect to the arc \( C \):

"The necessary and sufficient condition that \( \phi_1(\mathbf{z}) \) may be analytically continued by reflection with respect to the regular arc \( C \) of an analytic curve forming a portion of the boundary of the region for which \( \phi_1(\mathbf{z}) \) is defined is that \( \phi(\mathbf{z}) \) converges uniformly to real values along \( C \)." Townsend, p.255.

We may now apply this method of analytic continuation to a function which is given as defined inside of the region \( S \) formed by three arcs of circles \( C_1, C_2, C_3 \) cutting a circle \( C \) at right angles:
Let \( \phi_1(z) \) be an element of the function \( f(z) \) and we shall suppose that \( \phi_1(z) \) satisfies the conditions laid down in the above theorem (cf. Figure 8). We shall start the reflection using any arc, say \( C \); having done this and having obtained the region \( S \), into which \( \phi_1(z) \) is continued analytically, let us continue this process with an arc of the obtained region, say \( C' \). If we continue this process indefinitely it is possible to enlarge the region \( S \) originally given so as to include in the limit the entire region bounded by \( C \).

Let us continue this process indefinitely in an effort to continue \( \phi_1(z) \) outside of the originally given circle \( C \). It is seen that this is not possible. Hence we say that the circle \( C \) is a natural boundary of the function \( f(z) \), which function is obtained by performing the process of reflecting the element \( \phi_1(z) \) and its continuants an infinite number of times until the whole of the interior of the circle \( C \) is covered with the elements of the function \( f(z) \). The circle \( C \) constitutes a natural boundary of \( f(z) \), because no matter how many times the process of reflection is performed, it is impossible to get over the circumference of \( C \).
If the method of Schwarz be used in analytically continuing the given
element of a chosen function we see that the natural boundary is easy to
recognize, provided that one exists. It would be the curve, closed or not,
over which it is impossible to pass, no matter how many times the process
of reflection is performed. With this treatment of Schwarz's method we con-
clude our exposition of Analytic Continuation.
Section II

Having considered the matter of analytic continuation and the subject of natural boundaries in a somewhat general way, we are now in a position to give a theorem of Borel's which lays down a sufficiency condition to be satisfied by the exponents of a given power series in order that that power series may have the unit-circle as its natural boundary. After we have given his theorem we shall apply its results to the solution of several problems and also to a determination of the natural boundary of the THETA FUNCTIONS. The following is the theorem of Monsieur Borel which was published in Liouville's Journal de Mathématiques Pures et Appliquées for 1896:

"Sur les séries de Taylor admettant leur cercle de convergence comme coupure; par M. Émile Borel.

"Étant donné une série de Taylor, il est intéressant de savoir si elle peut être prolongée en quelque manière au-delà de son cercle de convergence ou si ce cercle est une coupure; à cette question peut se ramener la suivante non moins importante: une fonction de variable réelle donnée par son développement en série trigonométrique est-elle ou non analytique? Il est bien clair que ces questions sont de la nature de celles dont on ne peut espérer une solution complète, c'est-à-dire permettant sûrement de traiter un cas particulier quelconque: tout ce que l'on peut espérer, c'est, d'une part, transformer la condition nécessaire et suffisante qu'il est aisé d'enoncer et lui donner une forme plus immédiatement applicable -- il ne faut pas se dissimuler cependant qu'une telle transformation analytique laisse subsister entières les difficultés inhérentes à chaque cas --; d'autre part, indiquer
des règles précises donnant les conditions, soit nécessaires, soit suffisantes, mais ne s'appliquant chacune qu'à des cas particuliers. Je me propose indiquer ici une de ces transformations et une de ces règles. Je ferai usage, pour y parvenir, de la théorie des séries divergentes sommables, que j'ai récemment développée dans ce même Journal; en employant les expressions que j'ai introduites dans cette théorie, on a l'énoncé très simple que voici:

Pour qu'une série de Taylor n'admette pas son cercle de convergence comme coupure, il est nécessaire et suffisant qu'elle soit sommable en quelque région extérieure à ce cercle.

"De cette proposition générale, j'ai déduit comme application, le théorème particulier suivant: La série \[ \sum a_n \chi^{C_n} \]
dans laquelle les exposants \( C_n \) sont des entiers croissants et les coefficients \( a_n \), des nombres quelconques, admet son cercle de convergence comme coupure si le rapport \( \frac{C_{n+1} - C_n}{\sqrt{C_n}} \) est, à partir d'un certain rang, supérieur à un nombre fixe \( s \).

(Several pages of Borel's article have been omitted because they do not appertain to the matter of this thesis. They were intended to show that frequently a series permits of analytic continuation beyond the original circle of convergence, material which was covered in the first part of this thesis.)

"Nous nous contenterons pour le moment de ces remarques sur les points singuliers en général et nous bornerons à étudier un cas simple dans lequel on peut affirmer que le cercle de convergence est une coupure.
"Considérons d'abord, pour plus netteté, la série bien connue

\[ 1 + z + z^4 + z^9 + \cdots + z^n + \cdots \]

La fonction entière est ici

\[ u(a) = 1 + \frac{a^3}{1} + \frac{a^6}{1} + \cdots + \frac{a^n}{1} \]

Nous allons montrer que, quel que soit l'argument de \( z \), si son module dépasse l'unité, on ne saurait savoir \( \lim_{a \to 0} e^{-a} u(a) = 0 \). Désignons par \( n \) le module de \( z \), supposé plus grande que un; nous verrons que l'on peut donner à \( \alpha \) une série de valeurs réelles augmentant indéfiniment et pour chacune desquelles \( e^{-a} u(a) \) est supérieure à un nombre fixe. Posons, en effet, \( \alpha = n^2 \).

"Le terme de \( u(a) \) qui a le plus grand module, est évidemment alors

\[ \frac{a^n z^n}{n^2} \]

et son module est \( \frac{(n^2)^n}{n^2} \).

Nous allons, en retranchant de ce module les modules de tous les autres termes, obtenir une limite inférieure du module de \( u(a) \). Nous considérerons successivement les termes qui le précèdent et ceux qui le suivent et nous utiliserons la formule d'approximation

\[ p = \frac{p^p e^{-p}}{\sqrt{2\pi} \, p} \]

ce qui est légitime lorsque \( p \) est assez grand, ce que nous pouvons supposer.

"Prenons d'abord le terme de plus grand module et négligeons partout le facteur \( \sqrt{2\pi} \); nous obtiendrons pour sa valeur approchée \( \frac{1}{n} e^{n^2} \). Supposons maintenant \( p < n \) et considérons le terme \( \frac{(n^2)^n}{p^2} \). En négligeant toujours le facteur \( \sqrt{2\pi} \), nous aurons la valeur approchée

\[ \left( \frac{n}{p} \right)^2 e^p \]

"On voit aisément que les termes pour lesquels \( p \) est petit par rapport à \( n \) sont négligeables. Par exemple, les \( \frac{n}{2} \) termes pour lesquels \( p < \frac{n}{2} \)
sont inférieurs à celui qu'on obtient en faisant \( p = \frac{n}{2} \) et, par suite, leur somme inférieure à \( e^{n\left(\frac{1}{2} \log 2\right)} \) expression bien plus petite que \( \frac{1}{n} e^{n^2} \) dès que n est un peu grand. Si nous supposons \( p \) compris entre \( \frac{n}{2} \) et \( n \), nous aurons

\[
\log \frac{n}{p} = \log \left(1 + \frac{n-p}{p}\right) = \frac{n-p}{p} - \frac{(n-p)^2}{2p^2} + \frac{(n-p)^3}{3p^3} - \ldots
\]

et par suite

\[
\left(\frac{n}{p}\right)^{2p} = e^{2p(n-p) - (n-p)^2 + \frac{2(n-p)^3}{3p}} - \ldots
\]

La valeur approchée (a) devient, par suite,

\[
\frac{1}{p} e^{n^2} e^{-z(n-p)^2 + \frac{z(n-p)^3}{3p}} - \frac{1}{p} e^{(n-p)^2} - \ldots
\]

et, en supposant \( n-p < p \), c'est-à-dire \( p > \frac{n}{2} \), elle est inférieure à

\[
\frac{1}{p} e^{n^2-(n-p)^2}.
\]

"Un calcul analogue donnerait le même résultat en supposant \( p > n \) ; nous en concluons que le module de \( u(a) \) pour \( qn = n^2 \) est supérieur au produit de \( e^{n^2} \) par l'expression suivante

\[
\frac{1}{n} - \left(\frac{e^{-1}}{n-1} + \frac{e^{-2}}{n-2} + \ldots + \frac{e^{-n}}{n-n} + e^{-\frac{1}{n+1}} \right) - \left(\frac{e^{-1}}{n+1} + \frac{e^{-2}}{n+2} + \ldots \right)
\]

et il est manifeste que, pour \( n \) assez grand, cette expression est supérieure à \( \frac{1}{10n} \). Nous avons \( qn = n^2 \) ou \( d = \frac{n}{n} \) et \( p \) est supérieur à un; donc

\[
e^{-a} u(a) > \frac{1}{10n} e^{n^2 (1 - \frac{1}{n})}
\]

et cette expression peut évidemment devenir aussi grande que l'on veut, quelque voisin que soit \( p \) de l'unité.

"Nous en concluons que la série considérée n'est sommable en aucun point extérieur à son cercle de convergence et admet par suite ce cercle comme coupure. Ce résultat particulier surait pu être établi de bien d'autres manières; mais la démonstration que nous avons donnée peut visiblement être étendue à des fonctions bien plus générales. Considérons d'abord la
fonction
\[ \phi(z) = 1 + z^{c_1} + z^{c_2} + z^{c_3} + \cdots + z^{c_n} + \cdots \]
et supposons \( C_{n+1} - C_n > 3 \sqrt{C_n} \); la même démonstration réussira à for- tiori puisque les termes qui précéderont ou suivront le terme d'exposant \( C_n \) dans \( U(a) \) seront, pour \( a \in C_n \), plus petits que les termes analogues dans la série considérée en premier lieu, pour laquelle on a \( C_{n+1} - C_n = 2 \sqrt{C_n} + \)
la fonction \( \phi(z) \) admettra, par suite, son cercle de convergence comme coupure.

"Supposons maintenant que l'on ait \( C_{n+1} - C_n > \frac{1}{k} \sqrt{C_n} \)
k étant un nombre entier quelconque, et posons \( z = y^{\frac{1}{k}} \). Nous aurons
\[ \phi(z) = \Psi(y) = \sum y^{qk'C_n} = \sum y^{c_n} \]
en posant \( C'_n = qk'C_n \).

"On aura d'ailleurs \( C'_{n+1} - C'_n > 9k' \sqrt{C'_n} > 3 \sqrt{C'_n} \)
La fonction \( \Psi(y) \) admet, par suite, son cercle de convergence comme coupure; il en est évidemment de même de la fonction \( \phi(z) \) sous la seule condition \( C_{n+1} - C_n > \varepsilon \sqrt{C_n} \)
\( \varepsilon \) étant un nombre positif quelconque.

"Considérons maintenant la série \( f(z) = \sum a_n z^{c_n} \) les \( a_n \) étant des nombres quelconques tels que le rayon de convergence soit égal à l'unité; nous supposons de plus \( C_{n+1} - C_n > 3 \sqrt{C_n} \) (il suffirait évidemment que l'on ait \( C_{n+1} - C_n > \varepsilon \sqrt{C_n} \)). Le rayon de convergence étant égal à l'unité, on sait que, ou bien la suite \( \left| \sqrt[n]{a_n} \right| \), \( \cdots \cdots \), a pour limite un, ou bien elle tend vers plusieurs limites différentes, et un est la plus grande de ces limites. Dans tous les cas, on peut trouver dans cette suite une suite partielle ayant pour limite un \( \) (Cf. Hadamard - Essai sur l'étude des fonctions, etc., Journal de Mathématiques, 1892, p. 106); il suffira de recommencer précédents, mais en prenant seulement pour
termes de plus grand module des termes dont les coefficients fassent partie
de cette suite partielle; il n'y aura aucune difficulté à montrer que la
série n'est sommable en aucun point extérieur à son cercle de convergence
et admet, par suite, ce cercle comme coupure."

It is now our intention to apply the preceding result to a determina-
tion of the natural boundary of the THETA FUNCTIONS. We shall employ the
definition of the Theta Functions as infinite series rather than as infinite
products, for the former will be easier to handle in one work. The follow-
ing are the definitions given in Pierpont: Functions of a Complex Variable,
p. 429:

\[ \theta_0(u) = 1 + 2 \sum_{n=0}^{\infty} (-1)^n \cdot q^{n^2} \cos 2\pi n u \]

\[ \theta_1(u) = 2 \sum_{n=0}^{\infty} (-1)^n \cdot q^{(n+\frac{1}{2})^2} \cdot \mu_n (2n+1) \pi u \]

\[ \theta_2(u) = 2 \sum_{n=0}^{\infty} q^{(n+\frac{1}{2})^2} \cdot \cos (2n+1) \pi u \]

\[ \theta_3(u) = 1 + 2 \sum_{n=0}^{\infty} q^{n^2} \cos 2\pi n u \quad |q| < 1 \]

It is known that the absolute value of the Trigonometric Functions of
the Sine or Cosine can never exceed unity for real values of the argument,
hence we will simplify the work if in the above series we consider \( \theta_0(0) \),
\( \theta_2(0) \), \( \theta_3(0) \), and \( \theta'_3(0) \), since \( \theta_i(0) \) will equal zero on account of
the presence of \( \sin 0^+ \). (The derivative of a power series has the same
circle of convergence as the original series, hence we do not affect the
circle of convergence.) In order to facilitate the reading of the functions
when they have a zero argument we shall write \( \theta_0 \) for \( \theta_0(0) \) , \( \theta'_3 \) for \( \theta'_3(0) \),
etc.

We have, then, the following four functions to be considered:
The only restriction we need make on the quantity $\theta$ is that it satisfy the following inequality:

$$|\theta| < 1$$

If this is not the case, then the series given above are not convergent series and hence do not define functions. For a series to converge it is necessary that $\theta^n \to 0$ as $n \to \infty$ i.e. that $\lim_{n \to \infty} \theta^n = 0$. If $|\theta| \geq 1$ then the necessary condition for the convergence of infinite series is not met. Therefore the point $+1$ is a singular point of the function if $\theta = 1$.

Hence we may say that the unit circle is the circle of convergence for the $\theta$ functions. (What applied to $\theta^n$ will a fortiori apply to $\theta^{(n + \frac{1}{2})^2}$.)

The fact that the unit circle is the circle of convergence of the functions is, however, not a proof that that same circle is a natural boundary for them, or that they cannot be continued by a process of analytic continuation beyond the unit circle. In order to establish this fact we shall restate Borel's theorem and then apply it to each of the functions individually:

"The series $\sum a_n x^{C_n}$ in which the exponents $C_n$ are increasing integers and the coefficients $a_n$ are any numbers admits its circle of convergence as a natural boundary if the relationship $\frac{C_{n+1} - C_n}{\sqrt{C_n}}$ is, with the exception of a certain range, superior to a fixed number $\kappa$."

(a) $\theta_0 = 1 + z \sum_{n=0}^{\infty} (-i)^n \theta^{(n + \frac{1}{2})^2}$

(b) $\theta_2 = 2 \sum_{n=0}^{\infty} \theta^{(n + \frac{1}{2})^2}$

(c) $\theta_3 = 1 + 2 \sum_{n=0}^{\infty} \theta^{n^2}$

(d) $\theta_4 = 2 \pi \sum_{n=0}^{\infty} (-i)^n (2n+1) \theta^{(n + \frac{1}{2})^2}$
We may apply this same theorem of Borel's to a series that has been considered by Lerch in a short article which appeared in Acta Mathematica, pp. 87-88. The series considered is of the form

\[ P(x) = \sum_{n=0}^{\infty} a_n x^{m_n} = a_0 x^{m_0} + a_1 x^{m_1} + a_2 x^{m_2} + \ldots \]

in which the exponents \( m_n \) are such that each term is a divisor of all the following terms. The coefficients \( a_n \) may be real or complex numbers, such that their real parts form a divergent series; the essential condition is that the above series be convergent. It is apparent from an inspection of \( P(x) \) that as \( x \to 1, P(x) \to \infty \). Therefore the point \( +I \) is a singular point and hence the unit circle is the circle of convergence.

Let us first examine the general case in light of Borel's theorem:

\[
\lim_{n \to \infty} \frac{c_{n+1} - c_n}{\sqrt{c_n}} = \lim_{n \to \infty} \frac{m_{n+1} - m_n}{\sqrt{m_n}}
\]

But \( \frac{m_{n+1}}{m_n} = l \) according to the imposed conditions.

\[
\lim_{n \to \infty} \frac{\ln m_n - m_n}{\sqrt{m_n}} = \lim_{n \to \infty} \frac{m_n (l-1)}{\sqrt{m_n}} = \lim_{n \to \infty} \sqrt{m_n} (l-1) \to \infty > k
\]

Since the coefficients satisfy the requirements of Borel's theorem, the unit circle is the natural boundary of the function.

In particular we might consider two series whose exponents are such that they satisfy the requirement that each one be a divisor of all the succeeding ones:
The unit circle is obviously the circle of convergence. Applying Borel's theorem:

In the case of \( P_r(x) \):

\[
\lim_{n \to \infty} \frac{b^{n+1} - b^n}{\sqrt{b^n}} = \lim_{n \to \infty} \frac{b^n \cdot b - b^n}{\sqrt{b^n}} = \lim_{n \to \infty} \frac{b^n(b-1)}{\sqrt{b^n}} = \lim_{n \to \infty} b^{n/2}(b-1) \to \infty > k
\]

In the case of \( P_2(x) \):

\[
\lim_{n \to \infty} \frac{(n+1)! - n!}{\sqrt{n!}} = \lim_{n \to \infty} \frac{n!(n+1) - n!}{\sqrt{n!}} = \lim_{n \to \infty} \frac{n!(n+1-1)}{\sqrt{n!}} = \lim_{n \to \infty} n \cdot \sqrt{n!} \to \infty > k
\]

Hence each of the above series has the unit circle as a natural boundary.

A particular case of the series \( P_r(x) \) is the series in which the value of the constant \( b \) is equal to 2. This series is considered by Curtiss: "Analytic Functions of a Complex Variable, p.155, and by Whittaker and Watson:
Modern Analysis, p. 98. The methods of proof used by the two authors are essentially the same, so I have appended only one of them, that of Whittaker. Given the series:

\[ f(z) = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + z^3 + \ldots + z^n + \ldots \]

Applying Borel's theorem to the series we have the following results which very simply demonstrate that the unit circle is the natural boundary of the function (it is apparent from a consideration of the series and of the immediately preceding footnote that the unit circle is the circle of convergence of the function):

\[
\lim_{n \to \infty} \frac{2^{n+1} - 2^n}{\sqrt{2^n}} \leq \lim_{n \to \infty} \frac{2^n(2-1)}{\sqrt{2^n}} = \lim_{n \to \infty} 2^{n/2} \to \infty < k
\]

Since the ratio is greater than any preassigned arbitrary positive constant, the conclusion follows that the unit circle is the natural boundary of the function.

We might now consider a somewhat general type of power series which is of the following form:

\[ f(z) = \sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n + \ldots \]

The coefficients \( a_n \) may be any numbers whatever, subject only to the condition that the above series converge; we shall determine what conditions must be placed upon the positive exponent \( c \) so that the above series shall have the unit circle for its natural boundary. It is apparent from a consideration of \( f(z) \) that it can be convergent, in general, only for values of \( z \) less than unity; the unit circle would then be the circle of convergence of the function and the point \( +1 \) a singular point of the function. Applying
Borel's ratio:

\[
\frac{(n+1)^c - n^c}{\sqrt{n^c}} = n^c + cn^{c-1} + \frac{c(c-1)}{2!} n^{c-2} + \cdots + \frac{c(c-1) \cdots (c-n+1)}{n-1} n^{c-n+1} + \cdots - n^c
\]

\[
= C n^{c-1-\frac{c}{2}} + \frac{c(c-1)}{2!} n^{c-2-\frac{c}{2}} + \cdots
\]

\[
= C n^{\frac{c}{2}-1} + \frac{c(c-1)}{2!} n^{\frac{c}{2}-2} + \cdots
\]

We see that if the power to which \( n \) is to be raised is equal to or greater than 2 the series will have its circle of convergence as a natural boundary.

In particular if \( c=1 \), and \( a_0 = a_1 = a_2 = \cdots = a_n = \cdots = 1 \) we will have the expansion which has the form \( \delta(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \), and which we considered earlier in this work. Applying Borel's theorem we see that the coefficients do not meet the sufficiency condition that \( \sum z^n \) has the unit circle as its natural boundary, a result which is in accord with other work since it is possible to continue the original element outside its circle of convergence:
Thus it is again clear that the unit circle is not the natural boundary of the function.

If we make $a_0 = a_1 = a_2 = \ldots = a_n = \ldots = 1$ and $c = 2$ we have the function that is considered by Borel in the first part of his theorem and which is very similar to two of the THETA FUNCTIONS with zero argument. The obtained series would likewise be similar to the series considered by Fredholm in Comptes Rendus Mar. 1890, pp. 627-29, and again in Acta Mathematica xv, p. 279, in which $c = 2$ and $a_n = a^n$ $|a| < 1$

i.e. $f(z) = \sum_{n=0}^{\infty} a^n z^n = 1 + a z + a^2 z^2 + a^3 z^3 + \ldots + a^n z^n + \ldots$

The circle of convergence is obviously the unit circle.

\[ \lim_{n \to \infty} \frac{(n+1)^2 - n^2}{\sqrt{n^2}} = \lim_{n \to \infty} \frac{2n+1}{n} = \lim_{n \to \infty} 2 + \frac{1}{n} = 2 \gt k \]

Hence we may conclude that the unit circle is likewise the natural boundary of the function.

This same result was obtained by Fredholm in an entirely different manner. 

In virtue of the preceding general example on page 32 concerning \( \sum b^n \) we can with little difficulty solve a problem proposed in Forsyth: Theory of Functions, p. 163:

"Prove that the function \( f(\chi) = \sum_{n=0}^{\infty} 2^{-n} \chi^3 \) exists only within a circle of radius unity and centre at origin." (Poincare)

The above series is convergent for all positive values of \( \chi \) less than unity, and as \( \chi \to +1 \), \( f(\chi) \to \infty \). Hence the point \( \chi = +1 \) is a singular point of the function; the unit circle is therefore the circle of convergence. Applying Borel's ratio:

\[
\lim_{n \to \infty} \frac{3^{n+1} - 3^n}{\sqrt{3}^n} = \lim_{n \to \infty} 3^{\frac{n}{2}} (3-1) = \lim_{n \to \infty} 2 \cdot 3^{\frac{n}{2}} \to \infty > k
\]

The unit circle is therefore not only the circle of convergence, but it is likewise a natural boundary of the above function.

As a final application of the results of this theorem we may use it in solving several of the problems found in Bromwich: Theory of Infinite Series: pp. 501-2:

79. Prove \( \sum x^n \) has a singularity on every arc of the unit-circle, however small; and that the function cannot be continued beyond that circle.

80. Shew that the functions \( \sum x^n \), \( \sum x^n! \) tend to \( \infty \) as \( \chi \) approaches the points \( e^{(2\pi b/4m)} \), \( e^{(2\pi b/m)} \) respectively along the radii. Deduce that these functions cannot be continued along the radii.

82. Assuming Borel's theorem (Example 83) prove that the function \( \phi(\chi) = \sum p^n \chi^n \) cannot be continued beyond the unit circle although \( \phi(\chi), \phi'(\chi), \ldots \) all converge absolutely for every point on the circumference. (Fredholm.)

84. Prove by means of example 83 (i.e. Borel's Theorem) that the function \( f(\chi) = \sum_{n=1}^{\infty} \chi^n \)
has the unit circle as a natural boundary.

It will be noticed that these problems are but specific types of the more general examples which were considered on the immediately preceding pages. It was found that Borel's theorem was applicable to each of them and we showed that the unit circle was the natural boundary of the functions considered. Examples 79, 82 and 84 are all of the same general type and can be handled as one since little emphasis is placed upon the coefficients as long as they are such that the series converges when \( \chi \) is less than unity.

The theorem we have just considered permits of wide applications, general types of which we have already indicated. Further individual applications could be made to suggested problems but it is felt that they are covered in the general matter of the theorem.

THE END
A function \( f(z) \) of a complex variable is said to be analytic in a connected region \( R \) of the plane if it satisfies the following conditions:

a) To every point \( z \) of \( R \) there corresponds a definite value of \( f(z) \).

b) \( f(z) \) is a continuous function of \( z \) when the point \( z \) varies in \( R \), that is, when the absolute value of \( f(z+h) - f(z) \) approaches zero with the absolute value of \( h \).

c) At every point \( z \) of \( R \), \( f'(z) \) has a uniquely determined derivative \( f'(z) \); that is, to every point \( z \) corresponds a complex number \( f'(z) \) such that the absolute value of the difference

\[
\frac{f(z+h) - f(z)}{h} \to f'(z)
\]

approaches zero when \( |h| \) approaches zero. Given any positive number \( \varepsilon \), another positive number \( \eta \) can be found such that

d) \( |f(z+h) - f(z) - hf'(z)| \leq \varepsilon |h| \)

if \( |h| \) is less than \( \eta \).

The term holomorphic likewise is used in the same sense as above.

Goursat-Hedrick: Functions of a Complex Variable, p. 11

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2. Proof that we need but a finite number of operations:

"We can always choose the points \( z, z', z'', \ldots \) in such a way that the distance between any two consecutive points shall be greater than \( \frac{1}{2} \). On the other hand let \( S \) be the length of the path \( L \). The length of the broken line

\[
z, z', z'', \ldots, z_p\]

is always less than \( S \); hence we have \( p \frac{1}{2} + |z_p - b| < S \).

Let \( p \) be an integer such that \( (\frac{1}{2} + 1) \frac{1}{2} > S \). The preceding inequality shows that after \( p \) operations, at most, we shall come upon a point \( z_p \) of the path \( L \) whose distance from the point \( b \) will be less than \( \delta \); the point \( b \) will be in the interior of the circle of convergence \( C_p \) of the power series which represents the function \( f'(z) \) in the neighborhood of \( z_p \), and it will suffice to replace \( z \) by \( b \) in this series in order to have \( f(b) \). In the same way all the derivatives \( f'(b), f''(b), \ldots \) can be calculated."

Goursat-Hedrick, p. 197

3. An analytic function is defined by an aggregate of series composed of a primary series and its continuations; the separate series are called elements and the primary series the primary element.

Harkness-Morley, p. 154

4. If (7) were convergent in a circle of radius \( R+\alpha+S \), the series (4) would then be convergent in a circle of radius \( R+S \) about the origin as
center, which is contrary to the hypothesis that we made originally in which the radius was said to be R.

5. This usage is in keeping with that of Zoretti:
Leçons sur le Prolongement Analytique, p. 32:
"Ou bien la valeur de fonction en chacun des points de son domaine d'existence sera toujours la même quelque soit le chemin suivi pour atteindre ce point; en d'autres termes, la fonction n'aura qu'une valeur en chaque point; on dira qu'elle est uniforme. Ou bien il existe au moins un point pour lequel la fonction n'a pas la même valeur suivant le chemin qui aboutit en ce point; elle est multiforme. Ce dernier cas se subdivise encore en deux: ou bien le nombre des valeurs de la fonction en chaque point est borné, ou bien il ne l'est pas."

6. "An essential singularity may be contrasted with an ordinary singularity as follows: If the reciprocal of the function has a point for an ordinary point, this point is a pole, that is this point is a zero for the reciprocal of the original function; but when the value of the reciprocal of the function is not determinate at the point, then the point is an essential singularity of the function.
"An essential singularity is a point in whose neighborhood the function comes arbitrarily near to an arbitrary value an infinite number of times."
Burkhardt-Rasor: Theory of Functions of a Complex Variable p. 380

7. "If the coefficients of a power series \( \sum a_n x^n \) are all positive, (at any rate after a certain stage), the series has a singular point at the point \( x = R \), if \( R \) is the radius of the circle of convergence.
"The following proof is due to Landau:
"Suppose, if possible, that (for \( 0 < \rho < R \)) the series \( \sum \frac{\beta_n(x^n)}{n!} \) has a larger radius of convergence than \( R - \rho \); we can choose a real number \( \rho(> R) \) such that the last series converges for \( x = \rho \). Now this series (as in Art. 85) can be arranged as a double series which contains only positive terms; it will therefore remain convergent when summed as \( \sum a_n [n + (\rho - n)] \) .
That is, \( \sum a_n x^n \) will converge for \( x = \rho \), contrary to the original hypothesis; and so \( x = R \) must be singular point.
Bromwich: Infinite Series p. 253
8. By an analytic curve is meant one whose parametric equations are of the form \( X = \psi_i(t) \), \( t \) where the functions are real analytic functions of the real variable \( t \). An arc of such a curve would be regular if we added the condition that the derivatives \( \psi_i'(t), \psi_i'(t) \) are not simultaneously zero; that is if
\[
[\psi_i'(t)]^2 + [\psi_i'(t)]^2 \neq 0 \quad t_R < t < t_B
\]
Townsend, p. 253

9. In order to obtain \( \theta(t) \), it will be advantageous to use the infinite product definition of that Theta Function; the reader may find the work in Pierpont, p. 431.

10. Any numbers whatever subject only to the condition that the above series be convergent.

11. "Un Théorème de la Théorie des Série"
Extrait d'une lettre adressée à M. Mittag-Leffler par M. Lerch à Vinohrady.

"Soit donnée une série de nombres entiers positifs
\[ m_1, m_2, m_3, m_4, \ldots \]
dont chaque terme est un diviseur de tous les suivants, et soient
\[ c_1, c_2, c_3, c_4, \ldots \]
des quantités complexes dont les parties réelles sont respectivement
\[ \psi_1, \psi_2, \psi_3, \psi_4, \ldots \]
et qui sont supposées positives et telles que la série \( \sum c_r \) soit divergente.
Alors dans tous les cas où la série
\[
P(x) = \sum_{r=0}^{\infty} c_r x^{m_r}
\]
sera convergente pour chaque valeur de \( x \) moindre en valeur absolue que l'unité, elle définira une fonction de la variable \( x \) n'existant qu'à l'intérieur du fondamental cercle \( |x| \leq 1 \).

Car en posant
\[
X = e^{\pi i (\frac{z a}{m} + \omega)}
\]
ou $a$ est un nombre entier et $\alpha$ une quantité réelle et positive on a:

$$P(x) = \sum_{r=0}^{s-1} C_r e^{\frac{2a_m}{m^2} \pi i r - \alpha \pi m r} + \sum_{r=s}^{\infty} C_r e^{-\alpha \pi m r}$$

Or la série $\sum_r$ étant divergente et se composant de termes positifs il est aisé de voir que

$$\lim_{\alpha \to 0} \sum_{r=s}^{\infty} C_r e^{-\alpha \pi m r} = +\infty$$

d'où l'on a aussi

$$\lim_{\alpha \to 0} C_r e^{-\alpha \pi m r} = \infty$$

et par conséquent

$$\lim_{\alpha \to 0} P(e^{\pi i (\frac{2a}{m} + \alpha i)}) = \infty$$

Donc la fonction $P(x)$ croît indéfiniment quand $x$ s'approche d'une certaine manière des quantités de la forme $e^{\frac{2a}{m} \pi i}$ qui se présentent dans chaque partie de la circonférence $|x|=1$. Par conséquent, cette ligne-ci est une ligne singulière de la fonction $P(x)$.

Acta Mathematica x, pp.87-88. 1887

12. On a function to which the process of analytic continuation cannot be applied outside of the unit circle:

"Given the function

$$f(z) = 1 + 3^2 + 3^3 + 3^5 + 3^7 + \ldots + 3^{2^n} + \ldots$$

which converges in the interior of the unit circle with the center as the origin since it is clear that as $z \to 1-0$, $f(z) \to \infty$ and therefore $1$ is a singularity of $f(z)$.

But $f'(z) = \frac{f(z)}{z}$

and if $z \to 1-0$, $f'(z) \to \infty$ and so $f(z) = \infty$

Therefore the points for which $z = 1$ are singular; the point $-1$ is therefore a singular point of the function also.

Similarly since $f(z) = z^2 + z^3 + z^4 + \ldots$

we see that if $z^2 = 1$, then $z$ is a singular point of $f(z)$; and in general any root of the following equations

$$z^3 = 1, z^4 = 1, z^6 = 1, z^8 = 1, \ldots$$

is a singularity of $f(z)$. But all these points lie on the circle $|z| = 1$ and on any arc, however small there are an infinity of them. Any attempt to continue the function beyond the unit circle will therefore be frustrated by the existence of an unbroken line of singularities, beyond which it is impossible to pass.

In such a case $f(z)$ cannot be continued to any point outside the circle $|z| = 1$; such a function is a lacunary function, AND THE UNIT circle is the limiting circle of the function."

Whittaker and Watson, p. 98.
13. The above discussion is also verified in some work given by Zoretti:
Examinons rapidement ces exemples devenus classiques. Le premier est celui de Weierstrass relatif à la série:
\[ f(z) = \sum a_n z^n \]
qui définit dans le cercle \( |z| < 1 \) une fonction analytique. Si l'on examine la valeur de \( f(z) \) sur la circonférence de ce cercle, on trouve que c'est une fonction continue de l'arc dépourvue de dérivée (au moins si \( ab < 1 \)). Ce cercle est donc coupure essentielle de la fonction. M. Hadamard a étendu ces propriétés aux séries
\[ f(z) = \sum a_n z^n \]
dans lesquelles les entiers \( b_n \), pour les valeurs de l'indice supérieures à un certain entier \( p \), ont un plus grand commun diviseur qui croit indéfiniment avec \( p \). Le raisonnement est assez simple pour être reproduit: la fonction présente nécessairement un point singulier \( z_0 \) sur le cercle de convergence. Supprimons alors les \( p \) premiers termes de la série, ce qui revient à négliger une fonction holomorphe et par suite ne change pas les points singuliers. La nouvelle série, \( \phi(z) \), ne change pas si on change \( z \) en \( z \cdot e^{\frac{2\pi i}{p}} \), désignant le diviseur commun aux \( b_n \). Donc la fonction \( \phi \), et par suite \( \phi' \), admettent, outre le point \( z_0 \), les points singuliers
\[ z_0 \cdot e^{\frac{2\pi i}{p}} \]
c'est-à-dire, puisque \( \phi \) peut grandir indéfiniment, un ensemble partout dense de points du cercle et par suite tout le cercle."
Zoretti: Lecons sur le Prolongement Analytique p.91

14. "Permettez-moi de vous exposer un resultant assez remarquable qui a été trouvé par un de mes élèves, M. Fredholm; je vous prie de la communiquer aux Comptes Rendus, si vous trouvez cela opportun."
"Autant que je sache, toutes les fonctions qui n'existent que dans un certain domaine du plan et qui ont été étudiées jusqu'ici cessent d'exister parce que les fonctions elles-mêmes ou leurs dérivées deviennent discontinues sur la frontière. M. Fredholm a trouvé, dans un des champs les plus connus de l'Analyse, une fonction qui est continue, ainsi que toutes ses dérivées, sur toute la frontière qui limite le domaine d'existence de la fonction."
"Écrivez la fonction \( \theta \) sous la forme
\[ \sum_{v < z} e^{\nu t + \nu u} = \sum_{v < z} e^{\nu t + \nu u} + \sum_{v > z} e^{\nu t + \nu u} \]
et mettez
\[ \phi(t,u) = \sum_{v > z} e^{\nu t + \nu u} \]
Si la partie réelle de \( u \) est négative, la fonction est une fonction uniforme de \( t \) pour toutes les valeurs de \( t \), dont la partie réelle soit
negative. La Fonction, ainsi que toutes ses dérivées, sont des fonctions continues de $t$ sur l'axe imaginaire. Mais cet axe imaginaire forme la limite du domaine d'existence de la fonction. Pour voir cela, vous n'avez qu'à faire l'observation que la fonction $\phi(t,u)$ satisfait à l'égalité

$$\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial u^2},$$

et de mettre

$$\phi(t,u) = p(t-t_0)$$

$$= \phi(t_0,u) + \left( \frac{\partial \phi}{\partial t} \right)_{t=t_0} \frac{t-t_0}{u} + \left( \frac{\partial^2 \phi}{\partial t^2} \right)_{t=t_0} \frac{(t-t_0)^2}{u^2} + \ldots$$

$$= \phi(t_0,u) + \left( \frac{\partial^2 \phi}{\partial u^2} \right)_{t=t_0} \frac{t-t_0}{u} + \left( \frac{\partial^3 \phi}{\partial t^2 \partial u} \right)_{t=t_0} \frac{(t-t_0)^2}{u^2} + \ldots$$

où $t_0$ est une point sur l'axe imaginaire.

"D'après la théorème connu du Mme Kowalevski la série $p(t-t_0)$ ne peut être convergente à moins que $\phi(t_0,u)$ soit une fonction entière rationnelle ou transcendante de $u$. Cela n'a pas lieu, et la fonction $\phi(t,u)$ considérée comme fonction de $t$ n'existe donc, pourvu que $u$ soit une constante dont la partie réelle est négative, qu'à l'intérieur du domaine: partie réelle de $t > 0$.

"En mettant $e^t = x$, $e^x = a$, $|a| < 1$ vous obtenez un fonction de $x$ :

$$\sum_{r=0}^{\infty} a^r x^r$$

qui n'existe pour $|x| < 1$ et qui reste continue, ainsi que toutes ses dérivées, pour $|x| = 1$.

"Il est facile de voir qu'on peut beaucoup généraliser ce resultant obtenu par M. Fredholm."

Comptes Rendus Mar. 1890 p. 627
Acta Mathematica 1891 p. 279

N.B. Some corrections were made to the article which appeared in Comptes Rendus because of printing mistakes; the article given above embodies the corrections taken from Acta Mathematica.

Mme Kowalevski's article appeared in Crelle: Journal für Mathematik
lxxx. pp. 1-32
Goursat-Hedrick: Functions of a Complex Variable.

Pierpont: Functions of a Complex Variable.

Townsend: Functions of a Complex Variable.

Burkhardt-Rasor: Theory of Functions of a Complex Variable.

Curtiss: Analytic Functions of a Complex Variable.


Bromwich: Theory of Infinite Series.

Whittaker and Watson: Modern Analysis.


Zoretti, Ludovic: Leçons sur le Prolongement Analytique Professées à Collège de France.

Borel, Émile: Méthodes et Problèmes de Théorie des Fonctions.

MacRobert: Functions of a Complex Variable.

Pompieu: Sur la Continuité des Fonctions de Variables Complexes.

Comptes Rendus: March, 1890, p.627.

Comptes Rendus, October, 1896, p.548.

" " December, 1896, p.1051.

Liouville's Journal de Mathématiques Pures et Appliquées, 1896, p.441-61.


The following articles and books also contained information dealing with the subject matter of this paper. They were not used directly in its writing, however, although they were consulted.

Koch: Contributions à la Théorie du Prolongement d'une Fonction Analytique.

Klein-Fricke: Elliptic Modular Functions.

Carlson: Sur le Prolongement Analytique.


Borel: Leçons sur la Théorie de Fonctions.

Wilson: Advanced Calculus.

Ford: Automorphic Functions.

Crelle: Journal für Mathematik lxxx, pp.1-32.

Crelle: " " lxx, pp. 106-07.

Crelle: " " xcii, pp. 62-77.

Annales Scientifiques de L'École Normale, 1896, p. 367 et seq.

Liouville's Journal de Mathématiques Pures et Appliquées, 1892, p.102 etc.

Liouville's " " " " " " 1898, p. 317 etc.


Acta Mathematica xxii, p.65 et seq.

Acta Mathematica xxvii, p.70 et seq.

Comptes Rendus xci, pp.690-92

Comptes Rendus xciv, pp. 715-18

Comptes Rendus xcvi, pp. 1134-36

Comptes Rendus, December, 1895, p.1125.

Comptes Rendus, January, 1896, p.73.

" " April, 1896, p. 805.

Comptes Rendus, November, 1898, pp.607, 711, 751.

" " December, 1898, pp. 794, 1001.
The thesis "An Application of a Theorem of Borel on Natural Boundaries to the Theta-Zero Functions and Analogous Functions," written by Louis William Tordella, has been accepted by the Graduate School of Loyola University, with reference to form, and by the readers whose names appear below, with reference to content. It is, therefore, accepted as a partial fulfilment of the requirements for the degree of Master of Arts.

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